Natural Infinitesimals in Filter-Powers

Eduardo Ochs
`http://angg.twu.net/`

Start from the standard universe, **Set**, and construct the the universe of sequences", $\mathbf{Set}^{\mathbb{N}}$, and then the "semi-standard universe", $\mathbf{Set}^{\mathbb{N}}/\mathcal{N}$, in which the quotient by the filter of cofinite sets of naturals, "$/\mathcal{N}$", identifies sequences which differ only on finite sets of indices. Now generalize this a bit: a *filter-power*, $\mathbf{Set}^{\mathbb{I}}/\mathcal{F}$, is a universe of $\mathbb{I}$-indexed sequences modulo a quotient that identifies sequences when they coincide on sets of indices that are "$\mathcal{F}$-big".

If we substitute the filter $\mathcal{F}$ above by a (non-principal) ultrafilter $\mathcal{U}$ we get a "non-standard universe" (or: an "ultrapower"), $\mathbf{Set}^{\mathbb{I}}/\mathcal{U}$, whose logic is very close to the one of **Set** — it has exactly two truth-values — but in a $\mathbf{Set}^{\mathbb{I}}/\mathcal{U}$ we have infinitesimals (the equivalence classes of $\mathbb{I}$-sequences tending to 0), and we can use the "transfer theorems" of Non-Standard Analysis to move truths back and forth between **Set** and $\mathbf{Set}^{\mathbb{I}}/\mathcal{U}$.

Non-principal ultrafilters cannot be constructed explicitly, and to show that they exist we need the boolean prime ideal theorem, that is slightly weaker than the axiom of choice; this makes the infinitesimals of NSA quite hard to understand intuitively. On the other hand, the infinitesimals in a semi-standard universe like $\mathbf{Set}^{\mathbb{N}}/\mathcal{N}$ or $\mathbf{Set}^{\mathbb{R}}/\mathcal{R}_0$, where $\mathcal{R}_0$ is the filter of neighborhoods of $0 \in \mathbb{R}$, are very simple to describe — but the logic of a filter-power has more than two truth values.

We will show how "strictly calculational" proofs in NSA involving infinitesimals can be lifted through the quotient $\mathbf{Set}^{\mathbb{I}}/\mathcal{F} \to \mathbf{Set}^{\mathbb{I}}/\mathcal{U}$; and then, by choosing the right $\mathbb{I}$ and $\mathcal{F}$, and by using the "natural infinitesimals" — that are identity maps in disguise, modulo $\mathcal{F}$ — we get a straightforward translation of these strictly calculational proofs with infinitesimals into standard proofs in terms of limits and continuity.

One "archetypical example" will be discussed in detail: $\forall\omega \sim \infty\ \exists!\mathbf{o} \sim 0\ (1+\frac{1}{\omega})^{\omega} = e^a + \mathbf{o}$, where ω is an infinitely big natural number. The presentation should be accessible to people with basic knowledge of Calculus, Analysis, and Topology.