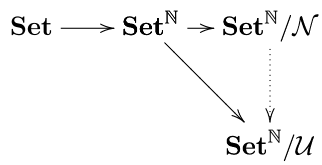
Index of the slides:

Non-Standard Analysis
The main idea:
Set is the “standard universe”,
$\mathbf{Set}^{\mathbb{N}}$ is the “universe of ($\mathbb{N}$-)sequences”,
$\mathbf{Set}^{\mathbb{N}}/\mathcal{N}$ is the “universe of $\mathbb{N}$-sequences modulo $\sim_{\mathcal{N}}$”,
$\mathbf{Set}^{\mathbb{N}}/\mathcal{U}$ is the “universe of $\mathbb{N}$-sequences modulo $\sim_{\mathcal{U}}$”,
where $\sim_{\mathcal{N}}$ is the equivalence relation induced by the filter $\mathcal{N}$,
and $\sim_{\mathcal{U}}$ is the equivalence relation induced by the ultrafilter $\mathcal{U}$,
where $\sim_{\mathcal{U}}$ has bigger classes than $\sim_{\mathbb{N}}$.



$\mathbf{Set} \to \mathbf{Set}^{\mathbb{N}}$ takes 4 to $(4, 4, 4, 4, \ldots)$,
$\mathbf{Set}^{\mathbb{N}} \to \mathbf{Set}^{\mathbb{N}}/\mathcal{N}$ takes $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$ to $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)/\mathcal{N}$, and
equivalence classes of sequences tending to zero will
behave as infinitesimals.

$\mathbf{Set}^{\mathbb{N}}/\mathcal{U}$ is a “non-standard universe”.
$\mathbf{Set}^{\mathbb{N}}$ and $\mathbf{Set}^{\mathbb{N}}/\mathcal{U}$ are quite similar —
they both obey the same first-order formulas (!!!)
(with bounded quantifiers and all constants standard)
and we have “transfer theorems” that let us “transfer truths”
from **Set** to $\mathbf{Set}^{\mathbb{N}}/\mathcal{U}$ and back.
And $\mathbf{Set}^{\mathbb{N}}/\mathcal{U}$ has infinitesimals!!!

Non-Standard Analysis (2)

The general case:
Set is the "standard universe",
$\mathbf{Set}^{\mathbb{I}}$ is the "universe of ($\mathbb{I}$-)sequences",
$\mathbf{Set}^{\mathbb{I}}/\mathcal{F}$ is the "universe of $\mathbb{I}$-sequences modulo $\sim_{\mathcal{F}}$",
$\mathbf{Set}^{\mathbb{I}}/\mathcal{U}$ is the "universe of $\mathbb{I}$-sequences modulo $\sim_{\mathcal{U}}$",
where $\sim_{\mathcal{F}}$ is the equivalence relation induced by the filter $\mathcal{F}$,
and $\sim_{\mathcal{U}}$ is the equivalence relation induced by the ultrafilter $\mathcal{U}$,
where $\sim_{\mathcal{U}}$ has bigger classes than $\sim_{\mathcal{F}}$.

$$\begin{array}{ccccc} \mathbf{Set} & \longrightarrow & \mathbf{Set}^{\mathbb{I}} & \longrightarrow & \mathbf{Set}^{\mathbb{I}}/\mathcal{F} \\ & & & \searrow & \vdots \\ & & & & \mathbf{Set}^{\mathbb{I}}/\mathcal{U} \end{array}$$

$\mathcal{F}$ is a filter on the index set $\mathbb{I}$,
$\mathcal{U}$ is an ultrafilter on $\mathbb{I}$, refining $\mathcal{F}$ (i.e., $\mathcal{F} \subset \mathcal{U}$).

Filters
Definition: $\mathcal{F} \subseteq \mathcal{P}(\mathbb{I})$ is a filter on $\mathbb{I}$ iff:

(i) $\mathbb{I} \in \mathcal{F}$,
(ii) $\mathcal{F}$ is closed by binary intersections,
(iii) $\mathcal{F}$ is "closed by supersets".

Our two archetypical filters:

$$\begin{array}{l} \mathcal{N} \subset \mathcal{P}(\mathbb{N}) \\ \mathcal{N} := \{\, I \subset \mathbb{N} \mid \mathbb{N} \backslash I \text{ is finite} \,\} \\ \mathcal{R}_0 \subset \mathcal{P}(\mathbb{R}) \\ \mathcal{R}_0 := \{\, I \subset \mathbb{R} \mid I \text{ contains an open neighborhood of } 0 \,\} \end{array}$$

$\mathcal{N}$ is the "filter of cofinites" (on $\mathbb{N}$),
$\mathcal{R}_0$ is the "filter of neighborhoods of 0" (in $\mathbb{R}$).

Define the following relation on $\mathbb{I}$-sequences:

$$a \sim_{\mathcal{F}} b \quad \Leftrightarrow \quad \{\, i \mid a_i = b_i \,\} \in \mathcal{F}$$

Prop: $\sim_{\mathcal{F}}$ is an equivalence relation $\Rightarrow$ $\mathcal{F}$ is a filter.

$$\begin{array}{lcl} a \sim_{\mathcal{F}} a & \Rightarrow & \mathcal{F} \ni \{\, i \mid a_i = a_i \,\} = \mathbb{I}, \\ a \sim_{\mathcal{F}} b \sim_{\mathcal{F}} c & \Rightarrow & \mathcal{F} \ni \{\, i \mid a_i = c_i \,\} \supseteq \{\, i \mid a_i = b_i \,\} \cap \{\, i \mid b_i = c_i \,\}, \end{array}$$

Look at this example (with $\mathbb{I} := \mathbb{R}$):
f is 0 in $(-2, 1)$, 1 elsewhere,
g is 0 everywhere,
h is 0 in $(-1, 2)$, -1 elsewhere,
h' is 0 in $(-1, 2)$, 1 in $(4, 5)$, -1 elsewhere;

f coincides with h exactly on $(-2, 1) \cap (-1, 2)$,
f coincides with h' on a bigger set — the above plus $(4, 5)$.

Prop: $\sim_{\mathcal{F}}$ is an equivalence relation $\Leftarrow$ $\mathcal{F}$ is a filter.

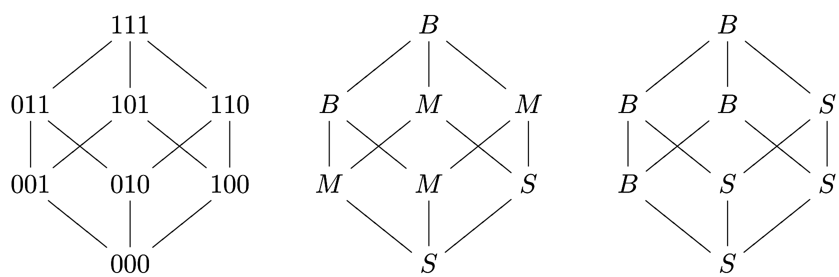
Proper filters, big/small/medium sets, and ultrafilters

Def: a filter $\mathcal{F}$ is *proper* when $\emptyset \notin \mathcal{F}$.
  $\mathcal{F}$ improper $\Leftrightarrow \emptyset \in \mathcal{F} \Leftrightarrow \mathcal{F} = \mathcal{P}(\mathbb{I}) \Leftrightarrow$
  $\Leftrightarrow$ all sequences are $\mathcal{F}$-equivalent.
  $\mathcal{N}$ is proper.

Def: $I \subset \mathbb{I}$ is *$\mathcal{F}$-big* when $I \in \mathcal{F}$.
  $\mathbb{N} + 4 = \{4, 5, 6, 7, \ldots\}$ is cofinite, and so $\mathcal{N}$-big.
Def: $I \subset \mathbb{I}$ is *$\mathcal{F}$-small* when $I \in \mathcal{F}$.
  $\{0, 1, 2, 3\}$ is finite, and so $\mathcal{N}$-small.
Def: $I \subset \mathbb{I}$ is *$\mathcal{F}$-medium* when I is neither $\mathcal{F}$-big, nor $\mathcal{F}$-small.
  $2\mathbb{N} = \{0, 2, 4, 6, ...\}$ is $\mathcal{N}$-medium.

A proper filter $\mathcal{F}$ divides $\mathcal{P}(\mathbb{I})$ in $\mathcal{F}$-big, $\mathcal{F}$-medium and $\mathcal{F}$-small sets.
Def: an *ultrafilter* is a filter $\mathcal{F}$ with no $\mathcal{F}$-medium sets.
  We will use $\mathcal{U}$ to denote ultrafilters.
  $\mathcal{N}$ is not an ultrafilter.

Two proper filters over $\mathbb{I} := \{\alpha, \beta, \gamma\}$:
The one at the right is an ultrafilter.



For $\mathcal{A} \subset \mathcal{P}(\mathbb{I})$,
Def: $\uparrow \mathcal{A} := \{\, A' \mid A \subseteq A' \subseteq \mathbb{I}, \text{for some } A \in \mathcal{A} \,\}$
  $\uparrow \mathcal{F} = \mathcal{F}$.
Def: $\downarrow \mathcal{A} := \{\, A' \mid A' \subseteq A, \text{for some } A \in \mathcal{A} \,\}$
  The set of $\mathcal{F}$-small sets is equal to its '$\downarrow$'.
Def: $\bigcap_{\text{fin}} \mathcal{A} := \{\, A_1 \cap \ldots \cap A_n \mid n \in \mathbb{N}, A_1, \ldots, A_n \in \mathcal{A} \,\}$
  where we define that $A_1 \cap \ldots \cap A_n = \mathbb{I}$ when $n = 0$.

Fact: for any $\mathcal{A} \subset \mathcal{P}(\mathbb{I})$,
  $\bigcap_{\text{fin}} \uparrow \mathcal{A} = \uparrow \bigcap_{\text{fin}} \mathcal{A}$ is a filter.

$\mathcal{N} = \uparrow \bigcap_{\text{fin}} \{\mathbb{N}, \mathbb{N}+1, \mathbb{N}+2, \mathbb{N}+3, \ldots\}$
$\mathcal{R}_0 = \uparrow \bigcap_{\text{fin}} \{(-1, 1),\ (-\frac{1}{2}, -\frac{1}{2}),\ (-\frac{1}{3}, -\frac{1}{3}), \ldots\}$

Cores and principal ultrafilters
The *core* of a filter $\mathcal{F}$ is $\bigcap \mathcal{F}$.
$\mathcal{N}$ has empty core.
$\mathcal{R}_0$ has core $= \{0\}$, but this can be "fixed" —
by removing $\{0\}$ from each $\mathcal{R}_0$-big set we get a
filter over $\mathbb{R} \backslash \{0\}$ — the filter of
"punctured neighborhoods" of $0 \in \mathbb{R}$, that has
empty core.

(By the way: $\mathcal{N}$ is a filter of punctured
neighborhoods of $\infty \in \mathbb{N}^*$ in $\mathbb{N}^* \backslash \{\infty\}$.)

Any ultrafilter refining $\mathcal{N}$ has empty core.
An ultrafilter with a non-empty core has a single point in its core.
An ultrafilter with a non-empty core is called "principal".
Principal ultrafilters are silly: if $\mathcal{U} = \uparrow \{a\}$
then the equivalence relation $\sim_{\mathcal{U}}$ pays attention only
to the index a, and $\mathbf{Set} \cong \mathbf{Set}^{\mathbb{I}}/\mathcal{U}$.

$$\begin{array}{ccccc}
\mathbf{Set} & \longrightarrow & \mathbf{Set}^{\mathbb{N}} & \to & \mathbf{Set}^{\mathbb{N}}/\mathcal{N} \\
 & & & \searrow & \vdots \\
 & & & & \mathbf{Set}^{\mathbb{N}}/\mathcal{U}
\end{array}$$

When $\mathcal{U}$ is non-principal
every infinite set in **Set**
gets new ("non-standard") elements
after the passage to $\mathbf{Set}^{\mathbb{I}}/\mathcal{U}$.

Interpreting some sentences

Take $\omega := (1, 2, 3, 4, \ldots)$ in $\mathbf{Set}^{\mathbb{N}}/\mathcal{N}$.
ω is bigger than any standard natural:
$\omega > 2 \equiv (\dashv, \dashv, \top, \top, \ldots) \sim_{\mathcal{N}} (\top, \top, \top, \top, \ldots) \equiv \top$

Take $\varepsilon := (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$ in $\mathbf{Set}^{\mathbb{N}}/\mathcal{N}$.
ε is smaller than any standard positive real:
$\varepsilon < \frac{1}{2} \equiv (\dashv, \dashv, \top, \top, \ldots) \sim_{\mathcal{N}} \top$.

$f(a)$ is $(f_1(a_1), f_2(a_2), f_3(a_3), \ldots)$.

$\forall a, b \in \mathbb{R}. ab = ba$

$\forall x \in (0, 1). x^2 \in (0, x)$

$\forall a, b \in \mathbb{R}. ab = 0 \supset (a = 0 \vee b = 0)$

Ultrafilters are evil
Take a denumerable family of sets of indices, $\mathcal{A} = \{A_1, A_2, A_3, \ldots\}$,
for example $\mathcal{A} := \{\mathbb{N}, 2\mathbb{N}, 3\mathbb{N}, 4\mathbb{N}, \ldots\}$.
Then $\uparrow \bigcap_{\text{fin}} \mathcal{A}$ is not a non-principal ultrafilter.
Let's see why.
Take $\mathcal{A}' := \{A_1, A_1 \cap A_2, A_1 \cap A_2 \cap A_3, \ldots\}$;
build $\mathcal{A}''$ from that by removing the repetitions.
In the non-trivial case, $\mathcal{A}'' = \{A''_1, A''_2, A''_3, \ldots\}$ is infinite.
Look at
$(\mathbb{I} \backslash A''_1) \cup (A''_2 \backslash A''_3) \cup (A''_4 \backslash A''_5) \cup \ldots$ and
$(A''_1 \backslash A''_2) \cup (A''_3 \backslash A''_4) \cup (A''_5 \backslash A''_6) \cup \ldots$ —
they are both medium sets.

Attempts to build non-principal explicitly are bound to fail.
To build non-principal ultrafilters we need a weak form of AC.
Halpern 1964: the “boolean prime ideal theorem” is independent from AC.

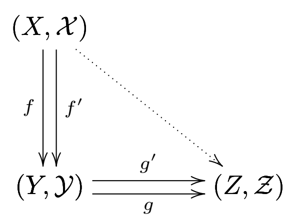
Partial functions with big domains
If $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ are filtered spaces —
i.e., $\mathcal{X}$ is a filter over X
and $\mathcal{Y}$ is a filter over Y —
then a partial function $f : X \to Y$ is said
to have ($\mathcal{X}$-)big domain when its domain is $\mathcal{X}$-big.

Shorter name: a "big partial function" is a
partial function with a big domain.
Even shorter: → "big function".

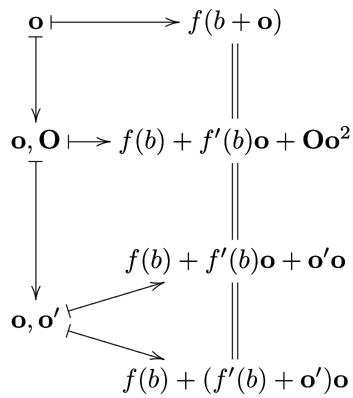
Filter-continuity
A partial function $f : X \to Y$ is *(filter-)continuous* when
the inverse image of every $\mathcal{Y}$-big set is $\mathcal{X}$-big.
(Being "big" is weaker than that: just $f^{-1}(Y) \in \mathcal{X}$.)

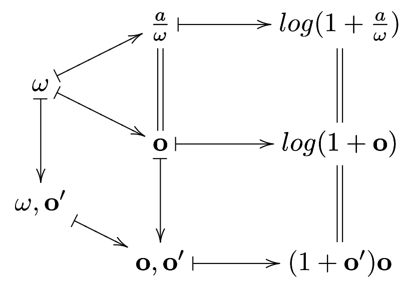
Two big functions f, g are *equivalent* when
they coincide on a big set.

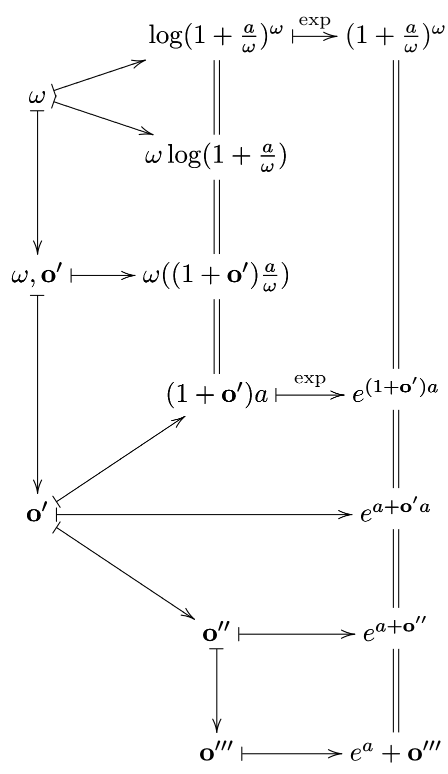
Big continuous functions compose.
Moreover: if $f \sim_{\mathcal{X}} f'$ and $g \sim_{\mathcal{Y}} g'$ are all big and continuous,
then $g \circ f \sim_{\mathcal{X}} g' \circ f'$ is big and continuous.



Diagram


$\mathbf{o}$
$f(b+\mathbf{o})$
$\mathbf{o},\mathbf{O}$
$f(b)+f'(b)\mathbf{o}+\mathbf{O}\mathbf{o}^2$
$f(b)+f'(b)\mathbf{o}+\mathbf{o}'\mathbf{o}$
$\mathbf{o},\mathbf{o}'$
$f(b)+(f'(b)+\mathbf{o}')\mathbf{o}$


$\frac{a}{\omega}$
$log(1+\frac{a}{\omega})$
ω
$\mathbf{o}$
$log(1+\mathbf{o})$
$\omega,\mathbf{o}'$
$\mathbf{o},\mathbf{o}'$
$(1+\mathbf{o}')\mathbf{o}$


$\log(1+\frac{a}{\omega})^\omega$
exp
$(1+\frac{a}{\omega})^\omega$
ω
$\omega\log(1+\frac{a}{\omega})$
$\omega,\mathbf{o}'$
$\omega((1+\mathbf{o}')\frac{a}{\omega})$
$(1+\mathbf{o}')a$
exp
$e^{(1+\mathbf{o}')a}$
$\mathbf{o}'$
$e^{a+\mathbf{o}'a}$
$\mathbf{o}''$
$e^{a+\mathbf{o}''}$
$\mathbf{o}'''$
$e^a+\mathbf{o}'''$

Filters are enough
Main theorem
Change of base
Filter-continuity is the same as continuity at the chosen point:

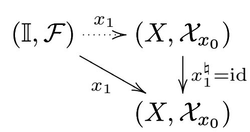
$$(\mathbb{R},\mathcal{R}_0)\to(X,\mathcal{X}_{x_0})$$

Filter-continuity is the same as infinitesimality:

$$(\mathbb{I},\mathcal{F})\to(\mathbb{R},\mathcal{R}_0)$$

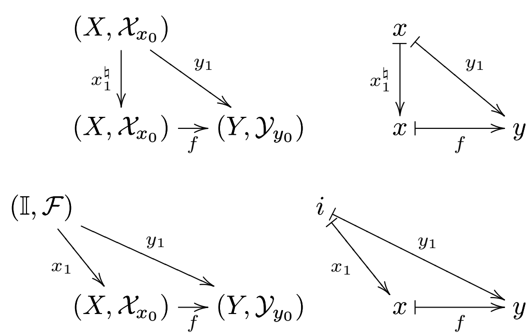
(general case: topological spaces)

Definition: the *natural infinitesimal* on a (standard) filtered space $(X,\mathcal{X}_{x_0})$, that we will denote by $x_1^\natural \stackrel{\natural}{\sim} x_0$, is the identity function $x_1^\natural = \mathrm{id} : (X,\mathcal{X}_{x_0}) \to (X,\mathcal{X}_{x_0})$; seen as an infinitesimal, it lives in $\mathbf{Set}^X/\mathcal{X}_{x_0}$. As it corresponds to the identity map, any other infinitesimal $x_1 \sim x_0$ — in the diagram below we take an x_1 living in $\mathbf{Set}^{\mathbb{I}}/\mathcal{F}$ — factors through $x_1^\natural$ it in a unique way; this suggests that there is a kind of "change of base" operation between filter-powers.



Now, for any $f:(X,\mathcal{X}_{x_0})\to(Y,\mathcal{Y}_{y_0})$ taking x_0 to y_0, this holds:

> **Key theorem:**
> (i) f is continuous at x_0
> $\Leftrightarrow$ (ii) for $(\mathbb{I},\mathcal{F}) := (X,\mathcal{X}_{x_0})$, $x_1^\natural \stackrel{\natural}{\sim} x_0$, we have $f(x_1^\natural)\sim f(x_0)$
> $\Leftrightarrow$ (iii) for all $(\mathbb{I},\mathcal{F})$ and $x_1\sim x_0$, we have $f(x_1)\sim f(x_0)$.



Proof: (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are obvious from what we've seen before — that the composite of continuous maps between filtered spaces is continuous. For $\neg$(i) $\Rightarrow$ $\neg$(ii), as f is not continuous at x_0, we can choose a $Y'\in\mathcal{Y}_{y_0}$ such that $f^{-1}(Y')\notin\mathcal{X}_{x_0}$; but then $y_1^{-1}(Y') = {x_1^\natural}^{-1}(f^{-1}(Y'))\notin\mathcal{X}_{x_0}$, and $f(x_1^\natural)\not\sim f(x_0)$.

For $\neg$(i) $\Rightarrow$ $\neg$(iii), take $(\mathbb{I}, \mathcal{F}) := (X, \mathcal{X}_{x_0})$, $x_1 := x_1^\natural$, and reuse the proof of $\neg$(i) $\Rightarrow$ $\neg$(ii).

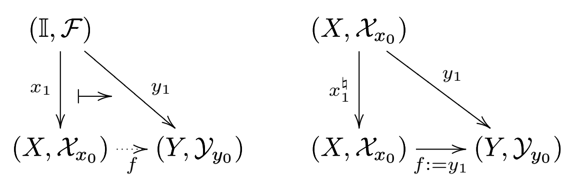
In texts about Non-Standard Analysis the infinitesimal characterization of continuity is presented in another form:

(i) f is continuous at x_0
$\Leftrightarrow$ (iv) for all $(\mathbb{I}, \mathcal{U})$ and $x_1 \sim x_0$, we have $f(x_1) \sim f(x_0)$.

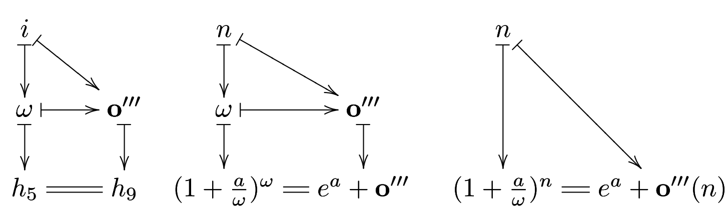
Clearly, (iii)$\Rightarrow$(iv); but to show that (iv) implies the rest we need to be in a universe with enough ultrafilters.

Each of the cells in the diagram in sec. 5 is an instance of the key theorem — maybe slightly disguised. For example, to prove that $g(b+\mathbf{o}) = (g'(b)+\mathbf{o}')\mathbf{o}$ we may start with $\frac{g(b+\mathbf{o})}{\mathbf{o}} - g'(b) = \mathbf{o}'$, for an infinitesimal $\mathbf{o} \neq 0$, i.e., $\lim_{\epsilon\to 0} \frac{g(b+\mathbf{o})}{\mathbf{o}}$.

What really matters, when we look at the diagrams, is that for any $(\mathbb{I}, \mathcal{F})$ and for any infinitesimal $x_1 : (\mathbb{I}, \mathcal{F}) \to (X, \mathcal{X}_{x_0})$ — maybe obeying some condition, like $\mathbf{o} \neq 0$ — there is a unique "adequate" infinitesimal $y_1 : (\mathbb{I}, \mathcal{F}) \to (Y, \mathcal{Y}_{y_0})$; we want to "represent" the operation $x_1 \mapsto y_1$ as a function $f : (X, \mathcal{X}_{x_0}) \to (Y, \mathcal{Y}_{y_0})$, and we can do that trivially by setting $(\mathbb{I}, \mathcal{F}) := (X, \mathcal{X}_{x_0})$, $x_1 := x_1^\natural$; then we can take $f := y_1$, and the f obtained in this way works in the general case.



Applying this idea to the composite of all cells in the example in sec. 5, we get this:



where $i \in (\mathbb{I}, \mathcal{F})$, $n, \omega \in (\mathbb{N}, \mathcal{N})$, and all the other "points" live in $(\mathbb{R}, \mathcal{R}_0)$. Note that the '$\mapsto$' arrows in this diagram do not stand for functions in the usual sense, but for functions between filtered spaces (not necessarily total). Incidentally, all of them are continuous.