Index of the slides:

Haskell

```
-- (find-libhugsfile "libraries/Hugs/Prelude.hs")

foldl            :: (a -> b -> a) -> a -> [b] -> a
foldl f z []      = z
foldl f z (x:xs)  = foldl f (f z x) xs

foldl1           :: (a -> a -> a) -> [a] -> a
foldl1 f (x:xs)   = foldl f x xs

-- foldl1 max [1, 3, 2, 4]
---> foldl max 1 [3, 2, 4]
---> foldl max (max 1 3) [2, 4]
---> foldl max (max (max 1 3) 2) [4]
---> foldl max (max (max (max 1 3) 2) 4) []
--->           (max (max (max 1 3) 2) 4)
--->           (max (max 3        2) 4)
--->           (max 3               4)
--->           4
```

Type theory: origins
(From the introduction to *25 Years of Automath*:)

> In order to prevent the paradoxes, Whitehead and Russell analysed the *vicious circles* present in all the known paradoxes. They came to the conviction that a hierarchy was necessary for a sound development of arithmetic and they proposed a type system: the *simple type theory*. It turned out that a refinement was necessary, which they called the *ramified theory of types*. This worked as they desired, albeit they needed an extra axiom, in order to "soften" the strictness of the typing hierarchy. Only with this *axiom of reducibility* they were able to incorporate full arithmetic, in particular the real numbers, based on Dedekind cuts.
>
> This idea of using types emerged quite naturally, once the vicious circles had been detected. In fact, one may say that types existed since early mathematics was developed: categories like 'natural number' and 'real number' in calculus, or 'point' and 'line' in geometry, grouped elements together in clusters with a common meaning or structure. In this sense, types were meant to emphasize the *similarities* between given entities. But at the same time, types can be of use in establishing *differences* between entities. The latter aspects turned out to be of great importance in combatting against the paradoxes.

Eilenbeg/MacLane 1945

From Ralf Krömer's *Tool and Object - A History and Philosophy of Category Theory* (p.65) on the reception of the first paper on Category Theory (Eilenberg/MacLane: "General Theory of Natural Equivalences", 1945):

> The readyness to write down and submit for publication a work almost completely concerned with conceptual clarification (and with the solution of some internal problems raised by the new concepts themselves) is a remarkable expression of courage. While (as Corry learned from Eilenberg, see [Corry 1996, 366 n.27]) Steenrod once stated concerning [1945] that "no paper had ever influenced his thinking more", P.A. Smith said that "he had never read a more trivial paper in his life". [Mac Lane 1988a, 334] writes, without mentioning a name: "One of our good friends (an admirer of Eilenberg) read the paper and told us privately that he thought that the paper was without any content".

Krömer, p.82, quoting Eilenberg/Steenrod 1952:

> The reader will observe the presence of numerous diagrams in the text. [...] Two paths connecting the same pair of vertices usually give the same homomorphism. This is called a *commutativity relation*. The combinatorially minded individual can regard it as a homology relation due to the presence of 2-dimensional cells adjoined to the graph. [...]
>
> The diagrams incorporate a large amount of information. [...] In the case of many theorems, the setting up of the correct diagram is the major part of the proof [1952, xi].

Type-checking and related problems

From Urzyczyn/Sorensen's "Lectures on the Curry-Howard Isomorphism" (online draft, p.89):

> 1. The type checking problem is to decide whether $\Gamma \vdash M : \tau$ holds, for a given context Γ, a term M and a type τ.
>
> 2. The type reconstruction problem, also called typability problem, is to decide, for a given term M, whether there exist a context Γ and a type τ, such that $\Gamma \vdash M : \tau$ holds, i.e., whether M is typable.
>
> 3. The type inhabitation problem, also called type emptiness problem, is to decide, for a given type τ, whether there exists a closed term M, such that $\Gamma \vdash M : \tau$ holds. (Then we say that τ is non-empty and has an inhabitant M).

Weaker systems
Why do we want weaker systems?
Things that can be constructed in them may have nicer properties:
Every reduction sequence terminates.
All terms of type $A \times B \to B \times A$ are the flip function
Each proof in Natural Deduction corresponds to a lambda-term
Every term is typed
Type-checking can be used to detect errors

Weaker systems should have more models:
If a judgment J
is true in a stronger system S
but false in a weaker system W
then there should be a model of W
in which J is not collapsed to "true".

How do we get intuition about weaker systems?
Stronger system: S, the universe of ZFC
Weaker system: W, where $\neg\neg P = P$ is not "true"
All constructions in W can be carried out in S.

Slides vs. brownie points
I have not been hired by UFF yet.
This is going to happen "at any moment" – since june.
http://angg.twu.net/concurso.html
I am (choose one):
- almost a university professor
- currently unemployed
- in unpayed holidays
- on strike

Articles would get me more brownie points from funding agencies,
but right now seminars and slides are more useful –

I need to discuss with local people, who are:
- algebraic geometers,
- (modal) logicians,
- computer scientists
- grad students who just had a CT course (that was "too algebraic")

Hyperdoctrines: subobjects and change-of-base

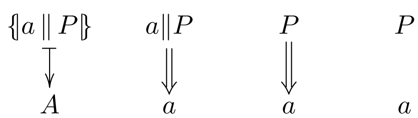
Our archetypal hyperdoctrine is this:

$$\begin{array}{rccc} \mathbf{Cod}: & \mathbf{Sub}(\mathbf{Set}) & \to & \mathbf{Set} \\ & (A' \rightarrowtail A) & \mapsto & A \end{array}$$

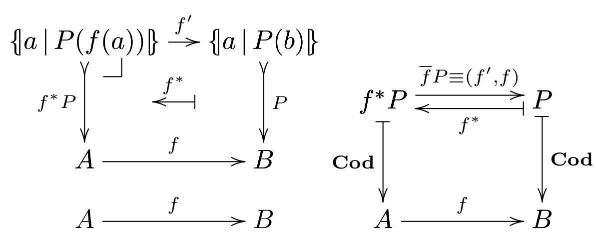
A linguistic trick:
everything will simplify A LOT if we pretend
that all subobject are subsets.
Another shorthand: 'P' instead of '$P(a)$'.

$$\begin{array}{rcl} (\{\!|a \,|\, P|\!\} \rightarrowtail A) & \mapsto & A \\ (\{\!|a \,|\, P|\!\} \rightarrowtail \{\!|a|\!\}) & \mapsto & \{\!|a|\!\} \\ \{\!|a \,\|\, P|\!\} & \mapsto & \{\!|a|\!\} \\ (a\|P) & \Rightarrow & a \end{array}$$

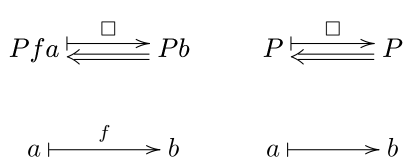
We will downcase the projection functor, **Cod**, to $(a\|P) \Rightarrow a$.
We will often draw it vertically and omit the '$a\|$' and the '$\Rightarrow$':



The change-of-base functors in $\mathbf{Cod} : \mathbf{Sub}(\mathbf{Set}) \to \mathbf{Set}$
are induced by pullbacks:



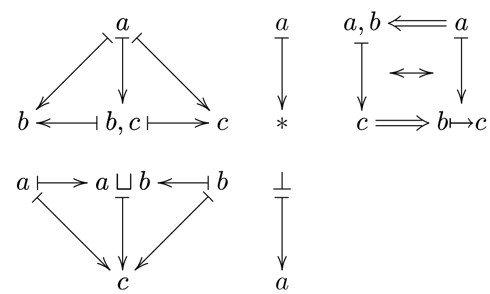
The map $\overline{f}P : \{\!|a \,\|\, P(f(a))|\!\} \to \{\!|b \,\|\, P(b)|\!\}$
corresponds to a pullback square in $\mathbf{Sub}(\mathbf{Set})$.
It is a *cartesian map* in the fibration $\mathbf{Cod} : \mathbf{Sub}(\mathbf{Set}) \to \mathbf{Set}$.
(We will see the abstract definition of cartesian maps later).
We will indicate cartesian maps with a '$\Box$'.
We will downcase the change-of-base diagram as this:



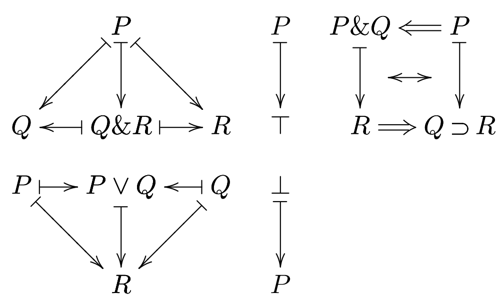
Hyperdoctrine: definition

Formally, a hyperdoctrine is a fibration for which:
(i) the base category is cartesian closed,
(ii) each fiber is bicartesian closed,
(iii) the change-of-base functors preserve the BiCCC structure modulo iso,
(iv) each change-of-base functor has a left adjoint
obeying Beck-Chevalley and Frobenius,
(v) each change-of-base functor has a right adjoint
obeying Beck-Chevalley and Frobenius.

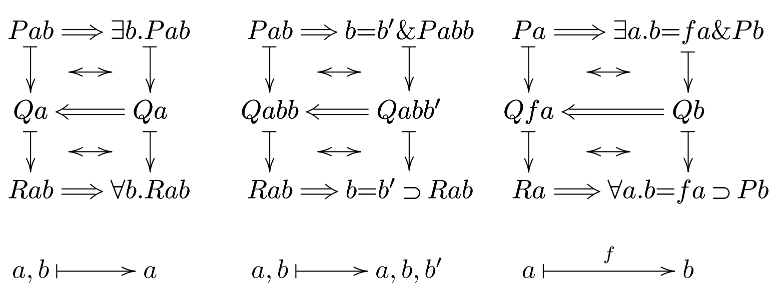
The base category is (bi)cartesian closed:



Each fiber is bicartesian closed:



Left adjoints ($\exists$/&=/$\exists$=) and right adjoints ($\forall/\supset=/\forall=$)
for change-of-base functors:



Quantifiers as adjoints: an example
'∃' and '∀' are left and right adjoint to
change-of-base along a projection $A \times B \to A$...
That this is true is far from obvious,
so let's look at an example. If
$A \equiv$ ••• ••• and
$B \equiv$: , then
$A \times B \equiv$ ••• •••/••• •••;
using the obvious positional notation with 0s and 1s
to denote subsets of $A \times B$ and A,
and for the right $P(a,b)$, $Q(a)$, $R(a,b)$, we have:

$$
\begin{array}{ccc}
\{\!|a,b \| P(a,b)|\!\} & \longmapsto & \{\!|a \| \exists b.P(a,b)|\!\} \\
\downarrow & \leftrightarrow & \downarrow \\
\{\!|a,b \| Q(a)|\!\} & \longleftarrow\!\shortmid & \{\!|a \| Q(a)|\!\} \\
\downarrow & \leftrightarrow & \downarrow \\
\{\!|a,b \| R(a,b)|\!\} & \longmapsto & \{\!|a \| \forall b.R(a,b)|\!\} \\
\\
A \times B & \longrightarrow & A
\end{array}
\qquad
\begin{array}{ccc}
\substack{100\,000 \\ 110\,000} & \longmapsto & 110\,000 \\
\downarrow & \leftrightarrow & \downarrow \\
\substack{111\,000 \\ 111\,000} & \longleftarrow\!\shortmid & 111\,000 \\
\downarrow & \leftrightarrow & \downarrow \\
\substack{111\,100 \\ 111\,110} & \longmapsto & 111\,100 \\
\\
A \times B & \longrightarrow & A
\end{array}
$$

Later we will see that '$\exists \dashv \pi^* \dashv \forall$' holds
even when the logic is just intuitionistic —
and we will use this to "define" ∃ and ∀ in hyperdoctrines...

Also, we will see in which sense
"hyperdoctrines are models of typed intuitionistic logic",
and we will construct some hyperdoctrines in which the logic
is not classical.

Preservations by change-of-base: overview
Start with $f : A \to B$,
and $\{\!|b \,\|\, P(b)|\!\}$ and $\{\!|b \,\|\, Q(b)|\!\}$.

There are two different categorical ways to build
an object that "deserves the name"
$\{\!|a \,\|\, P(f(a))\&Q(f(a))|\!\}$.

We want these two ways to be isomorphic,
and there is a natural way to build
one of the directions of the iso —
so the technical condition
"change-of-base functors preserve the BiCCC structure"
becomes, for each of the connectives $\top, \bot, \&, \vee, \supset$,
that this natural half-side of the iso
will have an inverse.

More precisely:

$$\begin{array}{ccc}
f^*\top_B & \bot_A & f^*(P \supset Q) \\
{\scriptstyle \natural=!}\downarrow\uparrow{\scriptstyle \mathsf{P}\top} & {\scriptstyle \natural=!}\downarrow\uparrow{\scriptstyle \mathsf{P}\bot} & {\scriptstyle \natural=\ldots}\downarrow\uparrow{\scriptstyle \mathsf{P}\supset} \\
\top_A & f^*\bot_B & f^*P \supset f^*Q
\end{array}$$

$$f^*P\&f^*Q \underset{\mathsf{P}\&}{\overset{\natural=\langle f^*\pi, f^*\pi'\rangle}{\leftrightarrows}} f^*(P\&Q)$$

$$f^*P \vee f^*Q \underset{\mathsf{P}\vee}{\overset{\natural=[f^*\iota, f^*\iota']}{\rightleftarrows}} f^*(P \vee Q)$$

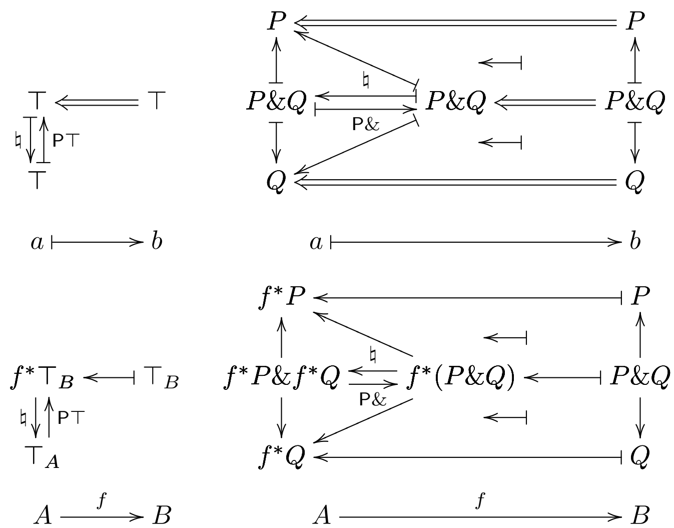
We will see the diagrams for these constructions in the next slides.
The natural half for '$\supset$', in particular, is a big construction.

The Beck-Chevalley conditions are similar: they say that the two
natural constructions for two objects that "deserve the same name"
are isomorphic — the natural half-side of an iso has an inverse.

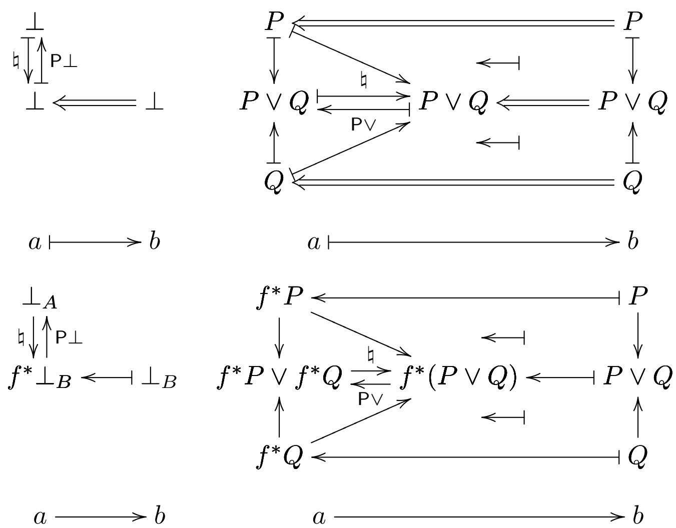
Frobenius conditions are more like distributivities.
The case where they are more natural will appear later,
in another hyperdoctrine, where the same structure will
interpret different operations. There it will be this:

$$b,(c,d) \underset{\mathsf{Frob}}{\overset{\natural}{\rightleftarrows}} (b,c),d$$

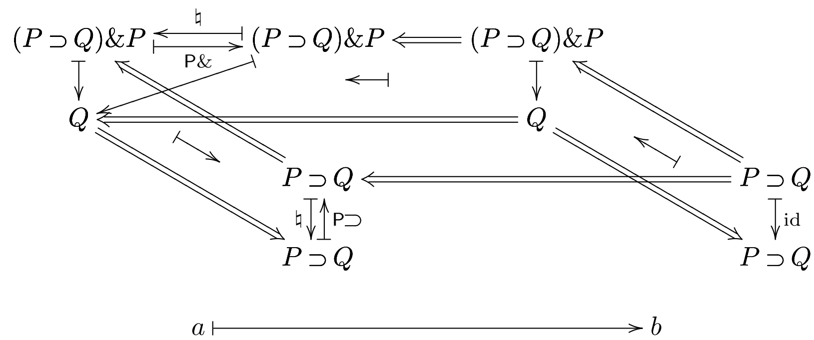
Preservation of 'true' and 'and'

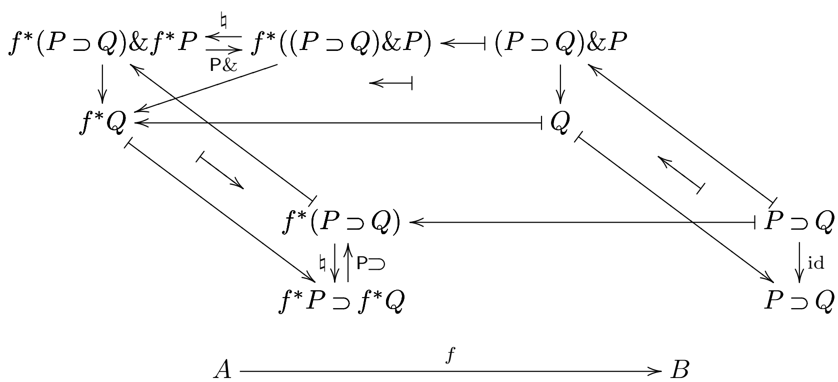


Preservation of 'false' and 'or'

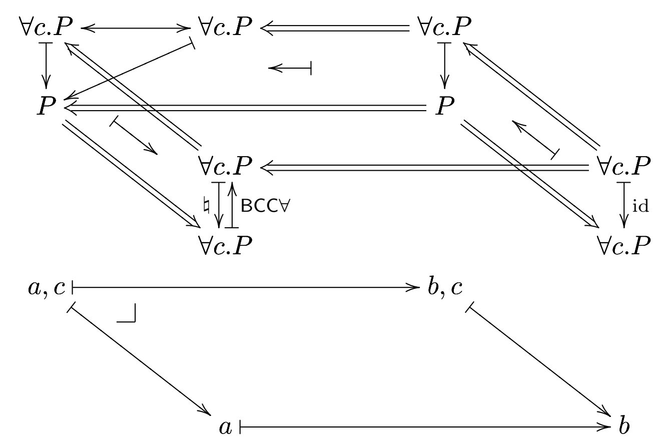


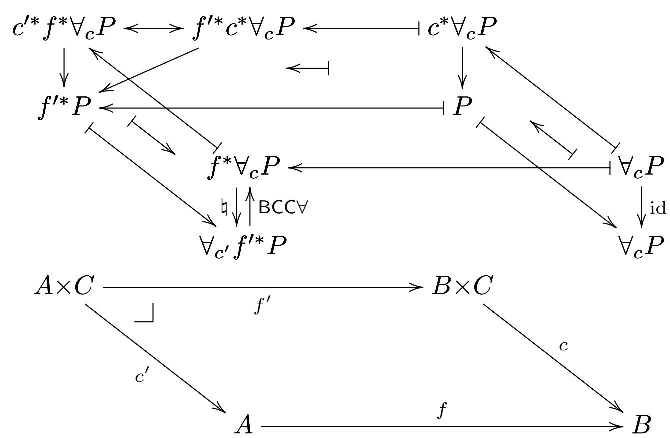
Preservation of 'implies'



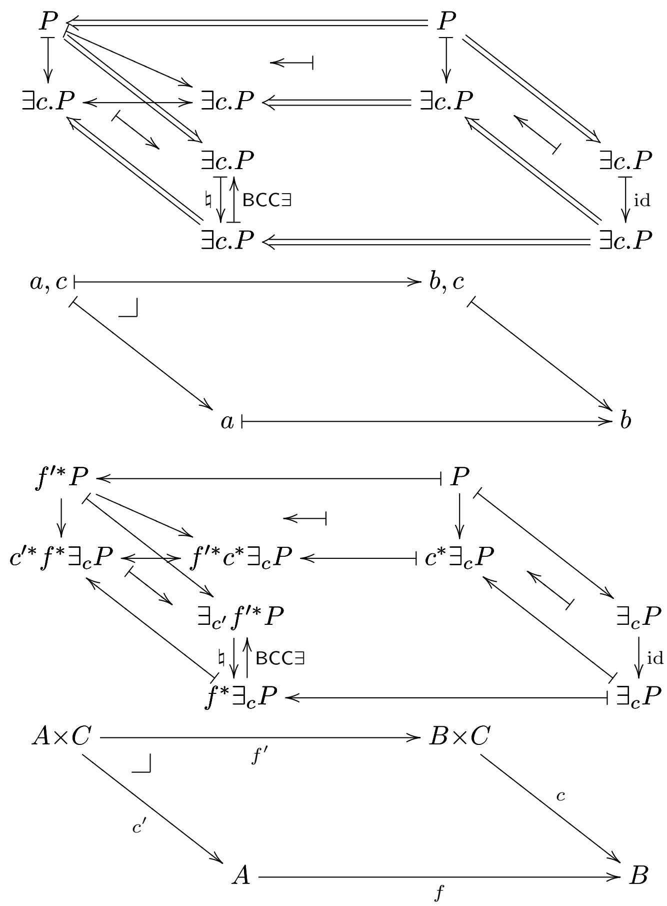


BCC for 'forall'

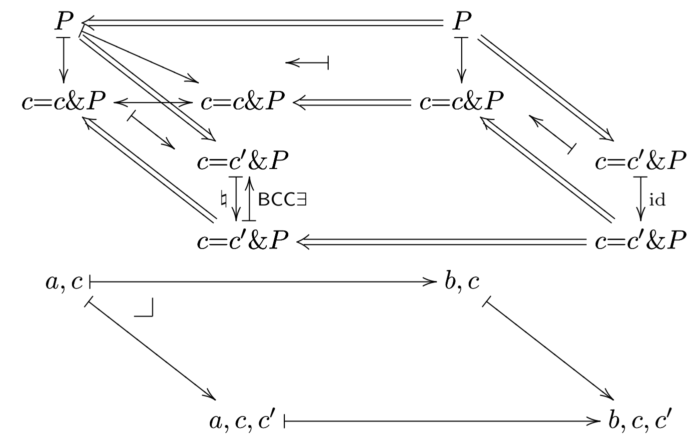


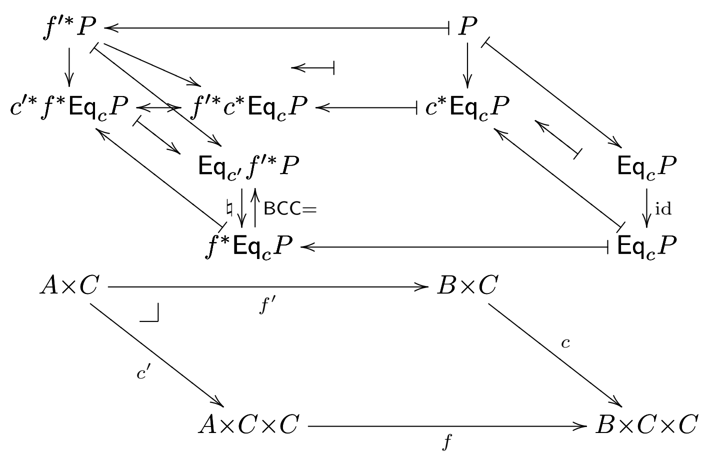


BCC for ‘exists’

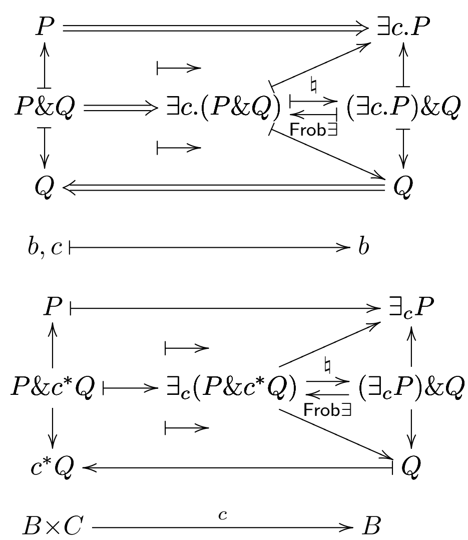


BCC for equality

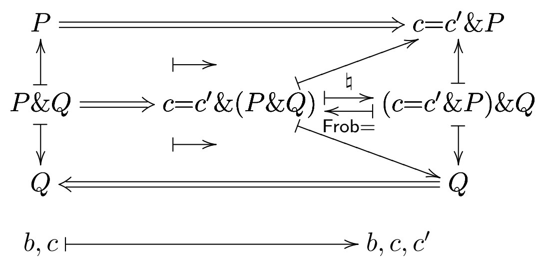


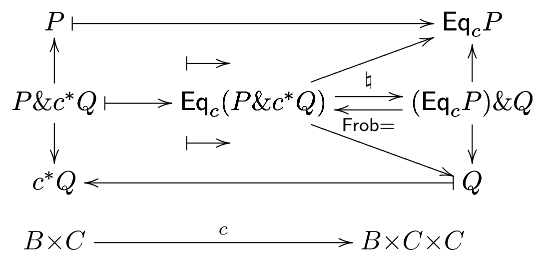


Frobenius for 'exists'

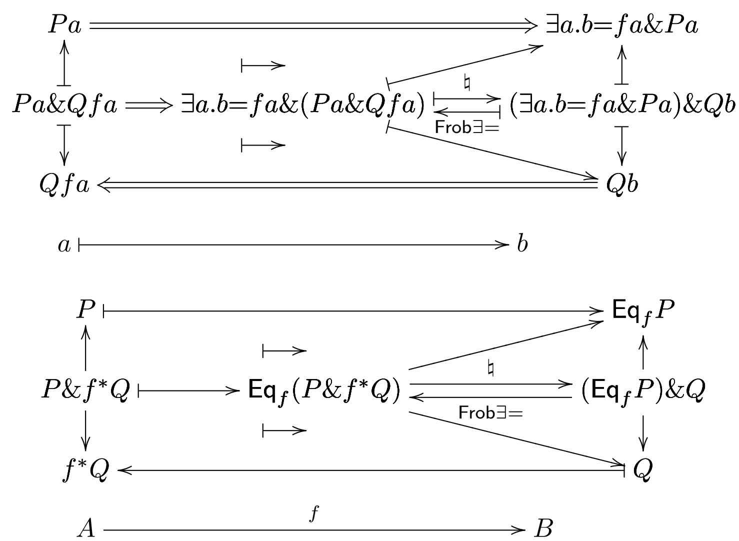


Frobenius for equality

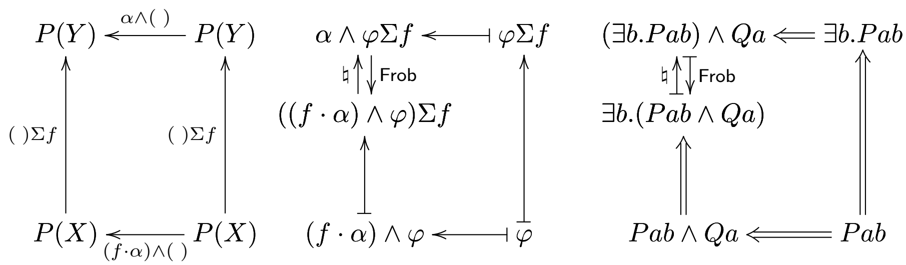


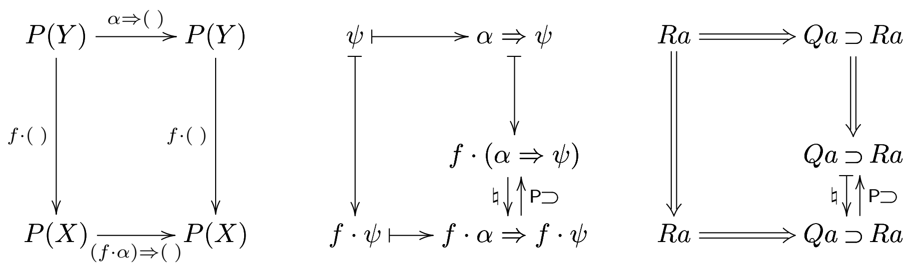


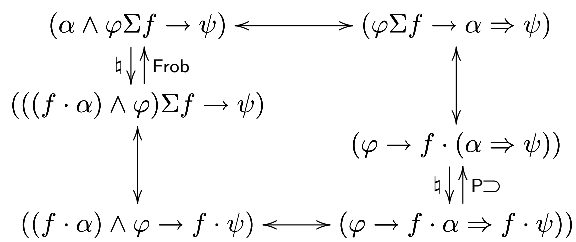
Frobenius for exists-equal

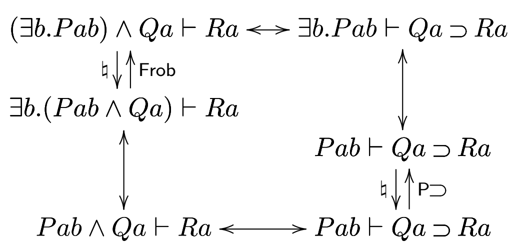


Frobenius is equivalent to preservation of ‘implies’

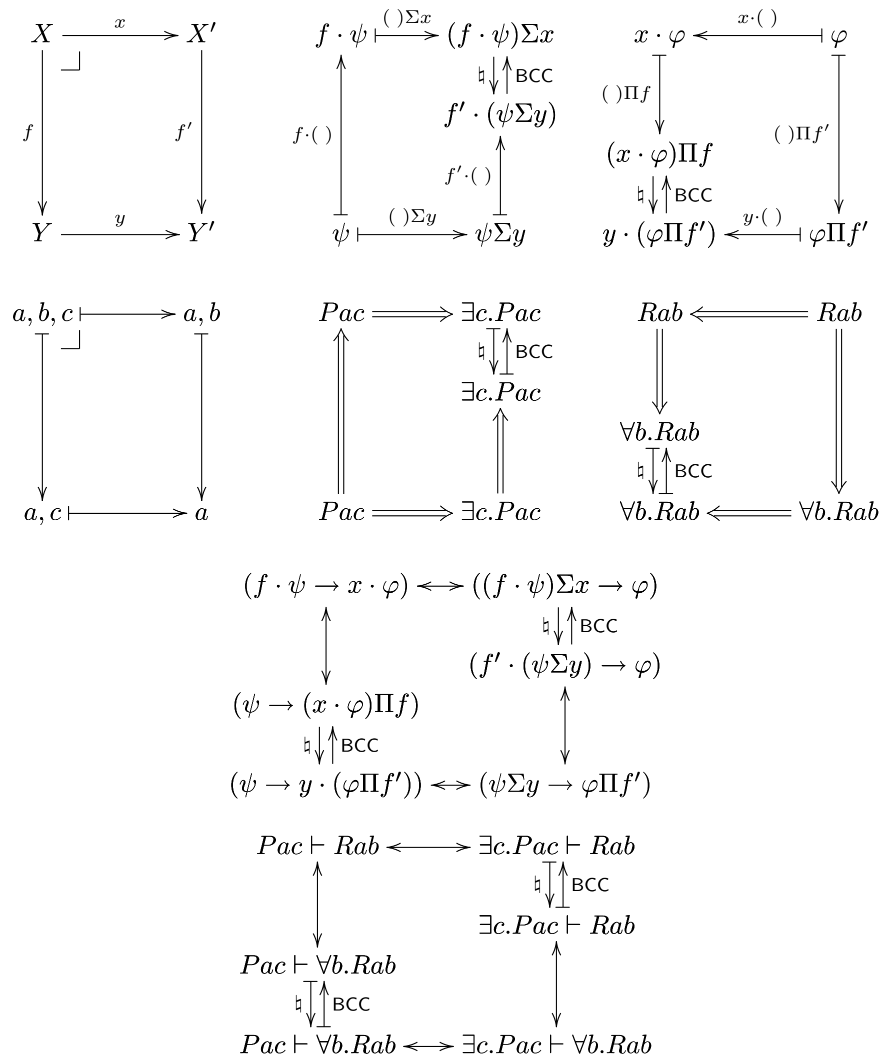








BCC for 'exists' holds iff BCC for 'forall' holds



From "constructions" to "intuitionistic proofs", and back
We have a notion of "categorical construction"
and a notion of "intuitionistic proof";
neither of them are very clear now —
but I will start with some examples,
and show how every "categorical construction"
in a hyperdoctrine can be translated to an
"intuitionistic proof", and how an
"intuitionistic proof" in a certain
typed intuitionistic logic can be translated
to a "categorical construction" in a hyperdoctrine...

Only after seeing these two translations in detail
we will start to think about adding new constants,
axioms, operations and rules to our
"typed intuitionistic logic", and how to add new
constants, operations, equalities, etc, to our
notion of hyperdoctrine.

Note: in this beginning all our categorical constructions
will look as if they were happening in Sub(Set)-¿Set —
because at this moment that is the only hyperdoctrine we know —
but the abstract diagrams that we will produce in the process
will later be seen to be appliable to every hyperdoctrine.

Similarly, all our intuitionistic proofs will look
as if they were just happening in a fragment of the full
language for sets and first-order predicates, but later
we will see that they will also apply to other "languages"
and "logics", with "models" different from Sub(Set)-¿Set...

But we need concrete examples to start with.

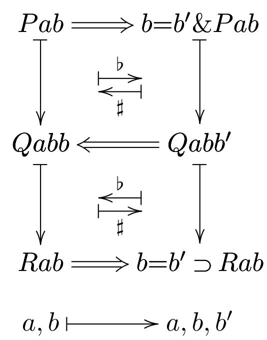
The adjunction functors as sequent rules

$$
\begin{array}{ccc}
P\&Q & \Longleftarrow & P \\
\downarrow & \overset{\flat}{\longleftarrow\!\!\dashv}\ \underset{\sharp}{\vdash\!\!\longrightarrow} & \downarrow \\
R & \Longrightarrow & Q \supset R
\end{array}
\qquad
\dfrac{P \vdash P'}{P\&Q \vdash P'\&Q}\ \&Q
\qquad
\dfrac{R \vdash R'}{Q \supset R \vdash Q \supset R'}\ Q\supset
$$

$$
\begin{array}{ccc}
(P, Q) & \Longrightarrow & P \vee Q \\
\downarrow & \overset{\flat}{\vdash\!\!\longrightarrow}\ \underset{\sharp}{\longleftarrow\!\!\dashv} & \downarrow \\
(R, R) & \Longleftarrow & R \\
\downarrow & \overset{\flat}{\longleftarrow\!\!\dashv}\ \underset{\sharp}{\vdash\!\!\longrightarrow} & \downarrow \\
(S, T) & \Longrightarrow & S\&T
\end{array}
\qquad
\dfrac{P \vdash P' \quad Q \vdash Q'}{P \vee Q \vdash P' \vee Q'}\ \vee
\qquad
\dfrac{R \vdash R'}{(R \vdash R'), (R \vdash R')}\ K
\qquad
\dfrac{S \vdash S' \quad T \vdash T'}{S\&T \vdash S'\&T'}\ \&
$$

$$
\begin{array}{ccc}
Pab & \Longrightarrow & \exists b.Pab \\
\downarrow & \overset{\flat}{\vdash\!\!\longrightarrow}\ \underset{\sharp}{\longleftarrow\!\!\dashv} & \downarrow \\
Qa & \Longleftarrow & Qa \\
\downarrow & \overset{\flat}{\longleftarrow\!\!\dashv}\ \underset{\sharp}{\vdash\!\!\longrightarrow} & \downarrow \\
Rab & \Longrightarrow & \forall b.Rab \\
a, b & \longmapsto & a
\end{array}
\qquad
\dfrac{Pab \vdash P'ab}{\exists b.Pab \vdash \exists b.P'ab}\ \exists
\qquad
\dfrac{Qa \vdash Q'a}{Qa \vdash Q'a}\ W
\qquad
\dfrac{Rab \vdash R'ab}{\forall b.Rab \vdash \forall b.R'ab}\ \forall
$$

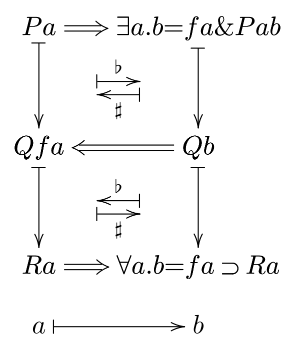
The adjunction functors as sequent rules (2)



$$\frac{Pab \vdash P'ab}{b{=}b'\&Pab \vdash b{=}b'\&Pab} = \&$$

$$\frac{Qabb' \vdash Q'abb'}{Qabb \vdash Q'abb} \ (b' := b)$$

$$\frac{Rab \vdash R'ab}{b{=}b' \supset Rab \vdash b{=}b' \supset R'ab} = \supset$$

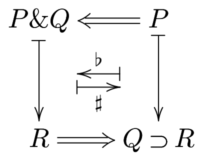


$$\frac{Pa \vdash P'a}{\exists a.b{=}fa\&Pb \vdash \exists a.b{=}fa\&.Pb} \ \exists =$$

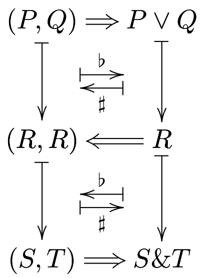
$$\frac{Qb \vdash Q'b}{Qfa \vdash Q'fa} \ (b := fa)$$

$$\frac{Ra \vdash R'a}{\forall a.b{=}fa \supset Ra \vdash \forall a.b{=}fa \supset R'a} \ \forall =$$

The adjunction maps as sequent rules

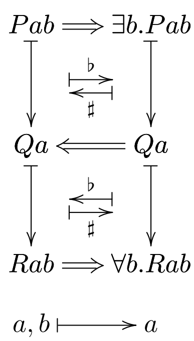


$$\frac{P\&Q \vdash R}{P \vdash Q \supset R}\ \mathrm{Cur}^\sharp \qquad \frac{P \vdash Q \supset R}{P\&Q \vdash R}\ \mathrm{Cur}^\flat$$



$$\frac{P \vdash R \quad Q \vdash R}{P \vee Q \vdash R}\ \vee^\flat \qquad \frac{P \vee Q \vdash R}{P \vdash R}\ \vee^\sharp_1 \quad \frac{P \vee Q \vdash R}{Q \vdash R}\ \vee^\sharp_2$$

$$\frac{R \vdash S \quad R \vdash T}{T \vdash S\&T}\ \vee^\sharp \qquad \frac{R \vdash S\&T}{R \vdash S}\ \vee^\flat_1 \quad \frac{R \vdash S\&T}{R \vdash T}\ \vee^\flat_2$$



$$\frac{Pab \vdash Qa}{\exists b.Pab \vdash Qa}\ \exists^\flat \qquad \frac{\exists b.Pab \vdash Qa}{Pab \vdash Qa}\ \exists^\sharp$$

$$\frac{Qa \vdash Rab}{Qa \vdash \forall b.Rab}\ \forall^\sharp \qquad \frac{Qa \vdash \forall b.Rab}{Qa \vdash Rab}\ \forall^\flat$$

The adjunction maps as sequent rules (2)

$$\begin{array}{ccc} Pab & \Longrightarrow & b{=}b'\&Pab \\ \downarrow & \overset{\flat}{\rightleftarrows}_{\sharp} & \downarrow \\ Qabb & \Longleftarrow & Qabb' \\ \downarrow & \overset{\flat}{\leftrightarrows}_{\sharp} & \downarrow \\ Rab & \Longrightarrow & b{=}b' \supset Rab \end{array}$$

$$a, b \longmapsto a, b, b'$$

$$\frac{Pab \vdash Qabb}{b{=}b'\&Pab \vdash Qabb'} = \&^{\flat} \qquad \frac{b{=}b'\&Pab \vdash Qabb'}{Pab \vdash Qabb} = \&^{\sharp}$$

$$\frac{Qabb \vdash Rab}{Qabb' \vdash b{=}b' \supset Rab} =\supset^{\sharp} \qquad \frac{Qabb' \vdash b{=}b' \supset Rab}{Qabb \vdash Rab} =\supset^{\flat}$$

$$\begin{array}{ccc} Pa & \Longrightarrow & \exists a.b{=}fa\&Pab \\ \downarrow & \overset{\flat}{\rightleftarrows}_{\sharp} & \downarrow \\ Qfa & \Longleftarrow & Qb \\ \downarrow & \overset{\flat}{\leftrightarrows}_{\sharp} & \downarrow \\ Ra & \Longrightarrow & \forall a.b{=}fa \supset Ra \end{array}$$

$$a \longmapsto b$$

$$\frac{Pa \vdash Qfa}{\exists a.b{=}fa\&Pab \vdash Qb} \ \exists =^{\flat} \qquad \frac{\exists a.b{=}fa\&Pab \vdash Qb}{Pa \vdash Qfa} \ \exists =^{\sharp}$$

$$\frac{Qfa \vdash Ra}{Qb \vdash \forall a.b{=}fa \supset Ra} =\supset^{\sharp} \qquad \frac{Qb \vdash \forall a.b{=}fa \supset Ra}{Qfa \vdash Ra} =\supset^{\flat}$$

The adjunction functors as ND proofs

$$\begin{array}{ccc} P\&Q & \Longleftarrow & P \\ \downarrow & \overset{\flat}{\longleftarrow}\ \underset{\sharp}{\longmapsto} & \downarrow \\ R & \Longrightarrow & Q\supset R \end{array}$$

$$\dfrac{\begin{matrix}\dfrac{P\&Q}{P}\ \&E_1\\ \vdots\\ P'\end{matrix} \qquad \dfrac{P\&Q}{Q}\ \&E_2}{P'\&Q}\ \&I$$

$$\dfrac{\begin{matrix}\dfrac{[Q]^1 \quad Q\supset R}{R}\ \supset E\\ \vdots\\ R'\end{matrix}}{Q\supset R'}\ \supset I;1$$

$$\begin{array}{ccc} (P,Q) & \Longrightarrow & P\vee Q \\ \downarrow & \overset{\flat}{\longmapsto}\ \underset{\sharp}{\longleftarrow} & \downarrow \\ (R,R) & \Longleftarrow & R \\ \downarrow & \overset{\flat}{\longleftarrow}\ \underset{\sharp}{\longmapsto} & \downarrow \\ (S,T) & \Longrightarrow & S\&T \end{array}$$

$$\dfrac{P\vee Q \qquad \dfrac{\begin{matrix}[P]^1\\ \vdots\\ P'\end{matrix}}{P'\vee Q'}\ \vee I_1 \qquad \dfrac{\begin{matrix}[Q]^1\\ \vdots\\ Q'\end{matrix}}{P'\vee Q'}\ \vee I_2}{P'\vee Q'}\ \vee E$$

$$\dfrac{\begin{matrix}\dfrac{S\&T}{S}\ \&E_1\\ \vdots\\ S'\end{matrix} \qquad \begin{matrix}\dfrac{S\&T}{T}\ \&E_2\\ \vdots\\ T'\end{matrix}}{S'\&T'}\ \&I$$

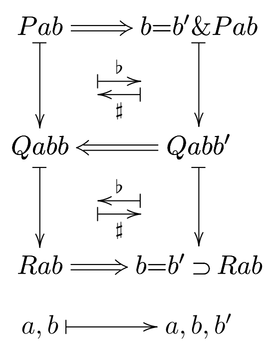
$$\begin{array}{ccc} Pab & \Longrightarrow & \exists b.Pab \\ \downarrow & \overset{\flat}{\longmapsto}\ \underset{\sharp}{\longleftarrow} & \downarrow \\ Qa & \Longleftarrow & Qa \\ \downarrow & \overset{\flat}{\longleftarrow}\ \underset{\sharp}{\longmapsto} & \downarrow \\ Rab & \Longrightarrow & \forall b.Rab \end{array}$$

$$a,b \longmapsto a$$

$$\dfrac{\exists b.Pab \qquad \dfrac{\begin{matrix}[Pab]^1\\ \vdots\\ P'ab\end{matrix}}{\exists b.P'ab}\ \exists I}{\exists b.P'ab}\ \exists E$$

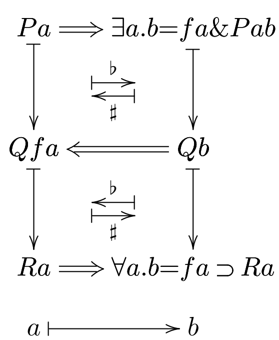
$$\dfrac{\begin{matrix}\dfrac{[b]^1 \quad \forall b.Rab}{Rab}\ \forall E\\ \vdots\\ R'ab\end{matrix}}{\forall b.R'ab}\ \forall I;1$$

The adjunction functors as ND proofs (2)



$$\dfrac{\dfrac{b{=}b'\&Pab}{b{=}b'}\,\&E_1 \qquad \begin{array}{c}\dfrac{b{=}b'\&Pab}{Pab}\,\&E_2\\ \vdots\\ P'ab\end{array}}{b{=}b'\&P'ab}\;?$$

$$\dfrac{\begin{array}{c}\dfrac{[b{=}b']^1 \quad b{=}b'\supset Rab}{Rab}\supset E\\ \vdots\\ R'ab\end{array}}{b{=}b'\supset Rab}\supset I;1$$



$$\dfrac{\exists a.fa=b\&Pa \qquad \dfrac{\dfrac{\dfrac{[fa=b\&Pa]^1}{fa=b}\,\&E_1 \qquad \begin{array}{c}\dfrac{[fa=b\&Pa]^1}{Pa}\,\&E_2\\ \vdots\\ P'a\end{array}}{fa=b\&P'a}\,\&I}{\exists a.fa=b\&P'a}\,\exists I}{\exists a.fa=b\&P'a}\,\exists E;1$$

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{[fa=b]^1 \qquad \dfrac{\forall a.fa=b\supset Ra}{fa=b\supset Ra}}{Ra}\\ \vdots\\ R'a\end{array}}{fa=b\supset R'a}\,1}{\forall a.fa=b\supset R'a}$$

The adjunction maps as ND proofs

$$\begin{array}{ccc} P\&Q & \Longleftarrow & P \\ \downarrow & \overset{\flat}{\longleftarrow}\ \underset{\sharp}{\longrightarrow} & \downarrow \\ R & \Longrightarrow & Q \supset R \end{array}$$

$$\dfrac{\begin{array}{c}\dfrac{P \quad [Q]^1}{P\&Q}\ \&I \\ \vdots \\ R\end{array}}{Q \supset R}\ (\supset I) \qquad \dfrac{\dfrac{P\&Q}{Q}\ \&E_2 \qquad \begin{array}{c}\dfrac{P\&Q}{P}\ \&E_1 \\ \vdots \\ Q \supset R\end{array}}{R}\ \supset E$$

$$\begin{array}{ccc} (P,Q) & \Longrightarrow & P \vee Q \\ \downarrow & \overset{\flat}{\longrightarrow}\ \underset{\sharp}{\longleftarrow} & \downarrow \\ (R,R) & \Longleftarrow & R \\ \downarrow & \overset{\flat}{\longleftarrow}\ \underset{\sharp}{\longrightarrow} & \downarrow \\ (S,T) & \Longrightarrow & S\&T \end{array}$$

$$\dfrac{P \vee Q \qquad \begin{array}{c}[P]^1 \\ \vdots \\ R\end{array} \qquad \begin{array}{c}[Q]^1 \\ \vdots \\ R\end{array}}{R}\ \vee E \qquad \begin{array}{c}\dfrac{P}{P \vee Q}\ \vee I_1 \\ \vdots \\ R\end{array} \qquad \begin{array}{c}\dfrac{Q}{P \vee Q}\ \vee I_2 \\ \vdots \\ R\end{array}$$

$$\dfrac{\begin{array}{c}R \\ \vdots \\ S\end{array} \quad \begin{array}{c}R \\ \vdots \\ T\end{array}}{S\&T}\ \&I \qquad \begin{array}{c}R \\ \vdots \\ \dfrac{S\&T}{S}\ \&E_1\end{array} \qquad \begin{array}{c}R \\ \vdots \\ \dfrac{S\&T}{T}\ \&E_2\end{array}$$

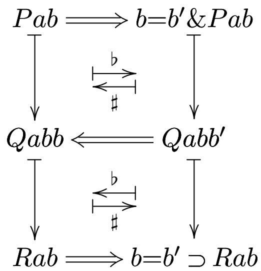
$$\begin{array}{ccc} Pab & \Longrightarrow & \exists b.Pab \\ \downarrow & \overset{\flat}{\longrightarrow}\ \underset{\sharp}{\longleftarrow} & \downarrow \\ Qa & \Longleftarrow & Qa \\ \downarrow & \overset{\flat}{\longleftarrow}\ \underset{\sharp}{\longrightarrow} & \downarrow \\ Rab & \Longrightarrow & \forall b.Rab \end{array}$$

$$a, b \longmapsto a$$

$$\dfrac{\exists b.Pab \qquad \begin{array}{c}[Pab]^1 \\ \vdots \\ Qa\end{array}}{Qa}\ \exists E \qquad \begin{array}{c}\dfrac{Pab}{\exists b.Pab}\ \exists I \\ \vdots \\ Qa\end{array}$$

$$\dfrac{Qa \qquad \begin{array}{c}[Qa]^1 \\ \vdots \\ Rab\end{array}}{\forall b.Rab}\ \forall I \qquad \dfrac{b \qquad \begin{array}{c}Qa \\ \vdots \\ \forall b.Rab\end{array}}{Rab}\ \forall E$$

The adjunction maps as ND proofs (2)

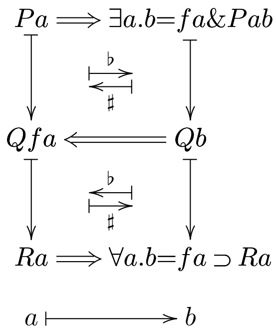


$$a, b \longmapsto a, b, b'$$

$$\dfrac{\dfrac{b{=}b'\&Pab}{b{=}b'}\,\&E_1 \qquad \begin{matrix}\dfrac{b{=}b'\&Pab}{Pab}\,\&E_2\\ \vdots \\ Qabb\end{matrix}}{Qabb'} = E \qquad\qquad \dfrac{\dfrac{}{b = b} = I \qquad \begin{matrix}\dfrac{[b{=}b']^1 \quad Pab}{b{=}b'\&Pab}\\ \vdots\\ Qabb'\end{matrix}}{Qabb}\; b' := b; 1$$

$$\dfrac{\begin{matrix}\dfrac{[b{=}b']^1 \quad Qabb'}{Qabb}\;?\\ \vdots \\ Rab\end{matrix}}{b{=}b' \supset Rab} \supset I; 1 \qquad\qquad \dfrac{\dfrac{}{b = b} = I \qquad Qabb \qquad \dfrac{[b{=}b']^1 \quad \begin{matrix}[Qabb']^1\\ \vdots\\ b{=}b' \supset Rab\end{matrix}}{Rab} \supset E}{Rab}\; b' := b; 1$$

The adjunction maps as ND proofs (3)



$$\dfrac{\exists a.b{=}fa\&Pa \qquad \dfrac{\dfrac{[b{=}fa\&Pa]^1}{b{=}fa} \qquad \begin{matrix}\dfrac{[b{=}fa\&Pa]^1}{Pa}\\ \vdots \\ Qfa\end{matrix}}{Qb}}{Qb}\;1 \qquad\qquad \dfrac{\dfrac{}{fa=fa} \qquad \dfrac{[b{=}fa]^1 \qquad \begin{matrix}\dfrac{\dfrac{[b{=}fa]^1 \quad Pa}{b{=}fa\&Pa}}{\exists a.b{=}fa\&Pa}\\ \vdots \\ Qb\end{matrix}}{Qfa}}{Qfa}$$

$$\dfrac{\dfrac{\begin{matrix}\dfrac{[fa=b]^1 \quad Qb}{Qfa}\\ \vdots \\ Ra\end{matrix}}{fa=b\supset Ra}\;1}{\forall a.fa=b\supset Ra} \qquad\qquad \dfrac{\dfrac{}{fa=fa} \qquad \dfrac{Qfa \qquad \dfrac{\dfrac{[Qfa']^2 \quad \begin{matrix}[Qb]^1\\ \vdots \\ \forall a.fa=b\supset Ra\end{matrix}}{\forall a.fa=fa'\supset Ra}\;1}{fa=fa\supset Ra}}{fa=fa\supset Ra}\;2}{Ra}$$

The Frobenius maps as ND proofs
The preservations of $\top, \bot, \&, \lor, \supset$ are all trivially true intuitionistically, as they are all of the form $\alpha \vdash \alpha$...

Same for Beck-Chevalley.

The $\natural$ arrows whose inverses are the Frobenius maps can be constructed from operations whose translations to Intuitionistic Logic we have already seen.

$$\exists_\pi(P\&\pi^*Q) \underset{\mathsf{Frob}\exists}{\overset{\natural}{\rightleftarrows}} (\exists_\pi P)\&Q$$

$$\exists b.(Pab\&Qa) \underset{\mathsf{Frob}\exists}{\overset{\natural}{\rightleftarrows}} (\exists b.Pab)\&Qa$$

$$\mathsf{Eq}_\delta(P\&\delta^*Q) \underset{\mathsf{Frob}=}{\overset{\natural}{\rightleftarrows}} (\mathsf{Eq}_\delta P)\&Q$$

$$b{=}b'\&(Pabb\&Qab) \underset{\mathsf{Frob}=}{\overset{\natural}{\rightleftarrows}} (b{=}b'\&Pabb)\&Qab$$

$$\exists{=}_f(P\&f^*Q) \underset{\mathsf{Frob}\exists=}{\overset{\natural}{\rightleftarrows}} (\exists{=}_f P)\&Q$$

$$\exists a.b{=}fa\&(Pa\&Qfa) \underset{\mathsf{Frob}\exists=}{\overset{\natural}{\rightleftarrows}} (\exists a.b{=}fa\&Pa)\&Qb$$

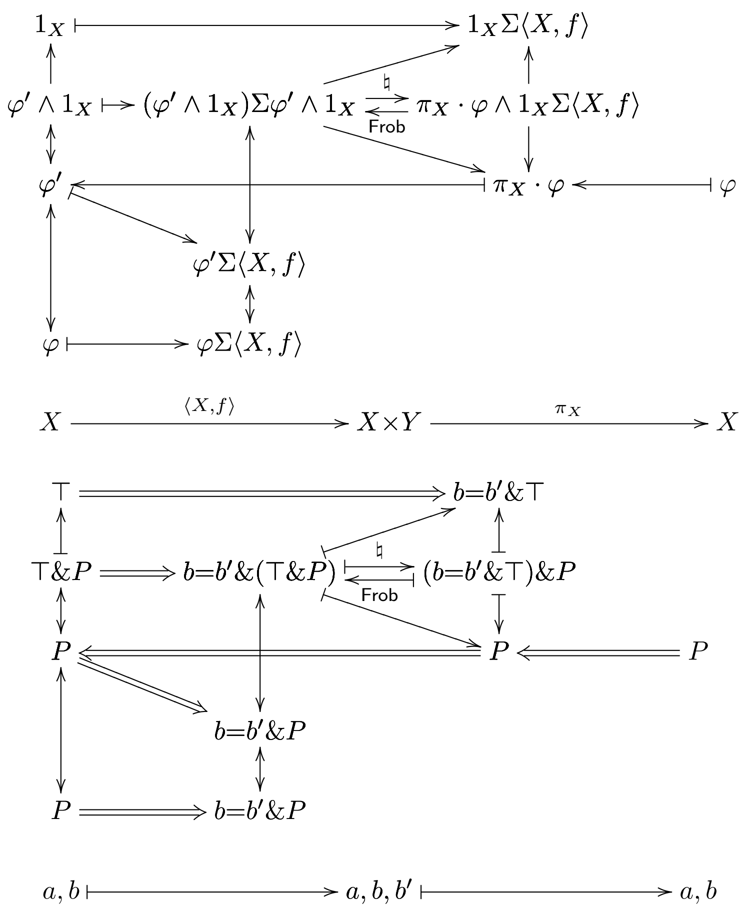
We need to show that the Frobenius maps "hold intuitionistically".

Here is a ND proof for the first case:
(the other cases are similar - and the trees are big)

$$\dfrac{\dfrac{(\exists b.Pab)\&Qa}{\exists b.Pab}\,\&E_1 \qquad \dfrac{\dfrac{[Pab]^1 \qquad \dfrac{(\exists b.Pab)\&Qa}{Qa}\,\&E_2}{Pab\&Qa}\,\&I}{\exists b.(Pab\&Qa)}\,\exists I}{\exists b.(Pab\&Qa)}\,\exists E;1$$

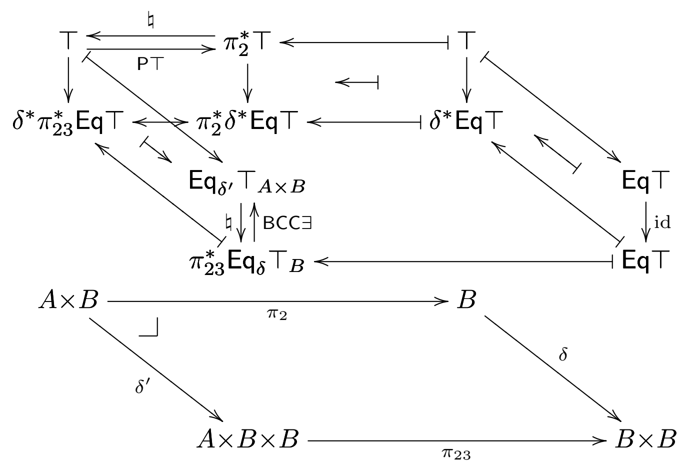
Frobenius for equality (2)
(Now we will start to see consequences of the structure)

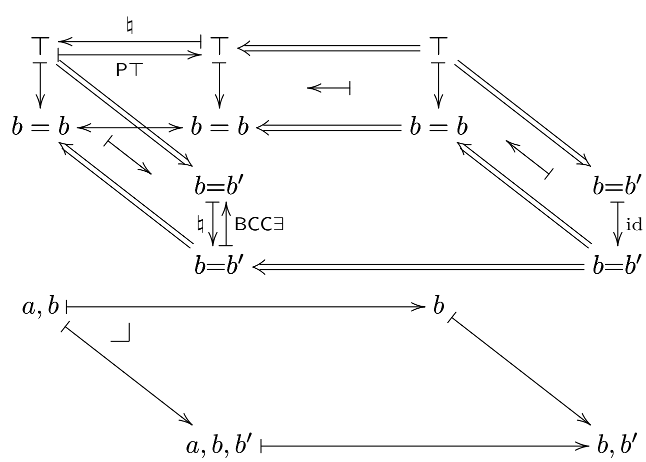
If we have $\mathsf{Eq}\top$ and Frobenius, then we have $\mathsf{Eq}P$:



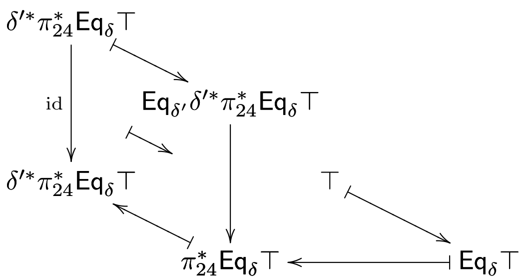
From now on we will usually write "$b{=}b'\&\top$" as just "$b{=}b'$".
Frobenius for equality implies that the four obvious definitions for $b{=}b'\&P$ are isomorphic.

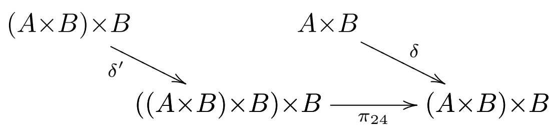
Dependent equality from simple equality

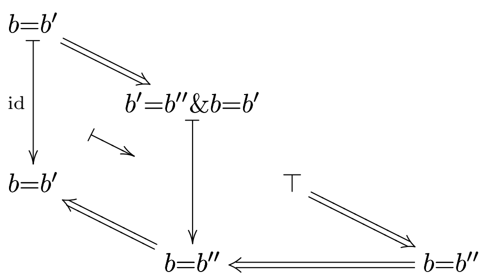


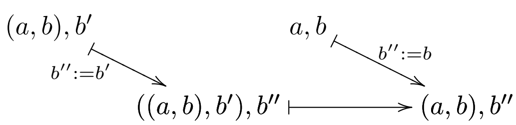


Transitivity of equality

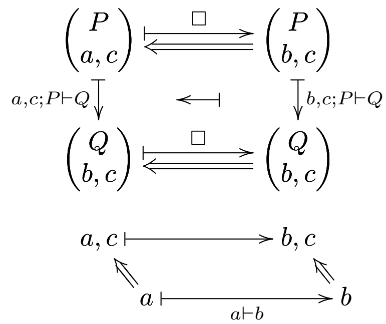




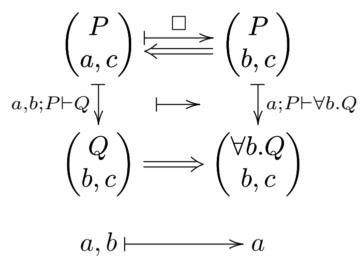


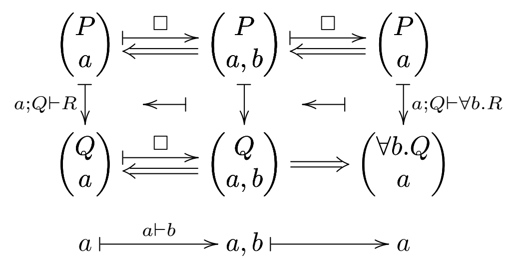


Hyperdoctrines: substitution, quantifiers, reflexivity
The substitution rule:

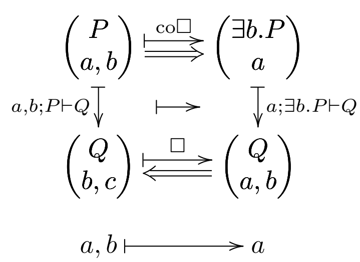


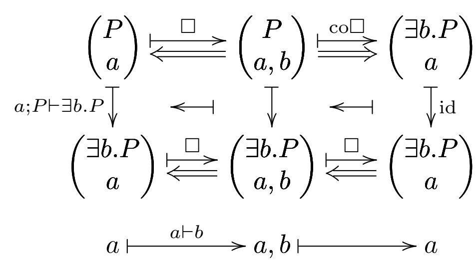
The '(∀I)' and '(∀E)' rules:



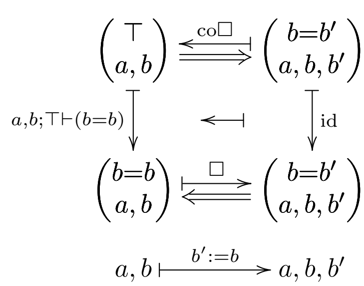


The '(∃E)' and '(∃I)' rules:

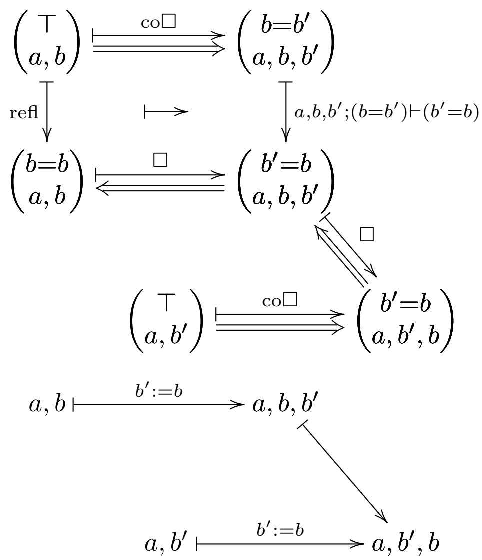




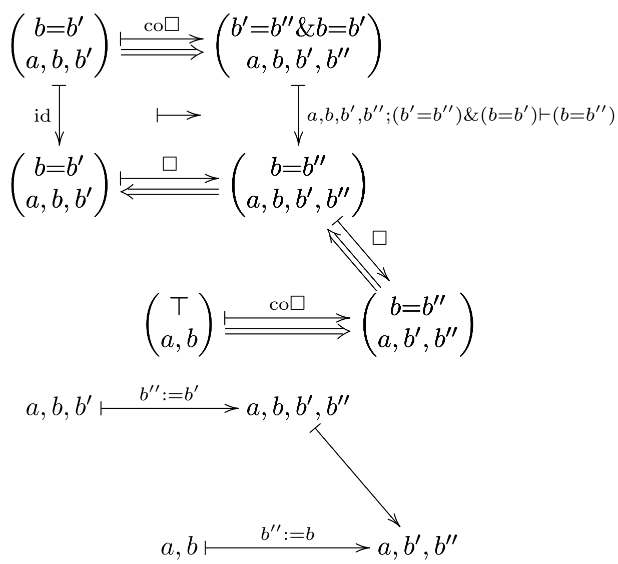
The "reflexivity" rule for equality (on variables):



Hyperdoctrines: symmetry, transitivity
The “symmetry” rule for equality (on variables):



The “transitivity” rule for equality (on variables):



Weak and strong exists-elim
There are two versions of $(\exists E)$, $(\exists E^-)$ and $(\exists E^+)$.
They are "equivalent enough", and $(\exists E^-)$ is much simpler categorically.
The rules, in natural deduction and sequent calculus forms:

$$\dfrac{\exists b.P(a,b) \qquad \begin{matrix}[P(a,b)]^1 \quad Q(a)\\ \vdots \\ R(a)\end{matrix}}{R(a)}\;1;(\exists E^+) \qquad \dfrac{a,b;P(a,b),Q(a)\vdash R(a)}{a;\exists b.P(a,b),Q(a)\vdash R(a)}\;(\exists E^+)$$

$$\dfrac{\exists b.P(a,b) \qquad \begin{matrix}[P(a,b)]^1\\ \vdots \\ Q(a)\end{matrix}}{Q(a)}\;1;(\exists E^-) \qquad \dfrac{a,b;P(a,b)\vdash Q(a)}{a;\exists b.P(a,b)\vdash Q(a)}\;(\exists E^-)$$

$(\exists E^-)$ is a particular case of $(\exists E^+)$ — take $Q(a) := \top$ in $(\exists E^+)$.

In the presence of $(\supset)$ the rule $(\exists E^-)$ implies $(\exists E^+)$:

$$\dfrac{Q(a) \qquad \dfrac{\exists b.P(a,b) \qquad \dfrac{\begin{matrix}[P(a,b)]^2 \quad [Q(a)]^1\\ \vdots\\ R(a)\end{matrix}}{Q(a)\supset R(a)}\;1;(\supset I)}{Q(a)\supset R(a)}\;2;(\exists E^+)}{R(a)}\;(\supset E)$$

$(\exists E^-)$, categorically:

$$\begin{array}{ccc} P(a,b) & \Rightarrow & \exists b.P(a,b)\\ \downarrow & \mapsto & \downarrow \\ Q(a) & \Longleftarrow & Q(a) \\ \\ a,b & \longmapsto & a\end{array}$$

$(\exists E^+)$, categorically:

$$\begin{array}{ccccccc} P(a,b)\&Q(a) & \Leftarrow & P(a,b) & \Longrightarrow & \exists b.P(a,b) & \Longrightarrow & (\exists b.P(a,b))\&Q(a)\\ \downarrow & \mapsto & \downarrow & \mapsto & \downarrow & \mapsto & \downarrow \\ R(a) & \Longrightarrow & Q(a)\supset R(a) & \Leftarrow & Q(a)\supset R(a) & \Longleftarrow & R(a) \\ \\ a,b & = & a,b & \longmapsto & a & = & a\end{array}$$

Note that $\mathbf{O}[\begin{pmatrix}Q(a)\supset R(a)\\ a,b\end{pmatrix}]$ has two constructions,
with a hidden iso between them.

Rules in ND and sequent calculus

The rules, in natural deduction form (($\forall E$) is wrong):

$$\dfrac{\begin{array}{cc} P & P \\ \vdots & \vdots \\ Q & R \end{array}}{Q\&R}\ (\&I)$$

$$\begin{array}{cc} P & \dfrac{Q\&R}{Q}\ (\&E_1) \\ & \vdots \\ & S \end{array} \qquad \begin{array}{cc} P & \dfrac{Q\&R}{R}\ (\&E_2) \\ & \vdots \\ & S \end{array}$$

$$\dfrac{\begin{array}{cc} P & [Q]^1 \\ & \vdots \\ & R \end{array}}{Q \supset R}\ 1; (\supset I)$$

$$\dfrac{\begin{array}{cc} P & P \\ \vdots & \vdots \\ Q & Q \supset R \end{array}}{R}\ (\supset E)$$

$$\dfrac{\begin{array}{c} P(a) \\ \vdots \\ Q(a,b) \end{array}}{\forall b.Q(a,b)}\ (\forall I)$$

$$\dfrac{\begin{array}{cc} a & Q(a) \\ \vdots & \vdots \\ f(a) & \forall b.R(a,b) \end{array}}{R(a,f(a))}\ (\forall E)$$

$$\dfrac{\begin{array}{c} Q(a) \\ \vdots \\ P(a,f(a)) \end{array}}{\exists b.P(a,b)}\ (\exists I)$$

$$\dfrac{\begin{array}{ccc} & [P(a,b)]^1 & Q(a) \\ & \vdots & \\ \exists b.P(a,b) & R(a) & \end{array}}{R(a)}\ (\exists E)$$

The rules, in sequent calculus form (($\forall E$) is wrong):

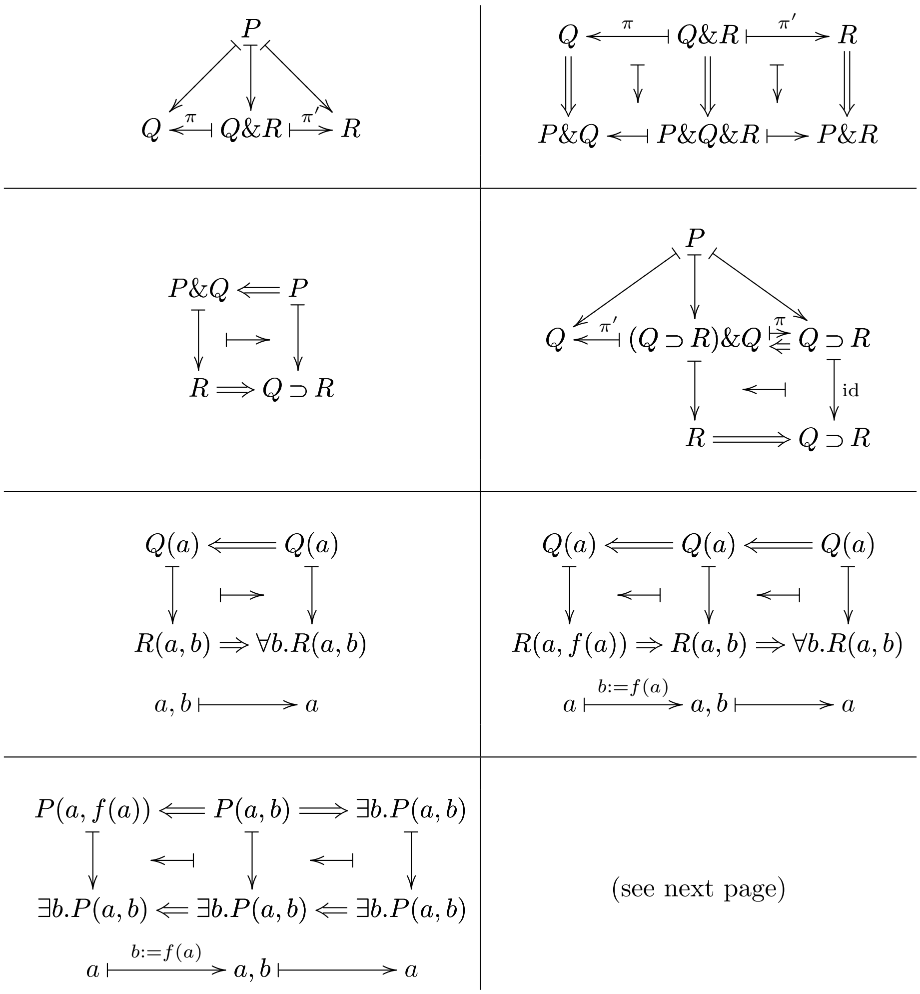
$$\dfrac{P \vdash Q \quad P \vdash R}{P \vdash Q\&R}\ (\&I) \qquad \dfrac{P, Q \vdash S}{P, Q\&R \vdash S}\ (\&E_1) \qquad \dfrac{P, R \vdash S}{P, Q\&R \vdash S}\ (\&E_2)$$

$$\dfrac{P, Q \vdash R}{P \vdash Q \supset R}\ (\supset I) \qquad \dfrac{P \vdash Q \quad P \vdash Q \supset R}{P \vdash R}\ (\supset E)$$

$$\dfrac{a, b; P(a) \vdash Q(a,b)}{a; P(a) \vdash \forall b.Q(a,b)}\ (\forall I) \qquad \dfrac{a \vdash f(a) \quad a; Q(a) \vdash \forall b.R(a,b)}{a; Q(a) \vdash R(a, f(a))}\ (\forall E)$$

$$\dfrac{a; Q(a) \vdash P(a, f(a))}{a; Q(a) \vdash \exists b.P(a,b)}\ (\exists I) \qquad \dfrac{a, b; P(a,b), Q(a) \vdash R(a)}{a; \exists b.P(a,b), Q(a) \vdash R(a)}\ (\exists E)$$

Rules, categorically
The rules, categorically (($\&E$) needs explanations, ($\forall E$) is wrong):



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