Just some trivial notes on Valéria de Paiva and Gavin Bierman's "On an Intuitionistic Modal Logic" (2000).
`http://www.cs.bham.ac.uk/~vdp/publications/studia.ps.gz`

Index of the slides:

“Box” commutes with “and”

$\Box(A \times B) \supset (\Box A \times \Box B)$ and $(\Box A \times \Box B) \supset \Box(A \times B)$:

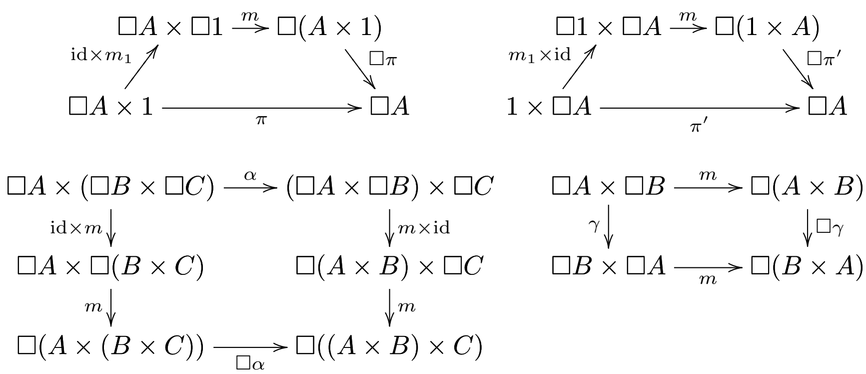
$$\dfrac{\dfrac{\Box(A \times B) \qquad \dfrac{\dfrac{[\Box(A \times B)]^1}{A \times B}}{A}}{\Box A}\,1 \qquad \dfrac{\Box(A \times B) \qquad \dfrac{\dfrac{[\Box(A \times B)]^1}{A \times B}}{B}}{\Box B}\,1}{\Box A \times \Box B}$$

$$\dfrac{\dfrac{\Box A \times \Box B}{\Box A} \qquad \dfrac{\Box A \times \Box B}{\Box B} \qquad \dfrac{\dfrac{[\Box A]^1}{A} \qquad \dfrac{[\Box B]^1}{B}}{A \times B}}{\Box(A \times B)}\,1$$

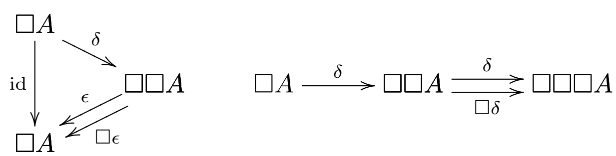
but the two maps between $\Box(A \times B) \leftrightarrows (\Box A \times \Box B)$ need not be inverses.

Categorical structure (1)

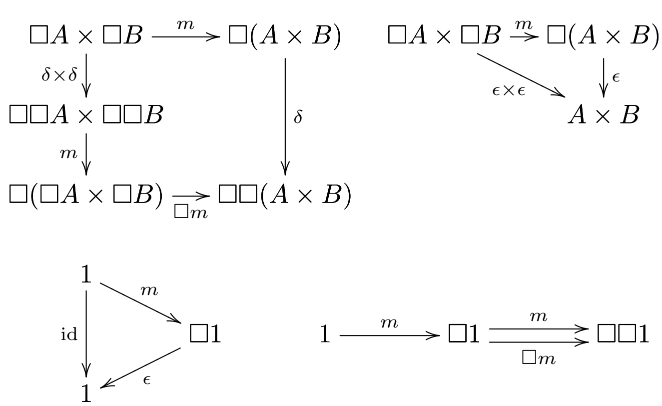
Monoidal functor:



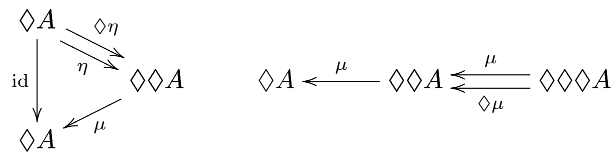
Comonad:



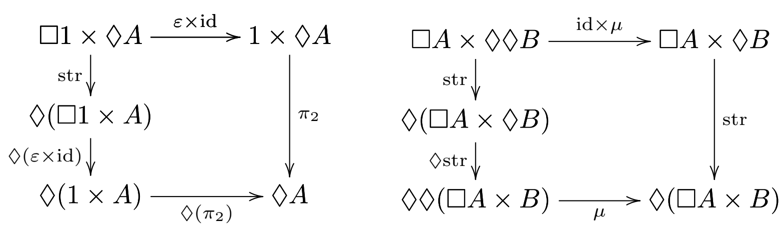
Monoidal comonad:

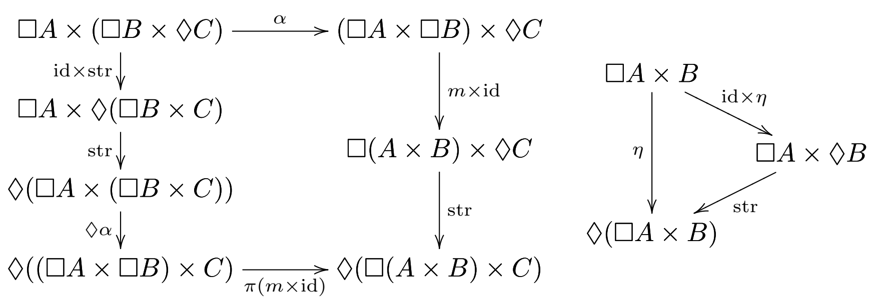


Categorical structure (2)
Monad:

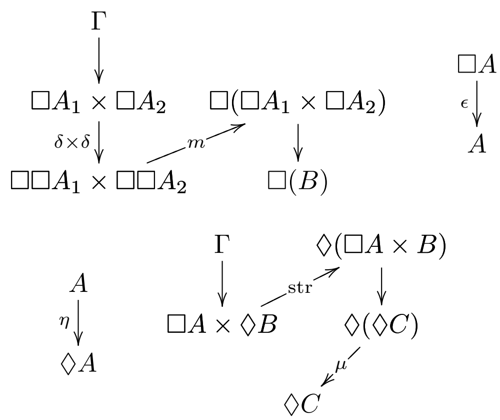


Strength:



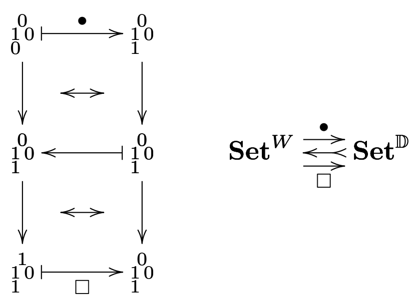


The natural deduction rules, categorically



Notes on Reyes & Zolfaghari's paper
Let W be a set of worlds, and let $\mathbb{D}$ be a DAG on W.
$\mathbb{D}$ induces a reflexive and transitive relation $R \subset W \times W$ —
$(\alpha \to \beta) \in \mathbb{D}$ iff $\alpha R \beta$ — "α sees β".
$\langle W, R \rangle$ is an S4-ish modal frame.
We can regard W and $\mathbb{D}$ as topological spaces:
W is discrete, $\mathbb{D}$ has the topology in which the open sets are the ones whose characteristic functions are non-decreasing.
The "identity" function $W \to \mathbb{D}$ is continuous.
Consider these toposes: $\mathbf{Set}^W$ and $\mathbf{Set}^{\mathbb{D}}$.
The truth-values of $\mathbf{Set}^W$ are the things that I call the "modal truth-values" on $\mathbb{D}$: they correspond to arbitrary functions $W \to \{0,1\}$.
The truth-values of $\mathbf{Set}^D$ are the things that I call the "intuitionistic truth-values" of $\mathbb{D}$: they correspond to non-decreasing functions $W \to \{0,1\}$.
There is a geometric morphism $\mathbf{Set}^W \to \mathbf{Set}^{\mathbb{D}}$.
'$\Box$' and '$\Diamond$' are endofunctors that act on the space of truth-values of $\mathbf{Set}^W$.
The intuitionistic truth-values are the ones that are stable by the action of $\Box$.

Hypothesis:



Beta-reduction rules

$$\dfrac{N : A \qquad \dfrac{\begin{array}{c}[x:A]^1 \quad \Gamma\\ \vdots\\ M : B\end{array}}{(\lambda x : A.M) : A \to B}\supset\mathcal{I};1}{(\lambda x:A.M)N : B}\supset\mathcal{E} \qquad\qquad \begin{array}{c} N:A \quad \Gamma \\ \vdots \\ M[x:=N]:B\end{array}$$

$$\dfrac{\dfrac{M:A \quad N:B}{\langle M,N\rangle : A\times B}\wedge\mathcal{I}}{\mathsf{fst}\langle M,N\rangle : A\times B}\wedge\mathcal{E}_1 \quad \rightsquigarrow_\beta \quad M:A$$

$$\dfrac{\dfrac{M:A}{\mathsf{inl}(M) : A+B}\vee\mathcal{I} \qquad \begin{array}{c}[x:A]^1 \quad \Gamma\\ \vdots\\ N : C\end{array} \qquad \begin{array}{c}[y:B]^1 \quad \Delta\\ \vdots\\ P : C\end{array}}{\mathsf{case}\ \mathsf{inl}(M)\ \mathsf{of}\ \mathsf{inl}(x) \to N \mid\mid \mathsf{inr}(y) \to P}\vee\mathcal{E} \quad \rightsquigarrow_\beta \quad \begin{array}{c} M:A \quad \Gamma \\ \vdots \\ N[x:=M]:C\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \vec{M} : \Box A\end{array} \qquad \begin{array}{c}[\vec{x}:\Box\vec{A}]^1\\ \vdots\\ N : B\end{array}}{\mathsf{box}\ N\ \mathsf{with}\ \vec{M}\ \mathsf{for}\ \vec{x} : \Box B}\Box\mathcal{I};1}{\mathsf{unbox}(\mathsf{box}\ N\ \mathsf{with}\ \vec{M}\ \mathsf{for}\ \vec{x} : \Box B)}\Box\mathcal{E} \quad \rightsquigarrow_\beta \quad \begin{array}{c} \Gamma \\ \vdots \\ \vec{M} : \Box\vec{A} \\ \vdots \\ N[\vec{x}:=\vec{M}]:B\end{array}$$

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \vec{M} : \Box A\end{array} \qquad \dfrac{\begin{array}{c}\Delta\\ \vdots\\ N : B\end{array}}{\Diamond N : \Diamond B}\Diamond\mathcal{I} \qquad \begin{array}{c}[\vec{x}:\Box\vec{A}\ y : B]^1\\ \vdots\\ P : \Diamond C\end{array}}{P[\vec{x}:=\vec{M}, \Diamond y := \Diamond N] : \Diamond C}\Diamond\mathcal{E};1 \quad \rightsquigarrow_\beta \quad \begin{array}{c} \Gamma \qquad \Delta \\ \vdots \qquad \vdots \\ \vec{M} : \Box A \quad N : B \\ \vdots \\ P[\vec{x}:=\vec{M}, y:=N]:\Diamond C\end{array}$$