

Just some trivial notes on Valéria de Paiva and Gavin  
Bierman's "On an Intuitionistic Modal Logic" (2000).  
<http://www.cs.bham.ac.uk/~vdp/publications/studia.ps.gz>

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**“Box” commutes with “and”**

$\Box(A \times B) \supset (\Box A \times \Box B)$  and  $(\Box A \times \Box B) \supset \Box(A \times B)$ :

$$\frac{\frac{\frac{\frac{\Box(A \times B)}{A \times B}}{A}}{\Box A} \quad 1 \quad \frac{\frac{\frac{\Box(A \times B)}{A \times B}}{B}}{\Box B} \quad 1}{\Box A \times \Box B}}$$

$$\frac{\frac{\frac{\Box A \times \Box B}{\Box A} \quad \frac{\Box A \times \Box B}{\Box B} \quad \frac{\frac{\Box A}{A} \quad \frac{\Box B}{B}}{A \times B}}{\Box(A \times B)} \quad 1}{\Box(A \times B)}$$

but the two maps between  $\Box(A \times B) \rightleftarrows (\Box A \times \Box B)$  need not be inverses.

**Categorical structure (1)**

Monoidal functor:

$$\begin{array}{ccc}
\begin{array}{ccc}
\Box A \times \Box 1 & \xrightarrow{m} & \Box(A \times 1) \\
\text{id} \times m_1 \nearrow & & \searrow \Box \pi \\
\Box A \times 1 & \xrightarrow{\pi} & \Box A
\end{array} & & 
\begin{array}{ccc}
\Box 1 \times \Box A & \xrightarrow{m} & \Box(1 \times A) \\
m_1 \times \text{id} \nearrow & & \searrow \Box \pi' \\
1 \times \Box A & \xrightarrow{\pi'} & \Box A
\end{array} \\
\\
\begin{array}{ccc}
\Box A \times (\Box B \times \Box C) & \xrightarrow{\alpha} & (\Box A \times \Box B) \times \Box C \\
\text{id} \times m \downarrow & & \downarrow m \times \text{id} \\
\Box A \times \Box(B \times C) & & \Box(A \times B) \times \Box C \\
m \downarrow & & \downarrow m \\
\Box(A \times (B \times C)) & \xrightarrow{\Box \alpha} & \Box((A \times B) \times C)
\end{array} & & 
\begin{array}{ccc}
\Box A \times \Box B & \xrightarrow{m} & \Box(A \times B) \\
\gamma \downarrow & & \downarrow \Box \gamma \\
\Box B \times \Box A & \xrightarrow{m} & \Box(B \times A)
\end{array}
\end{array}$$

Comonad:

$$\begin{array}{ccc}
\begin{array}{ccc}
\Box A & & \\
\text{id} \downarrow & \searrow \delta & \\
\Box A & & \Box \Box A \\
& \swarrow \epsilon & \nearrow \Box \epsilon \\
& \Box A & 
\end{array} & & 
\Box A \xrightarrow{\delta} \Box \Box A \xrightarrow[\Box \delta]{\delta} \Box \Box \Box A
\end{array}$$

Monoidal comonad:

$$\begin{array}{ccc}
\begin{array}{ccc}
\Box A \times \Box B & \xrightarrow{m} & \Box(A \times B) \\
\delta \times \delta \downarrow & & \downarrow \delta \\
\Box \Box A \times \Box \Box B & & \Box \Box(A \times B) \\
m \downarrow & & \downarrow \Box m \\
\Box(\Box A \times \Box B) & \xrightarrow[\Box m]{\delta} & \Box \Box(A \times B)
\end{array} & & 
\begin{array}{ccc}
\Box A \times \Box B & \xrightarrow{m} & \Box(A \times B) \\
& \searrow \epsilon \times \epsilon & \downarrow \epsilon \\
& & A \times B
\end{array} \\
\\
\begin{array}{ccc}
1 & & \\
\text{id} \downarrow & \searrow m & \\
1 & & \Box 1 \\
& \swarrow \epsilon & \nearrow \\
& 1 & 
\end{array} & & 
1 \xrightarrow{m} \Box 1 \xrightarrow[\Box m]{m} \Box \Box 1
\end{array}$$

### Categorical structure (2)

Monad:

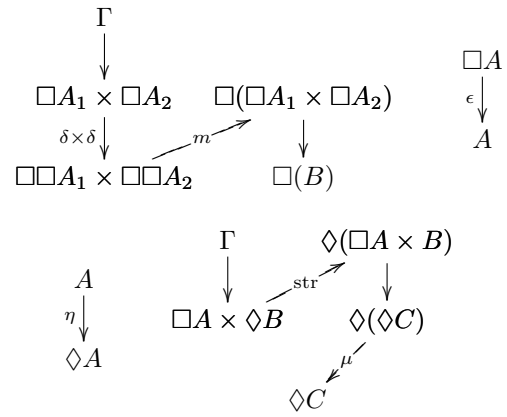
$$\begin{array}{ccc}
 \diamond A & \xrightarrow{\diamond \eta} & \diamond \diamond A \\
 \text{id} \downarrow & \searrow \eta & \\
 \diamond A & \xleftarrow{\mu} & \diamond \diamond A
 \end{array}
 \quad
 \diamond A \xleftarrow{\mu} \diamond \diamond A \xleftarrow[\diamond \mu]{\mu} \diamond \diamond \diamond A$$

Strength:

$$\begin{array}{ccc}
 \square 1 \times \diamond A & \xrightarrow{\varepsilon \times \text{id}} & 1 \times \diamond A \\
 \text{str} \downarrow & & \downarrow \pi_2 \\
 \diamond(\square 1 \times A) & & \diamond A \\
 \diamond(\varepsilon \times \text{id}) \downarrow & & \downarrow \diamond(\pi_2) \\
 \diamond(1 \times A) & \xrightarrow{\quad} & \diamond A
 \end{array}
 \quad
 \begin{array}{ccc}
 \square A \times \diamond \diamond B & \xrightarrow{\text{id} \times \mu} & \square A \times \diamond B \\
 \text{str} \downarrow & & \downarrow \text{str} \\
 \diamond(\square A \times \diamond B) & & \diamond(\square A \times B) \\
 \diamond \text{str} \downarrow & & \downarrow \mu \\
 \diamond \diamond(\square A \times B) & \xrightarrow{\quad} & \diamond(\square A \times B)
 \end{array}$$

$$\begin{array}{ccc}
 \square A \times (\square B \times \diamond C) & \xrightarrow{\alpha} & (\square A \times \square B) \times \diamond C \\
 \text{id} \times \text{str} \downarrow & & \downarrow m \times \text{id} \\
 \square A \times \diamond(\square B \times C) & & \square(A \times B) \times \diamond C \\
 \text{str} \downarrow & & \downarrow \text{str} \\
 \diamond(\square A \times (\square B \times C)) & & \diamond(\square(A \times B) \times C) \\
 \diamond \alpha \downarrow & & \downarrow \pi(m \times \text{id}) \\
 \diamond((\square A \times \square B) \times C) & \xrightarrow{\quad} & \diamond(\square(A \times B) \times C)
 \end{array}
 \quad
 \begin{array}{ccc}
 \square A \times B & \xrightarrow{\text{id} \times \eta} & \square A \times \diamond B \\
 \eta \downarrow & & \downarrow \text{str} \\
 \diamond(\square A \times B) & \xrightarrow{\quad} & \diamond(\square A \times B)
 \end{array}$$

The natural deduction rules, categorically



### Notes on Reyes & Zolfaghari's paper

Let  $W$  be a set of worlds, and let  $\mathbb{D}$  be a DAG on  $W$ .

$\mathbb{D}$  induces a reflexive and transitive relation  $R \subset W \times W$  —  
 $(\alpha \rightarrow \beta) \in \mathbb{D}$  iff  $\alpha R \beta$  — “ $\alpha$  sees  $\beta$ ”.

$\langle W, R \rangle$  is an S4-ish modal frame.

We can regard  $W$  and  $\mathbb{D}$  as topological spaces:

$W$  is discrete,  $\mathbb{D}$  has the topology in which the open sets are the ones whose characteristic functions are non-decreasing.

The “identity” function  $W \rightarrow \mathbb{D}$  is continuous.

Consider these toposes:  $\mathbf{Set}^W$  and  $\mathbf{Set}^{\mathbb{D}}$ .

The truth-values of  $\mathbf{Set}^W$  are the things that I call the “modal truth-values” on  $\mathbb{D}$ : they correspond to arbitrary functions  $W \rightarrow \{0, 1\}$ .

The truth-values of  $\mathbf{Set}^{\mathbb{D}}$  are the things that I call the “intuitionistic truth-values” of  $\mathbb{D}$ : they correspond to non-decreasing functions  $W \rightarrow \{0, 1\}$ .

There is a geometric morphism  $\mathbf{Set}^W \rightarrow \mathbf{Set}^{\mathbb{D}}$ .

‘ $\square$ ’ and ‘ $\diamond$ ’ are endofunctors that act on the space of truth-values of  $\mathbf{Set}^W$ .

The intuitionistic truth-values are the ones that are stable by the action of  $\square$ .

Hypothesis:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \begin{array}{c} 0 \\ 1 \end{array} 0 & \xrightarrow{\bullet} & \begin{array}{c} 0 \\ 1 \end{array} 0 \\
 \downarrow & \longleftrightarrow & \downarrow \\
 \begin{array}{c} 0 \\ 1 \end{array} 0 & \xleftarrow{\quad} & \begin{array}{c} 0 \\ 1 \end{array} 0 \\
 \downarrow & \longleftrightarrow & \downarrow \\
 \begin{array}{c} 1 \\ 1 \end{array} 0 & \xrightarrow{\square} & \begin{array}{c} 0 \\ 1 \end{array} 0
 \end{array} & & \mathbf{Set}^W \begin{array}{c} \xrightarrow{\bullet} \\ \xleftarrow{\quad} \\ \xrightarrow{\square} \end{array} \mathbf{Set}^{\mathbb{D}}
 \end{array}$$

**Beta-reduction rules**

$$\begin{array}{c}
[x : A]^1 \quad \Gamma \\
\vdots \\
M : B \\
\hline
N : A \quad (\lambda x : A.M) : A \rightarrow B \quad \supset \mathcal{I}; 1 \\
\hline
(\lambda x : A.M)N : B \quad \supset \mathcal{E}
\end{array}
\quad
\begin{array}{c}
N : A \quad \Gamma \\
\vdots \\
M[x := N] : B
\end{array}$$
  

$$\begin{array}{c}
M : A \quad N : B \\
\langle M, N \rangle : A \times B \quad \wedge \mathcal{I} \\
\hline
\text{fst}(M, N) : A \times B \quad \wedge \mathcal{E}_1 \quad \rightsquigarrow_{\beta} \quad M : A
\end{array}$$
  

$$\begin{array}{c}
M : A \quad [x : A]^1 \quad \Gamma \quad [y : B]^1 \quad \Delta \\
\text{inl}(M) : A + B \quad \vee \mathcal{I} \quad N : C \quad P : C \\
\hline
\text{case inl}(M) \text{ of } \text{inl}(x) \rightarrow N \parallel \text{inr}(y) \rightarrow P \quad \vee \mathcal{E} \\
\hline
\rightsquigarrow_{\beta} \quad \begin{array}{c} M : A \quad \Gamma \\ \vdots \\ N[x := M] : C \end{array}
\end{array}$$
  

$$\begin{array}{c}
\Gamma \quad [\vec{x} : \Box \vec{A}]^1 \\
\vdots \\
\vec{M} : \Box A \quad N : B \\
\hline
\text{box } N \text{ with } \vec{M} \text{ for } \vec{x} : \Box B \quad \Box \mathcal{I}; 1 \\
\hline
\text{unbox}(\text{box } N \text{ with } \vec{M} \text{ for } \vec{x} : \Box B) \quad \Box \mathcal{E} \\
\hline
\rightsquigarrow_{\beta} \quad \begin{array}{c} \Gamma \\ \vdots \\ \vec{M} : \Box \vec{A} \\ \vdots \\ N[\vec{x} := \vec{M}] : B \end{array}
\end{array}$$
  

$$\begin{array}{c}
\Gamma \quad \Delta \\
\vdots \quad \vdots \\
\vec{M} : \Box A \quad N : B \quad [\vec{x} : \Box \vec{A} \ y : B]^1 \\
\hline
\text{inl}(M) : \Box A + B \quad \diamond \mathcal{I} \quad P : \diamond C \\
\hline
P[\vec{x} := \vec{M}, \diamond y := \diamond N] : \diamond C \quad \diamond \mathcal{E}; 1 \\
\hline
\rightsquigarrow_{\beta} \quad \begin{array}{c} \Gamma \quad \Delta \\ \vdots \quad \vdots \\ \vec{M} : \Box A \quad N : B \\ \vdots \\ P[\vec{x} := \vec{M}, y := N] : \diamond C \end{array}
\end{array}$$