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The BiCCC structure and the classifier axiom
CCC:

$$\begin{array}{ccc}
 \begin{array}{c} a \\ \swarrow \quad \searrow \\ b \leftarrow \dashv b, c \dashv c \rightarrow c \end{array} & \begin{array}{c} a \\ \downarrow \\ * \end{array} & \begin{array}{c} a, b \Leftarrow a \\ \downarrow \qquad \Downarrow \\ c \Longrightarrow b \rightarrow c \end{array} \\
 \begin{array}{c} a \longrightarrow a \sqcup b \leftarrow b \\ \swarrow \quad \searrow \\ \vdash \end{array} & \begin{array}{c} \perp \\ \downarrow \\ a \end{array} &
 \end{array}$$

HA:

$$\begin{array}{ccc}
 \begin{array}{c} P \\ \swarrow \quad \searrow \\ Q \leftarrow \dashv Q \& R \dashv R \rightarrow R \end{array} & \begin{array}{c} P \\ \downarrow \\ \top \end{array} & \begin{array}{c} P \& Q \Leftarrow P \\ \downarrow \qquad \Downarrow \\ R \Longrightarrow Q \supset R \end{array} \\
 \begin{array}{c} P \longrightarrow P \vee Q \leftarrow Q \\ \swarrow \quad \searrow \\ \vdash \end{array} & \begin{array}{c} \perp \\ \downarrow \\ P \end{array} &
 \end{array}$$

Topos:

$$\begin{array}{ccc}
 \begin{array}{c} (a, b)|_c \longrightarrow b \\ \downarrow \quad \lrcorner \\ a \dashv \rightarrow c \end{array} & \begin{array}{c} a|_P \dashv \longrightarrow * \\ \downarrow \quad \lrcorner \\ a \dashv \xrightarrow{P} \omega \end{array} &
 \end{array}$$

The pre-hyperdoctrine structure

Hyperdoctrine:

$$\begin{array}{ccc}
 \left(\begin{matrix} P \\ a, b \end{matrix} \right) \xrightleftharpoons{\text{co}\square} \left(\begin{matrix} b=b' \& P \\ a, b, b' \end{matrix} \right) & \left(\begin{matrix} P \\ a, b \end{matrix} \right) \xrightleftharpoons{\text{co}\square} \left(\begin{matrix} \exists b. P \\ a \end{matrix} \right) \\
 \downarrow & \Leftrightarrow & \downarrow \\
 \left(\begin{matrix} P \\ a \end{matrix} \right) \xrightarrow{\square} \left(\begin{matrix} P \\ b \end{matrix} \right) & \left(\begin{matrix} Q[b, b] \\ a, b \end{matrix} \right) \xrightleftharpoons{\square} \left(\begin{matrix} Q[b, b'] \\ a, b, b' \end{matrix} \right) & \left(\begin{matrix} Q \\ a, b \end{matrix} \right) \xrightleftharpoons{\square} \left(\begin{matrix} Q \\ a \end{matrix} \right) \\
 \downarrow & & \downarrow \\
 \left(\begin{matrix} R \\ a, b \end{matrix} \right) \Longrightarrow \left(\begin{matrix} \forall b. R \\ a \end{matrix} \right) & &
 \end{array}$$

$$a \longrightarrow b$$

$$a, b \longrightarrow a, b, b'$$

$$a, b \longrightarrow a$$

LCCC:

$$\begin{array}{ccc}
 \left(\begin{matrix} c \\ a, b \end{matrix} \right) \xrightleftharpoons{\text{co}\square} \left(\begin{matrix} (b=b'), c \\ a, b, b' \end{matrix} \right) & \left(\begin{matrix} c \\ a, b \end{matrix} \right) \xrightleftharpoons{\text{co}\square} \left(\begin{matrix} b, c \\ a \end{matrix} \right) \\
 \downarrow & \Leftrightarrow & \downarrow \\
 \left(\begin{matrix} c \\ a \end{matrix} \right) \xrightarrow{\square} \left(\begin{matrix} c \\ b \end{matrix} \right) & \left(\begin{matrix} d \\ a, b \end{matrix} \right) \xrightleftharpoons{\square} \left(\begin{matrix} d \\ a, b, b' \end{matrix} \right) & \left(\begin{matrix} d \\ a, b \end{matrix} \right) \xrightleftharpoons{\square} \left(\begin{matrix} d \\ a \end{matrix} \right) \\
 \downarrow & & \downarrow \\
 \left(\begin{matrix} e \\ a, b \end{matrix} \right) \Longrightarrow \left(\begin{matrix} b \mapsto e \\ a \end{matrix} \right) & &
 \end{array}$$

$$a \longrightarrow b$$

$$a, b \longrightarrow a, b, b'$$

$$a, b \longrightarrow a$$

Two CCompC structures in a topos

The two CCompC structures in a topos:

$$\begin{array}{c}
 \begin{array}{ccccc}
 (P) & \xrightarrow{\quad} & (T) & \xrightarrow{\quad} & (Q) \\
 a & & b & & c \\
 \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow \\
 a & \longrightarrow & b & \longrightarrow & c|_Q
 \end{array}
 \quad
 \begin{array}{ccccc}
 (b) & \xrightarrow{\quad} & (*) & \xrightarrow{\quad} & (e) \\
 a & & c & & d \\
 \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow \\
 a & \longrightarrow & c & \longrightarrow & d, e
 \end{array}
 \end{array}$$

Cartesian morphisms project into pullbacks:

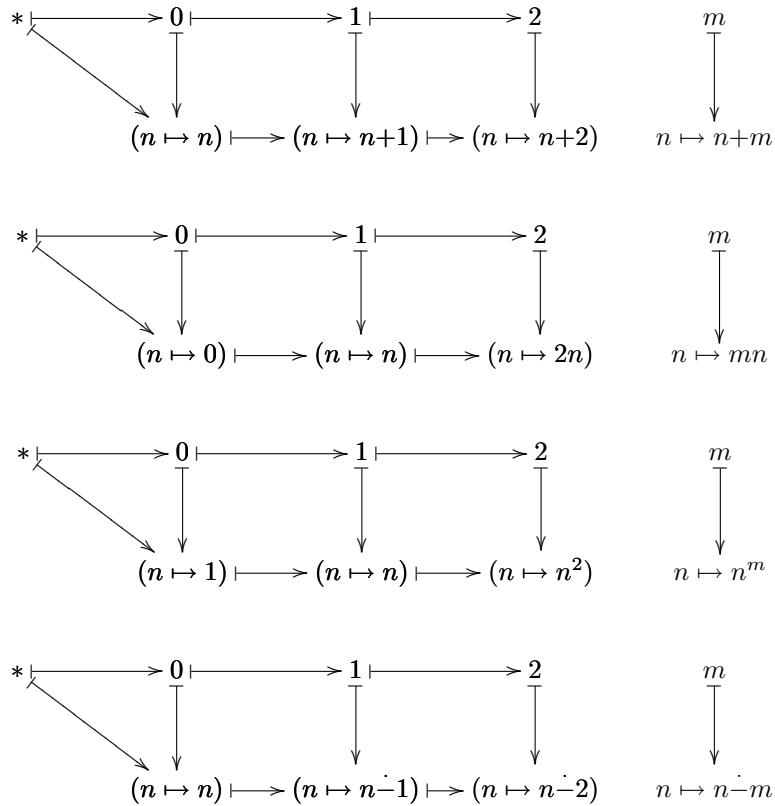
$$\begin{array}{c}
 \begin{array}{ccc}
 (P) & & (P) \\
 a & \Downarrow & b \\
 \Downarrow & \Downarrow & \Downarrow \\
 a|_P & \xrightarrow{\quad} & a \\
 & \lrcorner & \lrcorner \\
 & \Downarrow & \Downarrow \\
 & b|_P & \xrightarrow{\quad} & b
 \end{array}
 \quad
 \begin{array}{ccc}
 (c) & & (c) \\
 a & \Downarrow & b \\
 \Downarrow & \Downarrow & \Downarrow \\
 a, c & \xrightarrow{\quad} & a \\
 & \lrcorner & \lrcorner \\
 & \Downarrow & \Downarrow \\
 & b, c & \xrightarrow{\quad} & b
 \end{array}
 \end{array}$$

Cocartesian morphisms induce isos and epis:

$$\begin{array}{c}
 \begin{array}{ccc}
 (P) & & (\exists b.P) \\
 a, b & \Downarrow & a \\
 \Downarrow & \Downarrow & \Downarrow \\
 a, b|_P & \xrightarrow{\quad} & a, b \\
 & \lrcorner & \lrcorner \\
 & \Downarrow & \Downarrow \\
 & a|_{\exists b.P} & \xrightarrow{\quad} & a
 \end{array}
 \quad
 \begin{array}{ccc}
 (P) & & (b=b' \& P) \\
 a, b & \Downarrow & a \\
 \Downarrow & \Downarrow & \Downarrow \\
 a, b|_P & \xrightarrow{\quad} & a, b \\
 & \lrcorner & \lrcorner \\
 & \Downarrow & \Downarrow \\
 & a, b, b'|_{b=b' \& P} & \xrightarrow{\quad} & a, b, b'
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{ccc}
 (c) & & (b, c) \\
 a, b & \Downarrow & a \\
 \Downarrow & \Downarrow & \Downarrow \\
 (a, b), c & \xrightarrow{\quad} & a, b \\
 & \lrcorner & \lrcorner \\
 & \Downarrow & \Downarrow \\
 & a, (b, c) & \xrightarrow{\quad} & a
 \end{array}
 \quad
 \begin{array}{ccc}
 (c) & & ((b=b'), c) \\
 a, b & \Downarrow & a \\
 \Downarrow & \Downarrow & \Downarrow \\
 a, b, c & \xrightarrow{\quad} & a, b \\
 & \lrcorner & \lrcorner \\
 & \Downarrow & \Downarrow \\
 & a, b, b', (b=b'), c & \xrightarrow{\quad} & a, b, b'
 \end{array}
 \end{array}$$

Basic constructions with NNOs



NNOs: the morphism p'

Fact: in a topos with NNO the map $[0, s] : 1+N \rightarrow N$ is an iso.

First we need to define the arrow $s' : 1+N \rightarrow 1+N$,
using a factorization through a coproduct.

Note that s' takes the '*' in ' $* \sqcup n$ ' to 0, not to $*'$.
 $s' := [0; \kappa', s; \kappa']$.

$$\begin{array}{ccc}
 \begin{array}{ccc}
 1 & \xrightarrow{\kappa} & 1+N \\
 & \searrow 0 & \downarrow s' \\
 & N & N \\
 & \swarrow \kappa' & \searrow \kappa' \\
 & 1+N &
 \end{array}
 & \quad &
 \begin{array}{ccc}
 * & \longrightarrow & * \sqcup n \\
 \downarrow & & \downarrow \\
 0 & & n+1 \\
 \downarrow & & \downarrow \\
 *' \sqcup n+1 & &
 \end{array}
 \end{array}$$

Now we can define $p' : N \rightarrow 1+N$ by factoring (κ, s') through the NNO.
It is possible to show that p' and $[0, s]$ are inverses.

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 1 & \xrightarrow{0} & N & \xrightarrow{s} & N & \xrightarrow{s} & N & & N \\
 & \searrow \kappa & \downarrow p' & & \downarrow p' & & \downarrow p' & & \downarrow p' \\
 & 0 & 1+N & \xrightarrow{s'} & 1+N & \xrightarrow{s'} & 1+N & & 1+N \\
 & \downarrow [0,s] & & \downarrow [0,s] & & \downarrow [0,s] & & \downarrow [0,s] \\
 & N & N & \xrightarrow{s} & N & \xrightarrow{s} & N & & N
 \end{array}
 & \quad &
 \begin{array}{ccccc}
 * & \longrightarrow & 0 & \longrightarrow & 1 & \longrightarrow & 2 & & n \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 * \sqcup 0' & \longrightarrow & *' \sqcup 1' & \longrightarrow & *'' \sqcup 2' & \longrightarrow & * \sqcup n-1 & & n \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & 1 & \longrightarrow & 2 & & & & n
 \end{array}
 \end{array}$$

Define $n \mapsto n-1$ as $p'; [0, \text{id}]$,
The arrow $m \mapsto (n-m)$ of the previous page