Sheaves for Children

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Where sheaves stand

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 ____________________
|                    |
|  Category Theory   |    Cartesian Closed Categories,
|  ________________  |        Lambda-Calculus,
| |                | |      Intuitionistic Logic
| |  Topos Theory  | |
| |  ____________  | |
| | |            | | |        Modal Logic (S4)
| | |   Sheaves  | | |
| | |____________| | |          Topology
| |________________| |
|____________________|      Algebraic Geometry
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CT: Why?

Category Theory is fascinating (for some people!),
but (usually) too abstract...

The right level of abstraction
makes lots of proofs *almost* automatic:
proving something in CT means
constructing something (CT is constructive!), and
all “natural” constructions are equivalent (“coherence”).

More or less like this:

> *Let A and B be (arbitrary) sets.*
> *Then there is an “obvious” function* $\mathsf{flip} : A \times B \to B \times A$.

This *ought* to make some parts of CT *easy*!!!

(Long story... see “Internal Diagrams and Archetypal Reasoning in Category Theory”)

Why study CT *now*?

Public education in Brazil is being dismantled -
maybe we should be doing better things than studying
very technical & inaccessible subjects
with no research grants -

Category theory should be more accessible

Most texts about CT are for specialists in research universities...
Category theory should be more accessible.

To whom?...

- Non-specialists (in research universities)
- Grad students (in research universities)
- Undergrads (in research universities)
- Non-specialists (in conferences - where we have to be quick)
- Undergrads (e.g., in CompSci - in teaching colleges) - (“Children”)

ZSets

Take a finite, non-empty subset of $\mathbb{N}^2$;
translate it lowerleftwards as most as possible in $\mathbb{N}^2$,
until you get something that touches both axes.

Subsets of $\mathbb{N}^2$ obtained in this way are said to be "well-positioned", and we call them *ZSets*.

We can use a positional notation with bullets
to denote our favourite ZSets (unambiguously!)...

V	Vee		$\{(0,1),(2,1),(1,0)\}$
K	Kite		$\{(1,3),(0,2),(2,2),(1,1),(1,0)\}$
H	House		$\{(1,2),(0,2),(2,2),(0,1),(2,1)\}$

↑ Some of my favorite ZSets -
note that they have both short, one-letter names
and long, pronounceable names.

ZDAGs

An arrow between points of $\mathbb{N}^2$ that goes one unit down and 0/1/-1 units horizontally is called a *black pawn's move*.

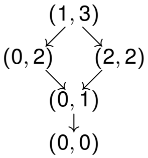
Take a ZSet, D, and draw all possible black pawns moves between its points; this gives us a set of arrows, BPM(D), that turns D, a ZSet, into a directed, acyclical, graph, $\mathbb{D}$, in a canonical way: $\mathbb{D} = (D, \text{BPM}(D))$.

Note the change of font!!!: $D \dashrightarrow \mathbb{D}$

Example:

K =  =

(1, 3)
(0, 2) (2, 2)
(0, 1)
(0, 0)

$\mathbb{K}$ = 

Truth-values

A function from a ZSet D to $\{0,1\}$ is a *modal truth-value*.

The positional notation gives us a way to write modal truth-values very compactly, and the points on a ZSet have a natural order - the "reading order", in which we read them line by line, left to right in each line.

This gives us a way to *read aloud* modal truth-values -
and to list all modal truth-values in order.

$$\begin{array}{ccc} & a & \\ \swarrow & & \searrow \\ b & & c \\ \searrow & & \swarrow \\ & d & \\ & \downarrow & \\ & e & \end{array} \qquad \begin{array}{ccc} & 0 & \\ \swarrow & & \searrow \\ 0 & & 1 \\ \searrow & & \swarrow \\ & 1 & \\ & \downarrow & \\ & 0 & \end{array} \quad = \quad \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \\ & 0 & \end{smallmatrix} \quad = \quad P \qquad \text{("Kite 00110")}$$

Notation: $\mathcal{P}(\mathbb{D})$ is the set of modal truth-values on $\mathbb{D}$

We use "$\mathcal{P}(\mathbb{D})$" because $\begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \\ & 0 & \end{smallmatrix}$ "is" $\{c,d\}$ ($\subset K$)

Intuitionistic truth-values

$$\begin{array}{ccc} & a & \\ \swarrow & & \searrow \\ b & & c \\ \searrow & & \swarrow \\ & d & \\ & \downarrow & \\ & e & \end{array} \qquad \begin{array}{ccc} & 0 & \\ \swarrow & & \searrow \\ 0 & & 1 \\ \searrow & & \swarrow \\ & 1 & \\ & \downarrow & \\ & 0 & \end{array} \quad = P \qquad \text{(“Kite 00110”)}$$

Now consider that each 1 in P is covered with (wet) black paint.
Then P (“Kite 00110”) is not *stable*, because the paint
of the 1 in position d will flow down into the 0 of position e,
and paint it black.

Stable modal truth-values are called *intuitionistic truth-values*.

Notation: $\downarrow P$ is P after letting the black paint flow down.

Example: $\downarrow {\scriptsize\begin{array}{c} 0 \\ 0\,1 \\ 1 \\ 0 \end{array}} = {\scriptsize\begin{array}{c} 0 \\ 0\,1 \\ 1 \\ 1 \end{array}}$

The order topology

$$\downarrow \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \\ & 0 & \end{smallmatrix} = \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix}$$

Notation: $\mathcal{P}(\mathbb{D})$ is the set of modal truth-values on $\mathbb{D}$
Notation: $\mathcal{O}(\mathbb{D})$ is the set of intuitionistic truth-values on $\mathbb{D}$

$$\downarrow : \mathcal{P}(\mathbb{D}) \to \mathcal{O}(\mathbb{D})$$

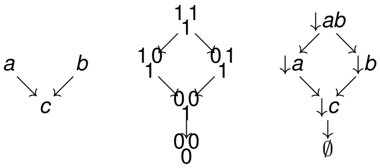
The topology $\mathcal{O}(\mathbb{D})$ is the *order topology* -
an arrow $\alpha \to \beta$ in $\mathbb{D}$ means that
if an open set contains α it has to contain β too.

(Order topologies are Alexandroff.)

Priming

Amazing fact: very often $\mathcal{O}(\mathbb{D})$ can be represented as a ZDAG too!
$\mathcal{O}(\mathbb{D})$ has a natural order:
$P \to Q$ means $P \leq Q$, $P \subseteq Q$, $P \vdash Q$,
and $\top = {}^{1}{}_{1}{}^{1}$ ("Top") is the terminal of the category...

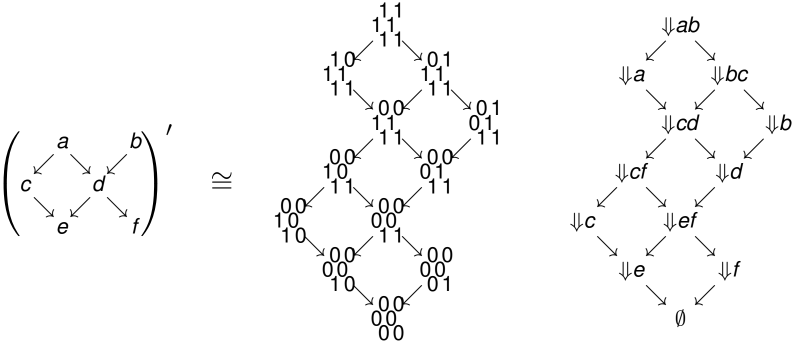
But if we draw $\mathcal{O}(\mathbb{D})^{\mathrm{op}}$ instead of $\mathcal{O}(\mathbb{D})$
we can see clearly how $\mathbb{D} \hookrightarrow \mathcal{O}(\mathbb{D})^{\mathrm{op}}$:



Note that $\downarrow ab = \downarrow\{a, b\} = \downarrow {}^{1}{}_{0}{}^{1} = {}^{1}{}_{1}{}^{1}$.
Def: $\mathbb{D}' = \mathcal{O}(\mathbb{D})^{\mathrm{op}}$.
$\mathbb{V}' \cong \mathbb{K}$ - and, by abuse of language, $\mathbb{V}' = \mathbb{K}$.

Unpriming

If $\mathbb{C}' = \mathbb{D}$ can we recover $\mathbb{C}$ from $\mathbb{D}$?
Better: if $\mathbb{D}$ is a ZDAG that is a Heyting algebra
can we find a $\mathbb{C} \subset \mathbb{D}$ such that $\mathbb{C}' = \mathbb{D}$?
Can we use that to determine quickly whether
an arbitrary $\mathbb{D}$ is a Heyting algebra?
Yes, yes, & yes!



Priming: theorems

We say that $\mathbb{D}$ is *3-thin* when $\bullet\bullet\bullet \not\subset \mathbb{D}$.

We say that $\mathbb{D}$ is *square-thin* when $\begin{smallmatrix}\bullet & \bullet\\ \bullet & \bullet\end{smallmatrix} \not\subset \mathbb{D}$.
We say that $\mathbb{D}$ is *thin* when it is both 3-thin and square-thin.
Fact: if $\mathbb{D}$ is 3-thin then $\mathbb{D}'$ is a ZDAG.
Fact: if $\mathbb{D}$ is thin then $\mathbb{D}'$ is a thin ZDAG.

Fact: every topology - whether planar or not - is a Heyting algebra - i.e., we can interpret T, F, $\land$, $\lor$, $\to$, $\neg$ on it, and every $\mathbb{D}'$ is a topology...

Priming gives us LOTS of Heyting algebras,
and lots of *planar* Heyting algebras!

Topological sheaves are defined on diagrams like $\mathbb{D} \hookrightarrow \mathbb{D}' \hookrightarrow \mathbb{D}''$.

Glueing locally-defined functions

Let U be $(-\infty, 1)$
and V be $(0, \infty)$...
Consider these open sets in $\mathbb{R}$,

$$
\begin{array}{ccc}
& (-\infty,\infty) & \\
\nearrow & & \nwarrow \\
(-\infty,1) & & (0,\infty) \\
\nwarrow & & \nearrow \\
& (0,1) & \\
& \uparrow & \\
& \emptyset &
\end{array}
\quad
\begin{array}{ccc}
& U \cup V & \\
\nearrow & & \nwarrow \\
U & & V \\
\nwarrow & & \nearrow \\
& U \cap V & \\
& \uparrow & \\
& \emptyset &
\end{array}
\quad
\begin{array}{ccc}
& X & \\
\nearrow & & \nwarrow \\
U & & V \\
\nwarrow & & \nearrow \\
& W & \\
& \uparrow & \\
& \emptyset &
\end{array}
\quad
\begin{array}{ccc}
& \mathcal{C}^\infty(X,\mathbb{R}) & \\
\swarrow & & \searrow \\
\mathcal{C}^\infty(U,\mathbb{R}) & & \mathcal{C}^\infty(V,\mathbb{R}) \\
\searrow & & \swarrow \\
& \mathcal{C}^\infty(W,\mathbb{R}) & \\
& \downarrow & \\
& \mathcal{C}^\infty(\emptyset,\mathbb{R}) &
\end{array}
\quad
\begin{array}{ccc}
& C(X) & \\
\swarrow & & \searrow \\
C(U) & & C(V) \\
\searrow & & \swarrow \\
& C(W) & \\
& \downarrow & \\
& C(\emptyset) &
\end{array}
$$

and the sheaf C of $\mathcal{C}^\infty$ functions from them to $\mathbb{R}$.

Upward arrows are *inclusions* (of an open set into another).
Downward arrows are *restrictions* (of domains).

Two functions f_U and f_V are *compatible*
if their restrictions to $U \cap V$ coincide.

Each compatible family $\{f_U, f_V\}$ in C has a unique glueing f_X.
Generalize that, and you get the definition of *sheafness*.

Sheafness

A compatible family $\{f_U, f_V\}$ is defined on $\{U, V\} = \begin{smallmatrix} & 0 & \\ 1 & & 1 \\ & 0 & \\ & 0 & \end{smallmatrix}$,
and it can be extended, using the restriction maps $\rho_{UW} : FU \to FW$ etc,
to a downward-closed compatible family $\{f_U, f_V, f_W, f_\emptyset\}$,
defined on $\{U, V, W, \emptyset\} = \begin{smallmatrix} & 0 & \\ 1 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix}$...

The "unique glueing" f_X of $\{f_U, f_V\}$ can be extended
to a downward-closed compatible family $\{f_X, f_U, f_V, f_W, f_\emptyset\}$,
defined on $\{X, U, V, W, \emptyset\} = \begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix}$.

The restriction

$$\{f_X, f_U, f_V, f_W, f_\emptyset\} \overset{\rho}{\mapsto} \{f_U, f_V, f_W, f_\emptyset\}$$

is trivial to define -
sheafness means that maps like these are bijections.

Topological sheafness

A closure operator:

$$\sqcup\{U, V, W, \emptyset\} = \{X, U, V, W, \emptyset\}$$

it takes the union $U \cup V \cup W \cup \emptyset = X$
and then all subsets of that.
It acts on $\mathbb{V}''$: $\quad \sqcup : {}^{\bullet\,\bullet}_{\ \bullet}{}'' \to {}^{\bullet\,\bullet}_{\ \bullet}{}''$

$$\sqcup \begin{smallmatrix} & 0 & \\ 1 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix} = \begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix}$$

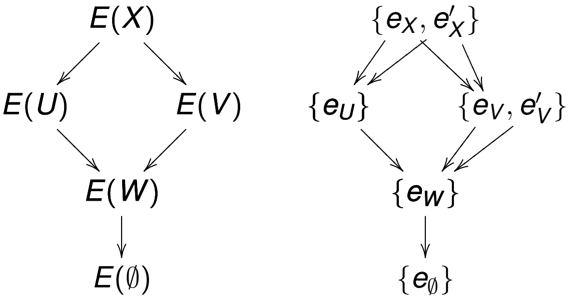
Which elements of $\mathbb{V}''$ are stable by $\sqcup$?
Only $\downarrow\{U, V\} \mapsto \downarrow\{X\}$ $(\begin{smallmatrix} & 0 & \\ 1 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix} \mapsto \begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix})$ and $\emptyset \mapsto \{\emptyset\}$ $(\begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & 0 & \\ & 0 & \end{smallmatrix} \mapsto \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & 0 & \\ & 1 & \end{smallmatrix})$
are *not* stable by $\sqcup$.

The stable elements of $\mathbb{V}''$ are these: $\begin{smallmatrix} & 1 & \\ & 0 & \\ 1 & & 1 \\ & 1 & \\ & 1 & \\ & 0 & \end{smallmatrix}$.

These diagrams of stable elements are what we need to define sheaves “in general”.

The evil presheaf

A presheaf F in $\mathbf{Set}^{\mathcal{O}(\mathbb{V})^{\mathrm{op}}}$
is simply a functor $F : \mathcal{O}(\mathbb{V})^{\mathrm{op}} \to \mathbf{Set}$.



The evil presheaf $E : \mathcal{O}(\mathbb{V})^{\mathrm{op}} \to \mathbf{Set}$, above,
fails to be in a sheaf in two ways:
the compatible family $\{e_U, e_V\}$ has two different glueings,
the compatible family $\{e_U, e'_V\}$ doesn't have a glueing.

Dual operations

Due to we being in a finite / planar / etc case,
several interesting operations have duals (adjoints):

- In finite DAGs the transitive-reflexive closure $(A, R) \mapsto (A, R^*)$ has an adjoint that keeps only the “essential arrows” of the graph;
- The “let the paint flow down” operation $\downarrow \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \\ & 0 & \end{smallmatrix} = \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix}$ has an adjoint $\begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix} \mapsto \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 0 & \\ & 0 & \end{smallmatrix}$ that returns the “generators”of an open set;
- Each closure operator like $\mathcal{U} \mapsto \sqcup \mathcal{U}$ has an adjoint that returns the smallest equivalent cover...

Where are the theorems?

Not here! Why???

Because this is “for children” -
we are focusing on the tools to let people check particular cases...
and this *complements* [Bell 1988] and my IDARCT paper,
that explains how to do theorems and archetypal cases in parallel

Slightly more advanced things:

- CCCs and Heyting Algebras; $(\land Q) \dashv (Q{\to})$
- presheaves of the form $\mathbf{Set}^{\mathbb{D}}$ as toposes
- the classifier object of a $\mathbf{Set}^{\mathbb{D}}$
- other modalities (besides $\sqcup$) in a $\mathbf{Set}^{\mathbb{D}}$
- all logical properties of modalities follow from three axioms
- operations on the lattice of modalities
- forcing
- sheafification
- geometric morphisms between toposes - I need help here =(

ZHAs

A *ZHA* is a ZSet that is a Heyting Algebra -
i.e., one whose points are intuitionistic truth-values,
and where we have the logical operations $\wedge$, $\vee$, $\to$, etc
defined on them, obeying the right equations...

...but this is how we want to *use* ZHAs.
Let's see a *visual* characterization of ZHAs.

(See the handouts)

ZHAs: definition

Def: a ZSet D is a ZHA iff it obeys these five conditions:

ZHA_1	The non-empty lines of D are sequential
ZHA_2	D has one top element and one bottom element
ZHA_{3L}	The left wall of D can be traversed by black pawns moves
ZHA_{3R}	The right wall of D can be traversed by black pawns moves
ZHA_4	Each line of D is made of consecutive same-parity points
ZHA_5	All points in each wide region of D have the same parity

(The bottlenecks of D divide it into regions with a top element and a bottom element. A region whose lines are a bottleneck, one or more non-bottlenecks, and another bottleneck is a "wide region").

(If these look like six conditions, let ZHA_3 be "both the left wall and the right wall of D can be traversed by black pawns moves")

Definitions for ZHAs

height_D	is the y coordinate of the uppermost points of D.
$\text{Line}_D(y)$	is all points of D whose y coordinate is the given y.
Lines_D	is the set of all ys such that $\text{Line}_D(y)$ is non-empty.
$\text{L}_D(y)$	is the x coordinate of the leftmost point in $\text{Line}_D(y)$.
$\text{R}_D(y)$	is the x coordinate of the rightmost point in $\text{Line}_D(y)$.
$\text{W}_D(y)$	is $\text{R}_D(y) - \text{L}_D(y)$.
B_D	is the set of all ys such that $\text{Lines}_D(y)$ has exactly one point; these points are the bottlenecks of D.
LW_D	is the left wall - the set of points of the form $(y, \text{L}_D(y))$
RW_D	is the right wall - the set of points of the form $(y, \text{R}_D(y))$
D^-	is the set of generators of D - formally, the points $(x, y) \in D$ such that exactly one of the points $(x-1, y-1)$, $(x, y-1)$, $(x+1, y-1)$ belongs to D.

$\text{L}_D, \text{R}_D, \text{W}_D : \text{Lines}_D \to \mathbb{N}$.
"Lines" here are horizontal lines in $\mathbb{N}^2$.

Definitions for ZHAs (2)

The line y is a bottleneck (of D) when $\text{Lines}_D(y)$ has exactly one point
The line y is a wide when $\text{Lines}_D(y)$ has more than one point

B_D	is the set of the (y coordinates of) bottlenecks of D.
CB_D	is the set of (y coords of pairs of) consecutive bottlenecks of D Formally: $(y_0, y_1) \in \text{CB}_D$ iff $([y_0, y_1] \cap \text{B}_D) = \{y_0, y_1\}$
CCB_D	is the of close consecutive bottlenecks of D Formally: $(y_0, y_1) \in \text{CCB}_D$ iff $(y_0, y_1) \in \text{CB}_D$ and $y_1 - y_0 = 1$
DCB_D	is the of distant consecutive bottlenecks of D Formally: $(y_0, y_1) \in \text{CCB}_D$ iff $(y_0, y_1) \in \text{CB}_D$ and $y_1 - y_0 \geq 2$
$\text{Reg}_D(y0, y1)$	is the region of all points $(x, y) \in D$ with $y_0 \leq y \leq y_1$.
WideRegs_D	is the set of all wide regions of D Formally: $\text{WideRegs}_D = \text{Reg}_D(\text{DCB}_D)$

ZHAs: definition (again, but this time formally)

Def: a ZSet D is a ZHA iff it obeys these conditions:

ZHA_1	The non-empty lines of D are sequential Formally: $\mathrm{Lines}_D = \{0, 1, ..., \mathrm{height}_D\}$
ZHA_2	D has one top element and one bottom element Formally: $\{0, \mathrm{height}_D\} \subset B_D$
ZHA_{3L}	The left wall of D can be traversed by black pawns moves Formally: $\mathrm{L}_D(y+1) - \mathrm{L}_D(y) \in \{-1, 0, 1\}$ whenever defined
ZHA_{3R}	The right wall of D can be traversed by black pawns moves Formally: $\mathrm{R}_D(y+1) - \mathrm{R}_D(y) \in \{-1, 0, 1\}$ whenever defined
ZHA_4	Each line of D is made of consecutive same-parity points Formally: $\mathrm{Line}_D(y) =$ $\{(a = \mathrm{L}_D(y), y), (a+2, y), (a+4, y), ..., (\mathrm{R}_D(y), y)\}$
ZHA_5	All points in each wide region of D have the same parity

D^- as a DAG (not necessarily a ZDAG)

For each ZDAG D its subset of generators, D^-
(the points of D with exactly one black pawn move going out)
has a natural partial order in it, obtained by restricting
the partial order in D...

$$(D, \mathrm{BPM}^*(D) \quad \mapsto \quad (D^-, \mathrm{BPM}(D)^* \cap (D^- \times D^-))$$

When D is a ZHA this order on D^- is generated by a
set of arrows that is very easy to draw -
draw an arrow from each point of the left wall to the next one,
draw an arrow from each point of the right wall to the next one,
draw inter-wall arrows (which will always be 45°) like this:

[See the handouts]

D^- as generators

This gives us a notion of stable subsets of D^-,
and thus a topology on D^-.
$(D^-, \mathcal{O}(D^-))$ is a topological space,
which lets us construct $(D^-)'$...

When D is a ZHA we have $(D^-)' \cong D$.
How?

We have
a natural function from D^- to D,
a natural function from $\mathcal{O}(D^-)$ to D,
a natural function from D to $\mathcal{O}(D^-)$,

[See the handouts]

ZHAs are sets of truth-values

Theorem 1: every ZHA D is isomorphic to $\mathcal{O}(D^-)$.
Theorem 2: every ZHA D is a Heyting Algebra.
Theorem 3: a ZSet D is a Heyting Algebra iff it is isomorphic to $\mathcal{O}(D^-)$.
Theorem 4: a ZSet D is a Heyting Algebra iff it is a ZHA "with some (or no) corners tucked in".

[To do: describe the iso in theorem 1]

For Further Reading I

J.L. Bell.
Toposes and Local Set Theories.
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E. Ochs.
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