Cálculo 2
PURO-UFF - 2016.1
P1 - 4/jul/2016 - Eduardo Ochs
Links importantes:
http://angg.twu.net/2016.1-C2.html (página do curso)
http://angg.twu.net/2016.1-C2/2016.1-C2.pdf (quadros)
http://angg.twu.net/LATEX/2016-1-C2-material.pdf
eduardoochs@gmail.com (meu e-mail)

1) **(Total: 1.0)** Calcule $\int \tan x \, dx$.

2) **(Total: 1.0)** Calcule $\int_{x=-1}^{x=2} x^{-4} \, dx$.

3) **(Total: 1.5)** Calcule $\int \cos^4 x \, dx$.

4) **(Total: 1.5)** Calcule $\int \frac{x^2}{x^2+x-2} \, dx$.

5) **(Total: 1.5)** Calcule $\int x \, e^x \cos x \, dx$.

6) **(Total: 2.0)**
6a) **(1.5 pts)** Calcule $\int_{x=0}^{x=1} \sqrt{4-x^2} \, dx$.
6b) **(0.1 pts)** Represente $\int_{x=0}^{x=1} \sqrt{4-x^2} \, dx$ graficamente.
6c) **(0.4 pts)** Mostre como calcular $\int_{x=0}^{x=1} \sqrt{4-x^2} \, dx$ pelo gráfico.

7) **(Total: 2.5)** Calcule $\int_{x=-1}^{x=2} |e^x - 1| \, dx$.

Método de Heaviside:
Se $f(x) = \frac{\alpha}{x-a} + \frac{\beta}{x-b} + \frac{\gamma}{x-c} = \frac{p(x)}{(x-a)(x-b)(x-c)}$,
então $\lim_{x\to a} f(x)(x-a) = \alpha = \frac{p(a)}{(a-b)(a-c)}$.

Substituição:

$$\begin{array}{l} g(h(x))|_{x=a}^{x=b} = \int_{x=a}^{x=b} g'(h(x))\frac{d\,h(x)}{dx}\,dx \\ \quad\;\; || \\ g(u))|_{u=h(a)}^{u=h(b)} = \int_{u=h(a)}^{u=h(b)} g'(u)\,du \end{array}$$

Fórmulas:

$$\begin{array}{lll} \int_{x=a}^{x=b} f(g(x))\frac{d\,g(x)}{dx}\,dx & \int f(g(x))\frac{d\,g(x)}{dx}\,dx & \\ = \int_{x=a}^{x=b} f(u)\frac{du}{dx}\,dx & = \int f(u)\frac{du}{dx}\,dx & \left[u=g(x)\right] \\ = \int_{u=g(a)}^{u=g(b)} f(u)\,du & = \int f(u)\,du & \left[u=g(x)\right] \end{array}$$

Substituição inversa:

$$\begin{array}{l} g(h(x))|_{x=h^{-1}(\alpha)}^{x=h^{-1}(\beta)} = \int_{x=h^{-1}(\alpha)}^{x=h^{-1}(\beta)} g'(h(x))\frac{d\,h(x)}{dx}\,dx \\ \quad\;\; || \\ g(u))|_{u=h(h^{-1}(\alpha))}^{u=h(h^{-1}(\beta))} = \int_{u=h(h^{-1}(\alpha))}^{u=h(h^{-1}(\beta))} g'(u)\,du \\ \quad\;\; || \\ g(u))|_{u=\alpha}^{u=\beta} = \int_{u=\alpha}^{u=\beta} g'(u)\,du \end{array}$$

Fórmulas:

$$\begin{array}{lll} \int_{u=\alpha}^{u=\beta} f(u)\,du & \int f(u)\,du & \\ = \int_{x=g^{-1}(\alpha)}^{x=g^{-1}(\beta)} f(u)\frac{du}{dx}\,dx & = \int f(u)\frac{du}{dx}\,dx & \left[\begin{smallmatrix} u=g(x) \\ x=g^{-1}(u) \end{smallmatrix}\right] \\ = \int_{x=g^{-1}(\alpha)}^{x=g^{-1}(\beta)} f(g(x))\frac{d\,g(x)}{dx}\,dx & = \int f(g(x))\frac{d\,g(x)}{dx}\,dx & \left[x=g^{-1}(u)\right] \end{array}$$

Substituição trigonométrica:

$$\begin{array}{lll} \int_{s=a}^{s=b} F(s,\sqrt{1-s^2})\,ds & \int F(s,\sqrt{1-s^2})\,ds & \\ = \int_{\theta=\operatorname{arcsen} a}^{\theta=\operatorname{arcsen} b} F(\operatorname{sen}\theta,\sqrt{1-\operatorname{sen}^2\theta})\frac{d\operatorname{sen}\theta}{d\theta}\,d\theta & = \int F(s,\sqrt{1-s^2})\frac{ds}{d\theta}\,d\theta & \left[\begin{smallmatrix} s=\operatorname{sen}\theta \\ \theta=\operatorname{arcsen}\theta \end{smallmatrix}\right] \\ = \int_{\theta=\operatorname{arcsen} a}^{\theta=\operatorname{arcsen} b} F(\operatorname{sen}\theta,\cos\theta)\cos\theta\,d\theta & = \int F(s,c)c\,d\theta & \left[\begin{smallmatrix} s=\operatorname{sen}\theta \\ c=\cos\theta \\ \theta=\operatorname{arcsen}\theta \end{smallmatrix}\right] \end{array}$$

$$\begin{array}{lll} \int_{z=a}^{z=b} F(z,\sqrt{z^2-1})\,dz & \int F(z,\sqrt{z^2-1})\,dz & \\ = \int_{\theta=\operatorname{arcsec} a}^{\theta=\operatorname{arcsec} b} F(\sec\theta,\sqrt{\sec^2\theta-1})\frac{d\sec\theta}{d\theta}\,d\theta & = \int F(z,\sqrt{z^2-1})\frac{dz}{d\theta}\,d\theta & \left[\begin{smallmatrix} z=\sec\theta \\ \theta=\operatorname{arcsec} z \end{smallmatrix}\right] \\ = \int_{\theta=\operatorname{arcsec} a}^{\theta=\operatorname{arcsec} b} F(\sec\theta,\tan\theta)\sec\theta\tan\theta\,d\theta & = \int F(z,t)zt\,d\theta & \left[\begin{smallmatrix} z=\sec\theta \\ \theta=\operatorname{arcsec} z \\ t=\tan\theta \end{smallmatrix}\right] \end{array}$$

$$\begin{array}{lll} \int_{t=a}^{t=b} F(t,\sqrt{1+t^2})\,dt & \int F(t,\sqrt{1+t^2})\,dt & \\ = \int_{\theta=\arctan a}^{\theta=\arctan b} F(\tan\theta,\sqrt{1+\tan^2\theta})\frac{d\tan\theta}{d\theta}\,d\theta & = \int F(t,\sqrt{1+t^2})\frac{dt}{d\theta}\,d\theta & \left[\begin{smallmatrix} t=\tan\theta \\ \theta=\arctan t \end{smallmatrix}\right] \\ = \int_{\theta=\arctan a}^{\theta=\arctan b} F(\tan\theta,\sec\theta)\sec^2\theta\,d\theta & = \int F(t,z)z^2\,d\theta & \left[\begin{smallmatrix} t=\tan\theta \\ \theta=\arctan t \\ z=\sec\theta \end{smallmatrix}\right] \end{array}$$

Gabarito (não revisado, contém erros de vários tipos!):

1)
$$\begin{aligned}
\int \tan\theta\,d\theta &= \int \tfrac{s}{c}\,d\theta \\
&= \int \tfrac{1}{c} s\,d\theta \\
&= -\int \tfrac{1}{c}\,dc \\
&= -\ln|c| \\
&= -\ln|\cos\theta|
\end{aligned}$$

2)
$$\begin{aligned}
\int x^{-4}\,dx &= \tfrac{x^{-3}}{-3} \\
\int_{x=-1}^{x=2} x^{-4}\,dx &= \int_{x=-1}^{x=0} x^{-4}\,dx + \int_{x=0}^{x=2} x^{-4}\,dx \\
&= \lim_{a\to 0^-} \int_{x=-1}^{x=a} x^{-4}\,dx + \lim_{b\to 0^+} \int_{x=b}^{x=2} x^{-4}\,dx \\
&= \lim_{a\to 0^-} \left(\tfrac{x^{-3}}{-3} \Big|_{x=-1}^{x=a} \right) + \lim_{b\to 0^+} \left(\tfrac{x^{-3}}{-3} \Big|_{x=b}^{x=2} \right) \\
&= \lim_{a\to 0^-} \left(\tfrac{a^{-3}}{-3} - \tfrac{(-1)^{-3}}{-3} \right) + \lim_{b\to 0^+} \left(\tfrac{2^{-3}}{-3} - \tfrac{b^{-3}}{-3} \right) \\
&= \left(\tfrac{-\infty}{-3} - \tfrac{-1}{-3} \right) + \left(\tfrac{1/8}{-3} - \tfrac{+\infty}{-3} \right)
\end{aligned}$$

3) Seja $E = e^{i\theta}$. Então
$$\begin{aligned}
\cos\theta &= \tfrac{E+E^{-1}}{2}, \\
(\cos\theta)^2 &= \tfrac{E^2+2+E^{-2}}{4}, \\
(\cos\theta)^4 &= \tfrac{E^4+4E^2+6+4E^{-2}+E^{-4}}{16} \\
&= \tfrac{1}{8}\tfrac{E^4+E^{-4}}{2} + \tfrac{1}{2}\tfrac{E^2+E^{-2}}{2} + \tfrac{3}{8} \\
&= \tfrac{1}{8}\cos 4\theta + \tfrac{1}{2}\cos 2\theta + \tfrac{3}{8} \\
\int (\cos\theta)^4\,d\theta &= \int \tfrac{1}{8}\cos 4\theta + \tfrac{1}{2}\cos 2\theta + \tfrac{3}{8}\,d\theta \\
&= \tfrac{1}{8}\tfrac{\operatorname{sen} 4\theta}{4} + \tfrac{1}{2}\tfrac{\operatorname{sen} 2\theta}{2} + \tfrac{3}{8}\theta \\
&= \tfrac{\operatorname{sen} 4\theta}{32} + \tfrac{\operatorname{sen} 2\theta}{4} + \tfrac{3}{8}\theta
\end{aligned}$$

4) $\frac{x^2}{x^2+x-2} = \frac{(x^2+x-2)-x+2}{x^2+x-2} = 1 + \frac{-x+2}{x^2+x-2} = 1 + \frac{-x+2}{(x+2)(x-1)}$
Se $\frac{-x+2}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$ então
$\lim_{x\to -2} \frac{-x+2}{x-1} = A = \frac{4}{-3}$ e
$\lim_{x\to 1} \frac{-x+2}{x+2} = B = \frac{1}{3}$.
Conferindo:
$$\begin{aligned}
\tfrac{A}{x+2} + \tfrac{B}{x-1} &= \tfrac{-4/3}{x+2} + \tfrac{1/3}{x-1} \\
&= \tfrac{-\frac{4}{3}(x-1)+\frac{1}{3}(x+2)}{(x+2)(x-1)} \\
&= \tfrac{-x+2}{(x+2)(x-1)}
\end{aligned}$$
Daí:
$$\begin{aligned}
\int \tfrac{x^2}{x^2+x-2}\,dx &= \int 1 + -4/3x + 2 + \tfrac{1/3}{x-1}\,dx \\
&= x - \tfrac{4}{3}\ln|x+2| + \tfrac{1}{3}\ln|x-1|
\end{aligned}$$

5) Sejam $c = \cos x$, $s = \operatorname{sen} x$. Então

$$\int \underbrace{e^x}_{f} \underbrace{c}_{g'} \, dx = \underbrace{e^x}_{f} \underbrace{s}_{g} - \int \underbrace{e^x}_{f'} \underbrace{s}_{g} \, dx$$

$$\int \underbrace{e^x}_{f} \underbrace{s}_{g'} \, dx = \underbrace{e^x}_{f} \underbrace{(-c)}_{g} - \int \underbrace{e^x}_{f'} \underbrace{(-c)}_{g} \, dx = -e^x c + \int e^x c \, dx$$

$$\int e^x c \, dx = e^x s - \int e^x s \, dx = e^x s - (-e^x c + \int e^x c \, dx) = e^x s + e^x c - \int e^x c \, dx$$
$$2 \int e^x c \, dx = e^x s + e^x c$$
$$\int e^x c \, dx = (e^x s + e^x c)/2$$

$$\int e^x s \, dx = -e^x c + \int e^x c \, dx = -e^x c + (e^x s + e^x c)/2 = (e^x s - e^x c)/2$$

$$\begin{array}{rcl}
\int \underbrace{x}_{f} \underbrace{e^x c}_{g'} \, dx & = & \underbrace{x}_{f} \underbrace{(e^x s + e^x c)/2}_{g} - \int \underbrace{1}_{f'} \underbrace{(e^x s + e^x c)/2}_{g} \, dx \\
& = & \frac{1}{2}(xe^x s + xe^x c) - \frac{1}{2} \int e^x s \, dx - \frac{1}{2} \int e^x c \, dx \\
& = & \frac{1}{2}(xe^x s + xe^x c) - \frac{1}{4}(e^x s - e^x c) - \frac{1}{4}(e^x s + e^x c) \\
& = & \frac{1}{2}(xe^x s + xe^x c) - \frac{1}{2} e^x s \\
& = & \frac{1}{2}(xe^x s + xe^x c - e^x s)
\end{array}$$

6)

$$\begin{array}{rcll}
\sqrt{4 - x^2} & = & \sqrt{4 - 4(x/2)^2} & \\
& = & \sqrt{4(1 - (x/2)^2)} & \\
& = & 2\sqrt{1 - (x/2)^2} & \\
\int \sqrt{4 - x^2} \, dx & = & \int 2\sqrt{1 - (x/2)^2} \, dx & \left[\begin{smallmatrix} s = x/2 \\ x = 2s \\ dx = 2ds \end{smallmatrix} \right] \\
& = & \int 2\sqrt{1 - s^2} \cdot 2 \, ds & \\
& = & 4 \int \sqrt{1 - s^2} \, ds & \left[\begin{smallmatrix} s = \operatorname{sen} \theta \\ \theta = \operatorname{arcsen} s \\ ds = \cos \theta d\theta \end{smallmatrix} \right] \\
& = & 4 \int \sqrt{1 - (\operatorname{sen} \theta)^2} \cos \theta \, d\theta & \\
& = & 4 \int (\cos \theta)^2 \, d\theta & \\
& = & 4 \int \frac{1 + \cos 2\theta}{2} \, d\theta & \\
& = & 4(\frac{\theta}{2} + \frac{\operatorname{sen} 2\theta}{4}) & \\
& = & 2\theta + \operatorname{sen} 2\theta &
\end{array}$$

7)

$$\begin{array}{rcl}
\int_{x=-1}^{x=2} |e^x - 1| \, dx & = & \int_{x=-1}^{x=0} |e^x - 1| \, dx + \int_{x=0}^{x=2} |e^x - 1| \, dx \\
& = & \int_{x=-1}^{x=0} 1 - e^x \, dx + \int_{x=0}^{x=2} e^x - 1 \, dx \\
& = & (x - e^x)|_{x=-1}^{x=0} + (e^x - x)|_{x=0}^{x=2} \\
& = & (0 - e^0) - (-1 - e^{-1}) + (e^2 - 2) - (e^0 - 0) \\
& = & -1 + 1 + e^{-1} + e^2 - 2 - 1 \\
& = & -3 + e^{-1} + e^2
\end{array}$$