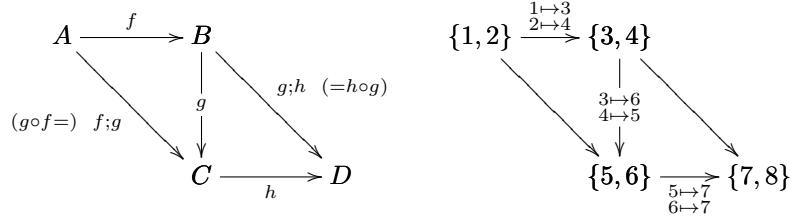


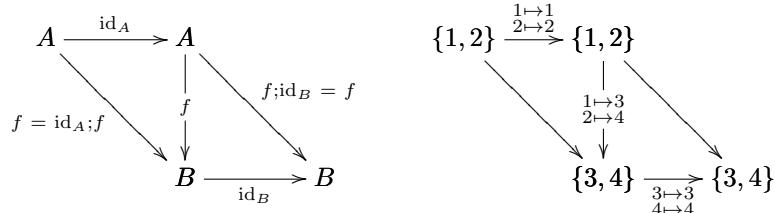
Composition  
(is associative)



$$\begin{aligned}
 ((h \circ g) \circ f)(a) &= (h \circ g)(f(a)) \\
 &= h(g(f(a))) \\
 &= h((g \circ f)(a)) \\
 &= (h \circ (g \circ f))(a) \\
 ((h \circ g) \circ f)(a) &= (h \circ (g \circ f))(a) \\
 (h \circ g) \circ f &= h \circ (g \circ f)
 \end{aligned}
 \quad
 \begin{array}{c}
 \xrightarrow[ ]{} \xrightarrow[ ]{} \xrightarrow[ ]{} \xrightarrow[ ]{} \xrightarrow[ ]{} \xrightarrow[ ]{} \\
 \xleftarrow[ ]{} \xleftarrow[ ]{} \xleftarrow[ ]{} \xleftarrow[ ]{} \xleftarrow[ ]{} \xleftarrow[ ]{} \\
 D \xleftarrow[ ]{} C \xleftarrow[ ]{} B \xleftarrow[ ]{} A
 \end{array}
 \quad
 \begin{array}{c}
 \xrightarrow[ ]{} \xrightarrow[ ]{} \xrightarrow[ ]{} \xrightarrow[ ]{} \xrightarrow[ ]{} \xrightarrow[ ]{} \\
 \xleftarrow[ ]{} \xleftarrow[ ]{} \xleftarrow[ ]{} \xleftarrow[ ]{} \xleftarrow[ ]{} \xleftarrow[ ]{} \\
 h \circ g \circ f \xleftarrow[ ]{} h \circ g \xleftarrow[ ]{} h \circ f \xleftarrow[ ]{} h \circ (g \circ f)
 \end{array}$$

Identities:

If  $f : A \rightarrow B$  then  $\text{id}_A ; f = f = f ; \text{id}_B$

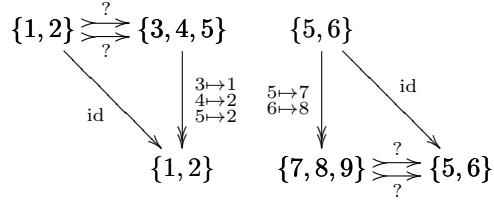


A theorem about lateral inverses:

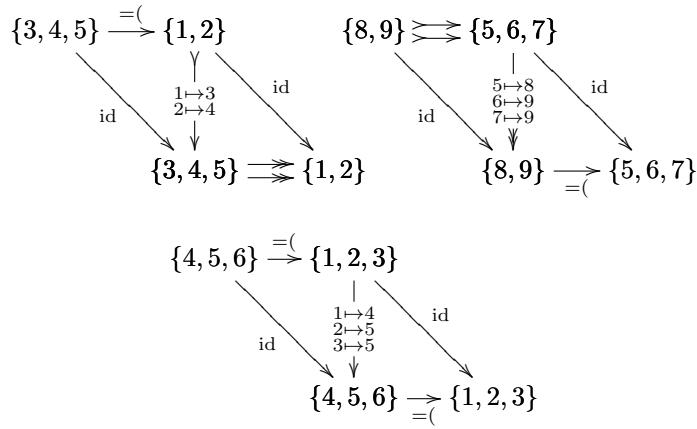
If  $f;g = \text{id}$  and  $g;h = \text{id}$  then  $f = h$

$$\begin{array}{ccc}
 B \xrightarrow[ ]{} A & & (f;g); h = \text{id}_B; h = h \\
 \downarrow g & \downarrow \text{id}_A = g; h & f; (g;h) = f; \text{id}_A = f \\
 B \xrightarrow[ ]{} A & & f = f; \text{id}_A \\
 & & = f; (g;h) \\
 & & = (f;g); h \\
 & & = \text{id}_B; h \\
 & & = h
 \end{array}$$

Multiple inverses



No inverses



### Products

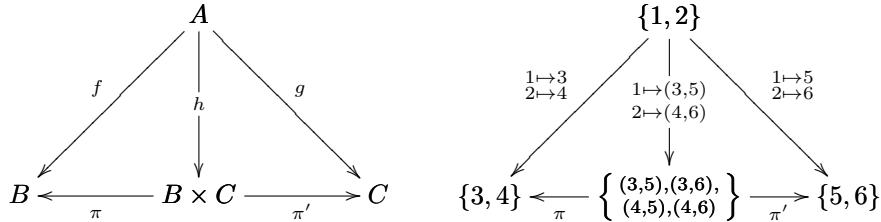
Property:  $\forall f, g. \exists! h. (f = h; \pi \& g = h; \pi')$

Solution:  $h = \lambda a : A. (f(a), g(a))$

Bijection:  $(f, g) \leftrightarrow h$

$(\rightarrow)$ :  $\lambda(f, g). (\lambda a : A. (f(a), g(a)))$

$(\leftarrow)$ :  $\lambda h. ((h; \pi), (h; \pi'))$



### Exponentials

Bijection:  $f \leftrightarrow g$

( $\rightarrow$ ) (“currying”):  $g := \text{cur } f := \lambda a : A. \lambda b : B. f(a, b)$

( $\leftarrow$ ) (“uncurrying”):  $f := \text{uncur } g := \lambda(a, b) : A \times B. ((g(a))(b))$

$$\begin{array}{ccc}
 A \times B & \xleftarrow{\lambda A.(A \times B)} & A \\
 \downarrow f & \Leftrightarrow & \downarrow g \\
 C & \xleftarrow{\lambda C.(B \rightarrow C)} & B \rightarrow C
 \end{array}
 \quad
 \begin{array}{c}
 \left\{ \begin{array}{l} (1,3),(1,4) \\ (2,3),(2,4) \end{array} \right\} \xleftarrow{\quad} \{1, 2\} \\
 \downarrow \begin{array}{l} (1,3) \mapsto 5 \\ (1,4) \mapsto 6 \\ (2,3) \mapsto 6 \\ (2,4) \mapsto 5 \end{array} \\
 \{5, 6\} \xleftarrow{\quad} \left\{ \begin{array}{l} \left\{ \begin{array}{l} (3,5), \\ (4,5) \end{array} \right\}, \left\{ \begin{array}{l} (3,5), \\ (4,6) \end{array} \right\}, \\ \left\{ \begin{array}{l} (3,6), \\ (4,5) \end{array} \right\}, \left\{ \begin{array}{l} (3,6), \\ (4,6) \end{array} \right\} \end{array} \right\} \\
 \downarrow \begin{array}{l} 1 \mapsto \left\{ \begin{array}{l} (3,5), \\ (4,6) \end{array} \right\} \\ 2 \mapsto \left\{ \begin{array}{l} (3,6), \\ (4,5) \end{array} \right\} \end{array}
 \end{array}$$

Properties:  $\text{cur uncur } f = f$ ,  $\text{uncur cur } g = g$

where:  $f \times f' := \langle \pi; f, \pi'; f' \rangle$ ,  $\text{uncur } g := (g \times \text{id}); \text{ev}$

solving type equations:

$$\frac{\pi \quad f \quad \pi' \quad f'}{\pi; f \quad \pi'; f'} \quad \frac{\pi : A \times ? \rightarrow A \quad \pi' : ? \times A' \rightarrow A' \quad f' : A' \rightarrow B'}{\pi; f : A \times ? \rightarrow B \quad \pi'; f' : ? \times A' \rightarrow B'} \quad \frac{\pi' : ? \times A' \rightarrow A' \quad f' : A' \rightarrow B'}{\langle \pi; f, \pi'; f' \rangle : A \times A' \rightarrow B \times B'} \text{ ren}$$

$$\frac{g \quad \text{id}}{g \times \text{id}} \quad \text{ev} \quad \frac{g : A \rightarrow (B \rightarrow C) \quad \text{id} : ? \rightarrow ?}{g \times \text{id} : A \times ? \rightarrow (B \rightarrow C) \times ?} \quad \text{ev} : (B \rightarrow C) \times B \rightarrow C$$

$$\frac{g \times \text{id} \quad \text{ev}}{(g \times \text{id}); \text{ev}} \text{ ren} \quad \frac{(g \times \text{id}); \text{ev} : A \times ? \rightarrow C}{\text{uncur } g : A \times ? \rightarrow C}$$