Expressions (and reductions)

$$\begin{array}{ccccc} 2\cdot 3+4\cdot 5 & \longrightarrow & 2\cdot 3+20 & & \\ \downarrow & & \downarrow & & \\ 6+4\cdot 5 & \longrightarrow & 6+20 & \longrightarrow & 26 \end{array}$$

$$\underbrace{\underbrace{2\cdot 3}_{6}+\underbrace{4\cdot 5}_{20}}_{26}$$

$$\begin{aligned} 2\cdot 3+4\cdot 5 &= 2\cdot 3+20 \\ &= 6+20 \\ &= 26 \end{aligned}$$

$$\begin{aligned} 2\cdot 3+4\cdot 5 &= 6+4\cdot 5 \\ &= 6+20 \\ &= 26 \end{aligned}$$

Subexpressions:

$$\underbrace{\underbrace{\underbrace{2}\cdot\underbrace{3}}+\underbrace{\underbrace{4}\cdot\underbrace{5}}}$$

Expressions with variables:

If $a = 5$ and $b = 2$, then:

$$\underbrace{\underbrace{(\underbrace{a}_{5}+\underbrace{b}_{2})}_{7}\cdot\underbrace{(\underbrace{a}_{5}-\underbrace{b}_{2})}_{3}}_{21}$$

If $a = 10$ and $b = 1$, then:

$$\underbrace{\underbrace{(\underbrace{a}_{10}+\underbrace{b}_{1})}_{11}\cdot\underbrace{(\underbrace{a}_{10}-\underbrace{b}_{1})}_{9}}_{99}$$

We know – by algebra, which is not for (tiny) children –
that $(a+b)\cdot(a-b) = a\cdot a - b\cdot b$ is true for all $a, b \in \mathbb{R}$
We know – without algebra – how to test
"$(a+b)\cdot(a-b) = a\cdot a - b\cdot b$"
for specific values of a and b...

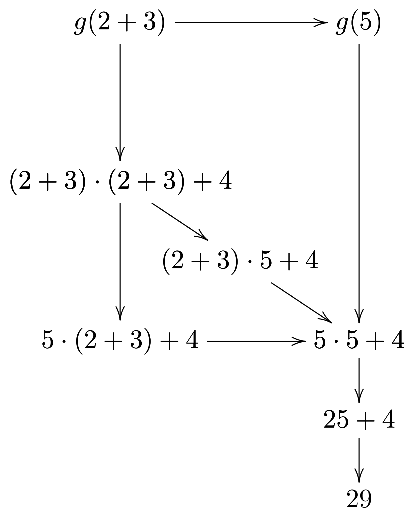
If $a = 5$ and $b = 2$, then:

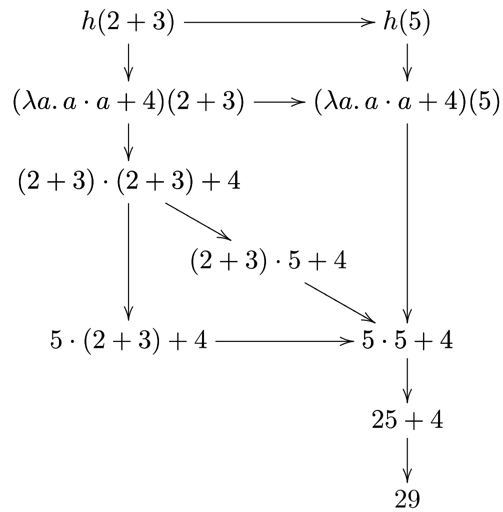
$$\underbrace{\underbrace{\underbrace{(\underbrace{a}_{5}+\underbrace{b}_{2})}_{7}\cdot\underbrace{(\underbrace{a}_{5}-\underbrace{b}_{2})}_{3}}_{21}=\underbrace{\underbrace{\underbrace{a}_{5}\cdot\underbrace{a}_{5}}_{25}-\underbrace{\underbrace{b}_{2}\cdot\underbrace{b}_{2}}_{4}}_{21}}_{\text{true}}$$

If $a = 10$ and $b = 1$, then:

$$\underbrace{\underbrace{\underbrace{(\underbrace{a}_{10}+\underbrace{b}_{1})}_{11}\cdot\underbrace{(\underbrace{a}_{10}-\underbrace{b}_{1})}_{9}}_{99}=\underbrace{\underbrace{\underbrace{a}_{10}\cdot\underbrace{a}_{10}}_{100}-\underbrace{\underbrace{b}_{1}\cdot\underbrace{b}_{1}}_{1}}_{99}}_{\text{true}}$$

A named function: $g(a) = a \cdot a + 4$
An unnamed function: $\lambda a.\, a \cdot a + 4$
Let $h = \lambda a.\, a \cdot a + 4$.
Then:





The usual notation for defining functions is like this:

$$\begin{array}{rrcl} f: & \mathbb{N} & \to & \mathbb{R} \\ & n & \mapsto & 2 + \sqrt{n} \end{array}$$

$$\begin{array}{rrcl} \text{(name)}: & \text{(domain)} & \to & \text{(codomain)} \\ & \text{(variable)} & \mapsto & \text{(expression)} \end{array}$$

It creates *named* functions
(with domains and codomains).

The usual notation for creating named functions
without specifying their domains and codomains
is just $f(n) = 2 + \sqrt{n}$.
Note that this is:

$$\begin{array}{cccc} f & (\, n \,) & = & 2 + \sqrt{n} \\ \text{(name)} & (\,\text{(variable)}\,) & = & \text{(expression)} \end{array}$$

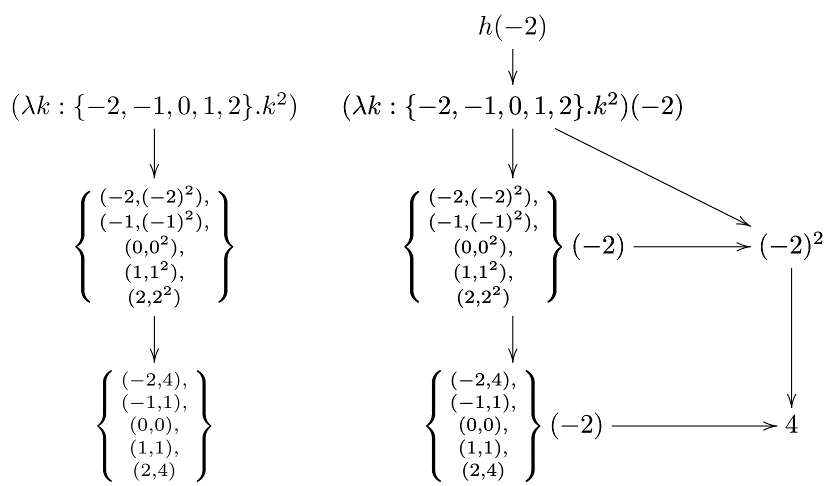
The *graph* of

$$h: \begin{array}{rcl} \{-2,-1,0,1,2\} & \to & \{0,1,2,3,4\} \\ k & \mapsto & k^2 \end{array}$$

is $\{(-2,4),(-1,1),(0,0),(1,1),(2,4)\}$.

We can think that a function *is* its graph,
and that a lambda-expression (with domain) reduces to a graph.
Then $h = \{(-2,4),(-1,1),(0,0),(1,1),(2,4)\}$
and $h(-2) = \{(-2,4),(-1,1),(0,0),(1,1),(2,4)\}(-2) = 4$.

Let $h := (\lambda k : \{-2,-1,0,1,2\}.k^2)$.
We have:



Note:
the graph of $(\lambda n : \mathbb{N}.n^2)$ has infinite points,
the graph of $(\lambda n : \mathbb{N}.n^2)$ is an infinite set,
the graph of $(\lambda n : \mathbb{N}.n^2)$ can't be written down *explicitly* without '...'s...

Mathematicians love infinite sets.
Computers hate infinite sets.
For mathematicians a function *is* its graph.
For computer scientists a function *is* is a finite program.
Computer scientists love 'λ's!

I love things like this: $\left\{ \begin{matrix} (3,30), \\ (4,40) \end{matrix} \right\}(3) = 30$

Types (introduction)

Let:
$A = \{1, 2\}$
$B = \{30, 40\}$.
If $f : A \to B$, then f is one of these
four functions:

$$\begin{smallmatrix}1\mapsto 30\\ 2\mapsto 30\end{smallmatrix}, \begin{smallmatrix}1\mapsto 30\\ 2\mapsto 40\end{smallmatrix}, \begin{smallmatrix}1\mapsto 40\\ 2\mapsto 30\end{smallmatrix}, \begin{smallmatrix}1\mapsto 40\\ 2\mapsto 40\end{smallmatrix}$$

or, in other notation,

$$\left\{\begin{smallmatrix}(1,30)\\ (2,30)\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}(1,30)\\ (2,40)\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}(1,40)\\ (2,30)\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}(1,40)\\ (2,40)\end{smallmatrix}\right\}$$

which means that:

$$f \in \left\{ \left\{\begin{smallmatrix}(1,30)\\ (2,30)\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}(1,30)\\ (2,40)\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}(1,40)\\ (2,30)\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}(1,40)\\ (2,40)\end{smallmatrix}\right\} \right\}$$

Let's use the notation "$A \to B$" for
"the set of all functions from A to B".

Then $(A \to B) = \left\{ \left\{\begin{smallmatrix}(1,30)\\ (2,30)\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}(1,30)\\ (2,40)\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}(1,40)\\ (2,30)\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}(1,40)\\ (2,40)\end{smallmatrix}\right\} \right\}$
and $f : A \to B$
means $f \in (A \to B)$.

In Type Theory and λ-calculus "$a : A$"
is pronounced "a is of type A", and the meaning
of this is *roughly* "$a \in B$".
(We'll see the differences between '$\in$' and ':' (much) later).

Note that:
1. if $f : A \to B$ and $a : A$ then $f(a) : B$
2. if $a : A$ and $b : B$ then $(a, b) : A \times B$
3. if $p : A \times B$ then $\pi p : A$ and $\pi' p : B$, where
'π' means 'first projection' and
'π'' means 'second projection';
if $p = (2, 30)$ then $\pi p = 2$, $\pi' p = 30$.

If $p : A \times B$ and $g : B \to C$, then:

$$\underbrace{(\underbrace{\pi \underbrace{p}_{:A\times B}}_{:A},\ \underbrace{\underbrace{g}_{:B\to C}\ \underbrace{(\pi' \underbrace{p}_{:A\times B}}_{:B})))}_{:C}}_{:A\times C}$$

Typed λ-calculus: trees

$A = \{1,2\}$
$B = \{3,4\}$
$C = \{30,40\}$
$D = \{10,20\}$
$A \times B = \left\{ \begin{matrix} (1,3),(1,4), \\ (2,3),(2,4) \end{matrix} \right\}$
$B \to C = \left\{ \left\{ \begin{smallmatrix} (3,30), \\ (4,30) \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} (3,30), \\ (4,40) \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} (3,40), \\ (4,30) \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} (3,40), \\ (4,40) \end{smallmatrix} \right\} \right\}$

If we know [the values of] a, b, f
then we know [the value of] $(a, f(b))$.
If $(a,b) = (2,3)$ and $f = \left\{ \begin{smallmatrix} (3,30), \\ (4,40) \end{smallmatrix} \right\}$
then $(a, f(b)) = (2, 30)$.
If we know the *types* of a, b, f
we know the type of $(a, f(b))$.
If we know the types of p, f
we know the type of $(\pi p, f(\pi' p))$.
If we know the types p, f
we know the type of $(\lambda p : A \times B.(\pi p, f(\pi' p)))$.

$$\dfrac{\dfrac{(a,b)}{a}\,\pi \qquad \dfrac{\dfrac{(a,b)}{b}\,\pi' \quad f}{f(b)}\,app}{(a, f(b))}\,pair \qquad \dfrac{\dfrac{(2,3)}{2}\,\pi \qquad \dfrac{\dfrac{(2,3)}{3}\,\pi' \quad \{(3,30),(4,40)\}}{30}\,app}{(2,30)}\,pair$$

$$\dfrac{\dfrac{(a,b) : A \times B}{a : A}\,\pi \qquad \dfrac{\dfrac{(a,b) : A \times B}{b : B}\,\pi' \quad f : B \to C}{f(b) : C}\,app}{(a, f(b)) : A \times C}\,pair$$

$$\dfrac{\dfrac{p : A \times B}{\pi p : A}\,\pi \qquad \dfrac{\dfrac{p : A \times B}{\pi' p : B}\,\pi' \quad f : B \to C}{f(\pi' p) : C}\,app}{(\pi p, f(\pi' p)) : A \times C}\,pair$$

$$\dfrac{\dfrac{\dfrac{p : A \times B}{\pi p : A}\,\pi \qquad \dfrac{\dfrac{p : A \times B}{\pi' p : B}\,\pi' \quad f : B \to C}{f(\pi' p) : C}\,app}{(\pi p, f(\pi' p)) : A \times C}\,pair}{(\lambda p : A \times B.(\pi p, f(\pi' p))) : A \times B \to A \times C}$$

Exercises

Let:
$A = \{1, 2\}$
$B = \{3, 4\}$
$C = \{30, 40\}$
$D = \{10, 20\}$
$f = \left\{ \begin{smallmatrix} (3,30), \\ (4,40) \end{smallmatrix} \right\}, \ f : B \to C$
$g = \left\{ \begin{smallmatrix} (1,10), \\ (2,30) \end{smallmatrix} \right\}, \ g : A \to D$

a) Check that:

$$(\lambda p : A \times B.(\pi p, f(\pi' p))) = \left\{ \begin{smallmatrix} ((1,3),(1,30)), \\ ((1,4),(1,40)), \\ ((2,3),(2,30)), \\ ((2,4),(2,40)) \end{smallmatrix} \right\}$$

b) Let $p = (2, 3)$. Evaluate $(g(\pi p), f(\pi' p))$.
c) Check that $(g(\pi p), f(\pi' p)) : D \times C$.
d) Suppose that $p : A \times B$. Check that

$$(\lambda p : A \times B.(g(\pi p), f(\pi' p))) : A \times B \to D \times C.$$

e) Evaluate $(\lambda p : A \times B.(g(\pi p), f(\pi' p)))$.

Here is another notation for checking types:

$$\underbrace{(\lambda \underbrace{p}_{:A\times B} : A\times B.\ \underbrace{(\underbrace{\pi \underbrace{p}_{:A\times B}}_{:A},\ \underbrace{\underbrace{f}_{:B\to C} \underbrace{(\pi' \underbrace{p}_{:A\times B}}_{:B})}_{:C})}_{:A\times C})}_{:A\times B\to A\times C}$$

Compare it with:

$$\dfrac{\dfrac{\dfrac{p : A\times B}{\pi p : A}\,\pi \qquad \dfrac{\dfrac{p : A\times B}{\pi' p : B}\,\pi' \qquad f : B\to C}{f(\pi' p) : C}\,app}{(\pi p, f(\pi' p)) : A\times C}\,pair}{(\lambda p : A\times B.(\pi p, f(\pi' p))) : A\times B\to A\times C}$$

Represente graficamente:
$A := \{(1,4),(2,4),(1,3)\}$
$B := \{(1,3),(1,4),(2,4)\}$
$C := \{(1,3),(1,4),(2,4),(2,4)\}$
$D := \{(1,3),(1,4),(2,3),(2,4)\}$
$h := \{(0,3),(1,2),(2,1),(3,0)\}$
$k := \{x : \{0,1,2,3\}; (x, 3-x)\}$
$m := \{y : \{0,1,2,3\}; (3-y, y)\}$

$f := (\lambda x : \{0,1,2\}.\, x \cdot x)$
$g := (\lambda a : \{0,1,2\}.\, 3 \cdot a)$
$q := (\lambda a : \{0,1,2\}.\, (x+1)^2 - 2x - 1)$

Tudo isto aqui a gente pode calcular com tabelas:
$\forall x : \{0,1,2,3\}.x < 2$
$\forall x : \{0,1,2,3\}.x^2 \leq 5$
$\forall x : \{0,1,2,3\}.x^2 < 10$
$\forall x : \{0,1,2,3\}.x \geq 2$
$\exists x : \{0,1,2,3\}.x^2 \geq 5$
$\{x : \{0,1,2,3\}; x^2\}$
$\{x : \{0,1,2,3\}, x \geq 2; x\}$
$\Sigma x : \{0,1,2,3\}.x^2$
$\Pi x : \{0,1,2,3\}.x + 1$
$\lambda x : \{0,1,2,3\}.5 - x$

$$
\begin{array}{rcl}
\forall x : \{0,1,2\}.P(x) & = & P(0) \& P(1) \& P(2) \\
\exists x : \{0,1,2\}.P(x) & = & P(0) \vee P(1) \vee P(2) \\
\{x : \{0,1,2\}; f(x)\} & = & \{f(0), f(1), f(2)\} \\
\Sigma x : \{0,1,2\}.f(x) & = & f(0) + f(1) + f(2) \\
\Pi x : \{0,1,2\}.f(x) & = & f(0) \cdot f(1) \cdot f(2) \\
\lambda x : \{0,1,2\}.f(x) & = & \{(0, f(0)), (1, f(1)), (2, f(2))\}
\end{array}
$$

Obs:

$$
\begin{array}{lcll}
\Sigma x : \{0,1,2,2,3\}.x^2 & \to & f(0) + f(1) + f(1) + f(2) & (?) \\
\Pi x : \{0,1,2,2,3\}.x + 1 & \to & f(0) \cdot f(1) \cdot f(1) \cdot f(2) & (?)
\end{array}
$$

If $P(x) = (x < 2)$ then

$$
\begin{array}{ccc}
\forall x : \{0,1,2\}.P(x) & \longrightarrow & P(0) \& P(1) \& P(2) \\
\downarrow & & \downarrow \\
\forall x : \{0,1,2\}.x < 2 & \longrightarrow & (0 < 2) \& (1 < 2) \& (2 < 2) \\
 & & \downarrow \\
 & & 1 \& 1 \& 0
\end{array}
$$