Planar Heyting Algebras for Children

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December 18, 2016

This paper shows a way to interpret (propositional) intuitionistic logic *visually* (see section 6).

The "for children" in the title has a precise, but somewhat unusual, meaning, that is explained in sec.24.

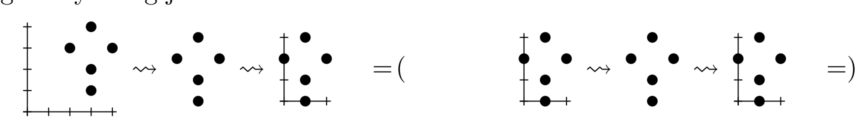
1 Positional notations

Definition: a *ZSet* is a finite, non-empty subset of $\mathbb{N}^2$ that touches both axes, i.e., that has a point of the form $(0, \_)$ and a point of the form $(\_, 0)$. We will often represent ZSets using a bullet notation, with or without the axes and ticks. For example:

$$K = \left\{ \begin{array}{ccc} & (1,3), & \\ (0,2), & & (2,2), \\ & (1,1), & \\ & (1,0) & \end{array} \right\} = \begin{array}{ccc} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{array} = \begin{array}{ccc} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{array}$$

We will use the ZSet above a lot in examples, so let's give it a short name: K ("kite").

The condition of touching both axes is what lets us represent ZSets unambiguously using just the bullets:



We can use a positional notation to represent functions *from* a ZSet. For example, if

$$\begin{array}{rrcl} f & : & K & \to & \mathbb{N} \\ & & (x,y) & \mapsto & x \end{array}$$

then

$$f = \left\{ \begin{array}{ccc} & ((1,3),1), & \\ ((0,2),0), & & ((2,2),2), \\ & ((1,1),1), & \\ & ((1,0),1) & \end{array} \right\} = \begin{array}{ccc} & 1 & \\ 0 & & 2 \\ & 1 & \\ & 1 & \end{array}$$

We will sometimes use λ-notation to represent functions compactly. For example:

$$\lambda(x,y){:}K.x = \left\{ \begin{array}{ccc} & ((1,3),1), & \\ ((0,2),0), & & ((2,2),2), \\ & ((1,1),1), & \\ & ((1,0),1) & \end{array} \right\} = \begin{array}{ccc} & 1 & \\ 0 & & 2 \\ & 1 & \\ & 1 & \end{array}$$

$$\lambda(x,y){:}K.y = \left\{ \begin{array}{ccc} & ((1,3),3), & \\ ((0,2),2), & & ((2,2),2), \\ & ((1,1),1), & \\ & ((1,0),0) & \end{array} \right\} = \begin{array}{ccc} & 3 & \\ 2 & & 2 \\ & 1 & \\ & 0 & \end{array}$$

The "reading order" on the points of a ZSet S "lists" the points of S starting from the top and going from left to right in each line. More precisely, if S has n points then $r_S : S \to \{1, \ldots, n\}$ is a bijection, and for example:

$$r_K = \begin{array}{ccc} & 1 & \\ 2 & & 3 \\ & 4 & \\ & 5 & \end{array}$$

Subsets of a ZSet are represented with a notation with '$\bullet$'s and '$\cdot$', and partial functions from a ZSet are represented with '$\cdot$'s where they are not defined. For example:

$$\begin{array}{ccc} & \bullet & \\ \cdot & & \bullet \\ & \bullet & \\ & \cdot & \end{array} \qquad \begin{array}{ccc} & 1 & \\ \cdot & & 3 \\ & 4 & \\ & \cdot & \end{array}$$

The *characteristic function* of a subset S' of a ZSet S is the function $\chi_{S'} : S \to \{0,1\}$ that returns 1 exactly on the points of S'; for example, $\begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 1 & \\ & 0 & \end{smallmatrix}$ is the characteristic function of $\begin{smallmatrix} & \bullet & \\ \cdot & & \bullet \\ & \bullet & \\ & \cdot & \end{smallmatrix} \subset \begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{smallmatrix}$. We will sometimes denote subsets by their characteristic functions because this makes them easier to "pronounce" by reading aloud their digits in the reading order — for example, $\begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 1 & \\ & 0 & \end{smallmatrix}$ is "one-zero-one-one-zero" (see sec.**??**).

2 ZDAGs

We will sometimes use the bullet notation for a ZSet S as a *shorthand* for one of the two DAGs induced by S: one with its arrows going up, the other one with them going down. For example: sometimes

$$\begin{array}{ccc} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{array} \quad \text{will stand for:} \quad \begin{array}{ccccc} & & \bullet & & \\ & \swarrow & & \searrow & \\ \bullet & & & & \bullet \\ & \searrow & & \swarrow & \\ & & \bullet & & \\ & & \downarrow & & \\ & & \bullet & & \end{array} = \begin{array}{ccccc} & & (1,3) & & \\ & \swarrow & & \searrow & \\ (0,2) & & & & (2,2) \\ & \searrow & & \swarrow & \\ & & (1,1) & & \\ & & \downarrow & & \\ & & (1,0) & & \end{array} =$$

$$\left(\left\{ \begin{array}{ccc} & (1,3), & \\ (0,2), & & (2,2), \\ & (1,1), & \\ & (0,0) & \end{array} \right\}, \left\{ \begin{array}{c} ((1,3),(0,2)),((1,3),(2,2)), \\ ((0,2),(1,1)),((2,2),(1,1)), \\ ((1,1),(0,0)) \end{array} \right\} \right)$$

Let's formalize this.

Consider a game in which black and white pawns are placed on points of $\mathbb{Z}^2$, and they can move like this:

$$\begin{array}{ccccc} & & \bullet & & \\ & \swarrow & \downarrow & \searrow & \\ \bullet & & \bullet & & \bullet \end{array} \qquad \begin{array}{ccccc} \circ & & \circ & & \circ \\ & \nwarrow & \uparrow & \nearrow & \\ & & \circ & & \end{array}$$

Black pawns can move from (x, y) to $(x + k, y - 1)$ and white pawns from (x, y) to $(x+k, y+1)$, where $k \in \{-1, 0, 1\}$. The mnemonic is that black pawns are "solid", and thus "heavy", and they "sink", so they move down; white pawns are "hollow", and thus "light", and they "float", so they move up.

Let's now restrict the board positions to a ZSet S. Black pawns can move from (x, y) to $(x+k, y-1)$ and white pawns from (x, y) to $(x+k, y+1)$, where $k \in \{-1, 0, 1\}$, but only when the starting and ending positions both belong to S. The sets of possible black pawn moves and white pawn moves on S can be defined formally as:

$$\begin{aligned} \mathsf{BPM}(S) &= \{\, ((x, y), (x', y')) \in S^2 \mid x - x' \in \{-1, 0, 1\}, y' = y - 1 \,\} \\ \mathsf{WPM}(S) &= \{\, ((x, y), (x', y')) \in S^2 \mid x - x' \in \{-1, 0, 1\}, y' = y + 1 \,\} \end{aligned}$$

...and now please forget everything else you expect from a game — like starting position, capturing, objective, winning... the idea of a "game" was just a tool to let us explain $\mathsf{BPM}(S)$ and $\mathsf{WPM}(S)$ quickly.

A *ZDAG* is a DAG of the form $(S, \mathsf{BPM}(S))$ or $(S, \mathsf{WPM}(S))$, where S is a ZSet.

A *ZPO* is partial order of the form $(S, \mathsf{BPM}(S)^*)$ or $(S, \mathsf{WPM}(S)^*)$, where S is a ZSet and the '*' denotes the transitive-reflexive closure of the relation.

Sometimes, when this is clear from the context, a bullet diagram like will stand for either the ZDAGs $(\ , \mathsf{BPM}(\ ))$ or $(\ , \mathsf{WPM}(\ ))$, or for the ZPOs $(\ , \mathsf{BPM}(\ )^*)$ or $(\ , \mathsf{WPM}(\ )^*)$ (sec.4), or even for the ZPOs seen as categories (section ___).

3 LR-coordinates

The *lr-coordinates* are useful for working on quarter-plane of $\mathbb{Z}^2$ that looks like $\mathbb{N}^2$ turned 45° to the left. Let $\langle l, r\rangle := (-l + r, l + r)$; then (the bottom part of) $\{\, \langle l, r\rangle \mid l, r \in \mathbb{N} \,\}$ is:

$$\begin{array}{ccccccccc} \langle 4,0\rangle & & \langle 3,1\rangle & & \langle 2,2\rangle & & \langle 1,3\rangle & & \langle 0,4\rangle \\ & \langle 3,0\rangle & & \langle 2,1\rangle & & \langle 1,2\rangle & & \langle 0,3\rangle & \\ & & \langle 2,0\rangle & & \langle 1,1\rangle & & \langle 0,2\rangle & & \\ & & & \langle 1,0\rangle & & \langle 0,1\rangle & & & \\ & & & & \langle 0,0\rangle & & & & \end{array} \quad = \quad \begin{array}{ccccccccc} (-4,4) & & (-2,4) & & (0,4) & & (2,4) & & (4,4) \\ & (-3,3) & & (-1,3) & & (1,3) & & (3,3) & \\ & & (-2,2) & & (0,2) & & (2,2) & & \\ & & & (-1,1) & & (1,1) & & & \\ & & & & (0,0) & & & & \end{array}$$

Sometimes we will write lr instead of $\langle l, r\rangle$. So:

$$\begin{array}{ccccccccc} 40 & & 31 & & 22 & & 13 & & 04 \\ & 30 & & 21 & & 12 & & 03 & \\ & & 20 & & 11 & & 02 & & \\ & & & 10 & & 01 & & & \\ & & & & 00 & & & & \end{array} \quad = \quad \begin{array}{ccccccccc} (-4,4) & & (-2,4) & & (0,4) & & (2,4) & & (4,4) \\ & (-3,3) & & (-1,3) & & (1,3) & & (3,3) & \\ & & (-2,2) & & (0,2) & & (2,2) & & \\ & & & (-1,1) & & (1,1) & & & \\ & & & & (0,0) & & & & \end{array}$$

Let $\mathbb{LR} = \{\, \langle l, r \rangle \mid l, r \in \mathbb{N} \,\}$.

4 ZHAs

A *ZHA* is a subset of $\mathbb{LR}$ "between a left and a right wall", as we will see.

A triple (h, L, R) is a "height-left-right-wall" when:
1) $h \in \mathbb{N}$
2) $L : \{0, \dots, h\} \to \mathbb{Z}$ and $R : \{0, \dots, h\} \to \mathbb{Z}$
3) $L(h) = R(h)$ (the top points of the walls are the same)
4) $L(0) = R(0) = 0$ (the bottom points of the walls are the same, 0)
5) $\forall y \in \{0, \dots, h\}.\, L(y) \leq R(y)$ ("left" is left of "right")
6) $\forall y \in \{1, \dots, h\}.\, L(y) - L(y-1) = \pm 1$ (the left wall makes no jumps)
7) $\forall y \in \{1, \dots, h\}.\, R(y) - R(y-1) = \pm 1$ (the right wall makes no jumps)

The *ZHA generated by* a height-left-right-wall (h, L, R) is the set of all points of $\mathbb{LR}$ with valid height and between the left and the right walls. Formally:

$$\mathsf{ZHAG}(h, L, R) = \{\, (x, y) \in \mathbb{LR} \mid y \leq h, L(y) \leq x \leq R(y) \,\}.$$

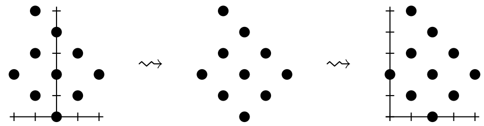
A *ZHA* is a set of the form $\mathsf{ZHAG}(h, L, R)$, where the triple (h, L, R) is a height-left-right-wall.

Here is an example of a ZHA (with the white pawn moves on it):

$$\begin{array}{ccccccccccccccc}
 & & (-3,9) & & & & & & & & & L(9)=-3 & R(9)=-3 & L(9)=R(9) & h=9 \\
 & \nearrow & & \nwarrow & & & & & & & & & & & \\
(-4,8) & & & & (-2,8) & & & & & & & L(8)=-4 & R(8)=-2 & & \\
 & \nwarrow & & \nearrow & & & & & & & & & & & \\
 & & (-3,7) & & & & & & & & & L(7)=-3 & R(7)=-3 & & \\
 & & & \nwarrow & & & & & & & & & & & \\
 & & & & (-2,6) & & & & & & & L(6)=-2 & R(6)=-2 & & \\
 & & & & & \nwarrow & & & & & & & & & \\
 & & & & & & (-1,5) & & & & & L(5)=-1 & R(5)=-1 & & \\
 & & & & & \nearrow & & \nwarrow & & & & & & & \\
 & & & & (-2,4) & & & & (0,4) & & & L(4)=-2 & R(4)=0 & & \\
 & & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & & & & \\
 & & (-3,3) & & & & (-1,3) & & & & (1,3) & L(3)=-3 & R(3)=1 & & \\
 & & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & & & & \\
 & & & & (-2,2) & & & & (0,2) & & & L(2)=-2 & R(2)=0 & & \\
 & & & & & \nwarrow & & \nearrow & & \nwarrow & & & & & \\
 & & & & & & (-1,1) & & & & (1,1) & L(1)=-1 & R(1)=1 & & \\
 & & & & & & & \nwarrow & & \nearrow & & & & & \\
 & & & & & & & & (0,0) & & & L(0)=0 & R(0)=0 & L(0)=R(0)=0 &
\end{array}$$

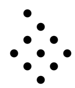
We will see later (section 6) that ZHAs (with white pawn moves) are Heyting Algebras.

We can use a bullet notation to denote ZHAs, but look at what happens when we start with a ZHA, erase the axes, and then add the axes back using the convention from sec. 1:



we get a ZSet whose bottom point is $(2, 0)$, but the bottom point of our original ZHA was $(0, 0)$... let's refine that convention. From this point on, it will be: when it is clear from the context that a bullet diagram represents a ZHA, then the $(0, 0)$ is its bottom point; otherwise the $(0, 0)$ is the point that makes the diagram fit in $\mathbb{N}^2$ and touch both axes.

The new convention also applies to functions from ZHAs, and for partial functions and subsets. For example:

$B =$  (a ZHA)

$$\lambda(x, y){:}B.x = \begin{array}{ccccc} & \text{-1} & & & \\ & & 0 & & \\ & \text{-1} & & 1 & \\ \text{-2} & & 0 & & 2 \\ & \text{-1} & & 1 & \\ & & 0 & & \end{array}$$

$$\lambda\langle l, r\rangle{:}B.l = \begin{array}{ccccc} & 3 & & & \\ & & 2 & & \\ & 2 & & 1 & \\ 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{array} \qquad \lambda\langle l, r\rangle{:}B.r = \begin{array}{ccccc} & 2 & & & \\ & & 2 & & \\ & 1 & & 2 & \\ 0 & & 1 & & 2 \\ & 0 & & 1 & \\ & & 0 & & \end{array}$$

We will often denote ZHAs by the identity function on them:

$$\lambda\langle l, r\rangle{:}B.\langle l, r\rangle \;=\; \lambda lr{:}B.lr \;=\; \begin{array}{ccccc} & 32 & & & \\ & & 22 & & \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{array} \qquad B \;=\; \begin{array}{ccccc} & 32 & & & \\ & & 22 & & \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{array}$$

Note that we are using the compact notation from the end of section 3: 'lr' instead of '$\langle l, r\rangle$'.

5 Propositional calculus

A *PC-structure* is a tuple

$$L = (\Omega, \leq, \top, \bot, \wedge, \vee, \rightarrow, \leftrightarrow, \neg),$$

where:
Ω is the "set of truth values",
$\leq$ is a relation on Ω,
$\top$ and $\bot$ are two elements of Ω,
$\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ are functions from $\Omega \times \Omega$ to Ω,
$\neg$ is a function from Ω to Ω.

Classical Logic "is" a PC-structure, with $\Omega = \{0, 1\}$, $\top = 1$, $\bot = 0$, $\leq = \{(0, 0), (0, 1), (1, 0)\}$, $\wedge = \left\{ \begin{smallmatrix} ((0,0),0),((0,1),0), \\ ((1,0),0),((1,1),1) \end{smallmatrix} \right\}$, etc.

PC-structures let us interpret expressions from Propositional Calculus, and let us define a notion of *tautology*. For example, in Classical Logic,

- $\neg\neg P \leftrightarrow P$ is a tautology because it is valid (i.e., it yields $\top$) for all values of P in Ω,
- $\neg(P \wedge Q) \to (\neg P \vee \neg Q)$ s a tautology because it is valid for all values of P and Q in Ω,
- but $P \vee Q \to P \wedge Q$ is *not* a tautology, because when $P = 0$ and $Q = 1$ the result is not $\top$:

$$\underbrace{\underbrace{\underbrace{P}_{0} \vee \underbrace{Q}_{1}}_{1} \to \underbrace{\underbrace{P}_{0} \wedge \underbrace{Q}_{1}}_{0}}_{0}$$

6 Propositional calculus in a ZHA

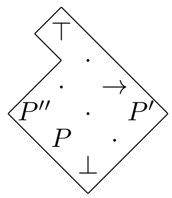
Let Ω be the set of points of a ZHA and $\leq$ the default partial order on it. The *default meanings for* $\top, \bot, \wedge, \vee, \to, \leftrightarrow, \neg$ are these ones:

$$\begin{array}{rcl}
\langle a,b\rangle \leq \langle c,d\rangle & := & a \leq c \wedge b \leq d \\
\langle a,b\rangle \geq \langle c,d\rangle & := & a \geq c \wedge b \geq d \\[4pt]
\langle a,b\rangle \mathsf{\ above\ } \langle c,d\rangle & := & a \geq c \wedge b \geq d \\
\langle a,b\rangle \mathsf{\ below\ } \langle c,d\rangle & := & a \leq c \wedge b \leq d \\
\langle a,b\rangle \mathsf{\ leftof\ } \langle c,d\rangle & := & a \geq c \wedge b \leq d \\
\langle a,b\rangle \mathsf{\ rightof\ } \langle c,d\rangle & := & a \leq c \wedge b \geq d \\[4pt]
\mathsf{valid}(\langle a,b\rangle) & := & \langle a,b\rangle \in \Omega \\
\mathsf{ne}(\langle a,b\rangle) & := & \mathsf{if\ valid\ }(\langle a,b+1\rangle) \mathsf{\ then\ ne}(\langle a,b+1\rangle) \mathsf{\ else\ } \langle a,b\rangle \mathsf{\ end} \\
\mathsf{nw}(\langle a,b\rangle) & := & \mathsf{if\ valid\ }(\langle a+1,b\rangle) \mathsf{\ then\ nw}(\langle a+1,b\rangle) \mathsf{\ else\ } \langle a,b\rangle \mathsf{\ end} \\[4pt]
\langle a,b\rangle \wedge \langle c,d\rangle & := & \langle \mathsf{min}(a,c), \mathsf{min}(b,d)\rangle \\
\langle a,b\rangle \vee \langle c,d\rangle & := & \langle \mathsf{max}(a,c), \mathsf{max}(b,d)\rangle \\[4pt]
\langle a,b\rangle \to \langle c,d\rangle & := & \begin{array}{llll}
\mathsf{if} & \langle a,b\rangle \mathsf{\ below\ } \langle c,d\rangle & \mathsf{then} & \top \\
\mathsf{elseif} & \langle a,b\rangle \mathsf{\ leftof\ } \langle c,d\rangle & \mathsf{then} & \mathsf{ne}(\langle a,b\rangle \wedge \langle c,d\rangle) \\
\mathsf{elseif} & \langle a,b\rangle \mathsf{\ rightof\ } \langle c,d\rangle & \mathsf{then} & \mathsf{nw}(\langle a,b\rangle \wedge \langle c,d\rangle) \\
\mathsf{elseif} & \langle a,b\rangle \mathsf{\ above\ } \langle c,d\rangle & \mathsf{then} & \langle c,d\rangle \\
\mathsf{end} & & &
\end{array} \\
\top & := & \mathsf{sup}(\Omega) \\
\bot & := & \langle 0,0\rangle \\
\neg\langle a,b\rangle & := & \langle a,b\rangle \to \bot \\
\langle a,b\rangle \leftrightarrow \langle c,d\rangle & := & (\langle a,b\rangle \to \langle c,d\rangle) \wedge (\langle c,d\rangle \to \langle a,b\rangle)
\end{array}$$

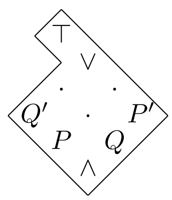
Let Ω be the ZHA at the top left in the figure below. Then, with the default meanings for the connectives neither $\neg\neg P \leftrightarrow P$ nor $\neg(P \wedge Q) \to (\neg P \vee \neg Q)$ are tautologies, as there are valuations that make them yield results different than

$\top = 32$:

$$\begin{array}{ccccc}
32 & & & & \\
& 22 & & & \\
21 & & 12 & & \\
20 & 11 & & 02 & \\
& 10 & & 01 & \\
& & 00 & &
\end{array}$$



$$\underbrace{(\underbrace{\neg\underbrace{\neg\underbrace{P}_{10}}_{02}}_{20}) \to \underbrace{P}_{10}}_{12}$$



$$\underbrace{\underbrace{\neg(\underbrace{\underbrace{P}_{10} \land \underbrace{Q}_{01}}_{00})}_{32} \to \underbrace{(\underbrace{\neg\underbrace{P}_{10}}_{02} \lor \underbrace{\neg\underbrace{Q}_{01}}_{20})}_{22}}_{22}$$

So: *some* classical tautologies are not tautologies in this ZHA.

The somewhat arbitrary-looking definition of '→' will be explained at the end of the next section.

7 Heyting Algebras

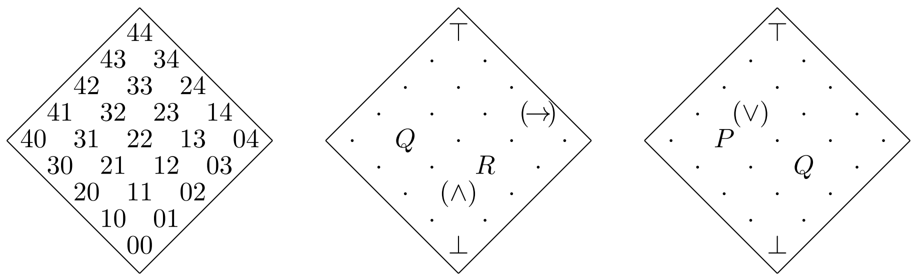
A *Heyting Algebra* is a PC-structure

$$H = (\Omega, \leq_H, \top_H, \bot_H, \land_H, \lor_H, \to_H, \leftrightarrow_H, \neg_H),$$

in which:
1) $(\Omega, \leq_H)$ is a partial order
2) $\top_H$ is the top element of the partial order
3) $\bot_H$ is the bottom element of the partial order
4) $P \leftrightarrow_H Q$ is the same as $(P \to_H Q) \land_H (Q \to_H P)$
5) $\neg_H P$ is the same as $P \to_H \bot_H$
6) $\forall P, Q, R \in \Omega.\ (P \leq_H (Q \land_H R)) \leftrightarrow ((P \leq_H Q) \land (P \leq_H R))$
7) $\forall P, Q, R \in \Omega.\ ((P \lor_H Q) \leq_H R) \leftrightarrow ((P \leq_H R) \land (Q \leq_H R))$
8) $\forall P, Q, R \in \Omega.\ (P \leq_H (Q \to_H R)) \leftrightarrow ((P \land_H R) \leq_H R)$
6') $\forall Q, R \in \Omega.\, \exists! Y \in \Omega.\, \forall P \in \Omega.\ (P \leq_H Y) \leftrightarrow ((P \leq_H Q) \land (P \leq_H R))$
7') $\forall P, Q \in \Omega.\, \exists! X \in \Omega.\, \forall R \in \Omega.\ (X \leq_H R) \leftrightarrow ((P \leq_H R) \land (Q \leq_H R))$
8') $\forall Q, R \in \Omega.\, \exists! Y \in \Omega.\, \forall P \in \Omega.\ (P \leq_H Y) \leftrightarrow ((P \land_H R) \leq_H R)$

The conditions 6', 7', 8' say that there are unique elements in Ω that "behave as" $Q \land_H R$, $P \lor_H Q$ and $Q \to_H R$ for given P, Q, R; the conditions 6,7,8 say that $Q \land_H R$, $P \lor_H Q$ and $Q \to_H R$ are exactly the elements with this behavior.

The positional notation on ZHAs is very helpful for visualizing what the conditions 6',7',8',6,7,8 mean. Let Ω be the ZDAG on the left below:



we will see that

a) if $Q = 31$ and $R = 12$ then $Q \wedge_H R = 11$,
b) if $P = 31$ and $Q = 12$ then $P \vee_H Q = 32$,
c) if $Q = 31$ and $R = 12$ then $Q \rightarrow_H R = 14$.

Let's see each case separately — but, before we start, note that in 6, 7, 8, 6', 7', 8' we work part with truth values in Ω and part with standard truth values. For example, in 6, with $P = 20$, we have:

$$\underbrace{\underbrace{(\underbrace{P}_{20} \leq_H \underbrace{(\underbrace{Q}_{31} \wedge_H \underbrace{R}_{12})}_{11})}_{0} \leftrightarrow \underbrace{((\underbrace{\underbrace{P}_{20} \leq_H \underbrace{Q}_{31}}_{1}) \wedge (\underbrace{\underbrace{P}_{20} \leq_H \underbrace{R}_{12}}_{0}))}_{0}}_{1}$$

a) Let $Q = 31$ and $R = 12$. We want to see that $Q \wedge_H R = 11$, i.e., that

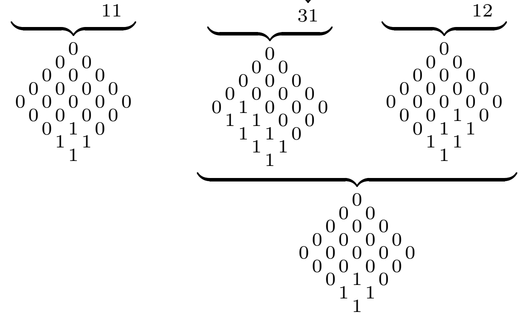
$$\forall P \in \Omega.\ \ (P \leq_H Y) \leftrightarrow ((P \leq_H Q) \wedge (P \leq_H R))$$

holds for $Y = 11$ and for no other $Y \in \Omega$. We can visualize the behavior of $P \leq_H Q$ for all 'P's by drawing $\lambda P{:}\Omega.(P \leq_H Q)$ in the positional notation; then we do the same for $\lambda P{:}\Omega.(P \leq_H R)$ and for $\lambda P{:}\Omega.((P \leq_H Q) \wedge (P \leq_H R))$. Suppose that the full expression, '$\forall P{:}\Omega.\ \_\_$', is true; then the behavior of the left side of the '$\leftrightarrow$', $\lambda P{:}\Omega.(P \leq_H Y)$, has to be a copy of the behavior of the right side, and that lets us find the only adequate value for Y.

The order in which we calculate and draw things is below, followed by the results themselves:

$$\underbrace{(P \leq_H \underbrace{Y}_{(7)})}_{(6)} \leftrightarrow \underbrace{((\underbrace{P \leq_H \underbrace{Q}_{(1)}}_{(3)}) \wedge (\underbrace{P \leq_H \underbrace{R}_{(2)}}_{(4)}))}_{(5)}$$

$$(P \leq_H \underbrace{Y}_{11}) \leftrightarrow ((P \leq_H \underbrace{Q}_{31}) \land (P \leq_H \underbrace{R}_{12}))$$



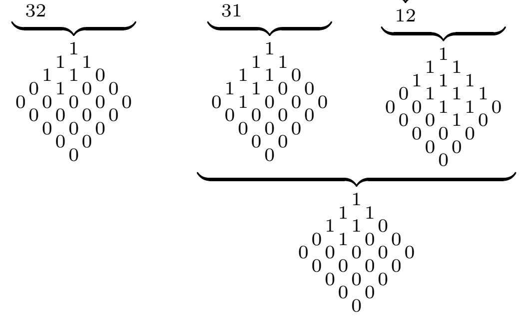
b) Let $P = 31$ and $Q = 12$. We want to see that $P \lor_H Q = 32$, i.e., that

$$\forall R{:}\Omega.\ \ (X \leq_H R) \leftrightarrow ((P \leq_H R) \land (Q \leq_H R))$$

holds for $X = 32$ and for no other $X \in \Omega$. We do essentially the same as we did in (a), but now we calculate $\lambda R{:}\Omega.(P \leq_H R)$, $\lambda R{:}\Omega.(Q \leq_H R)$, and $\lambda R{:}\Omega.((P \leq_H R) \land (Q \leq_H R))$. The order in which we calculate and draw things is below, followed by the results themselves:

$$\underbrace{(\underbrace{X}_{(7)} \leq_H R)}_{(6)} \leftrightarrow \underbrace{(\underbrace{(\underbrace{P}_{(1)} \leq_H R)}_{(3)} \land \underbrace{(\underbrace{Q}_{(2)} \leq_H R)}_{(4)})}_{(5)}$$

$$(\underbrace{X}_{32} \leq_H R) \leftrightarrow ((\underbrace{P}_{31} \leq_H R) \land (\underbrace{Q}_{12} \leq_H R))$$



c) Let $Q = 31$ and $R = 12$. We want to see that $Q \to_H R = 14$, i.e., that

$$\forall P{:}\Omega.\ \ (P \leq_H Y) \leftrightarrow ((P \land_H Q) \leq_H R)$$

holds for $Y = 14$ and for no other $Y \in \Omega$. Here the strategy is slightly different. We start by visualizing $\lambda P{:}\Omega.(P \land_H Q)$, which is a function from Ω to Ω, not a function from Ω to $\{0,1\}$ like the ones we were using before. The order in which we calculate and draw things is below, followed by the results:

$$\underbrace{(P \le_H \underbrace{Y}_{(6)})}_{(5)} \leftrightarrow \underbrace{(\underbrace{(P \land_H \underbrace{Q}_{(1)})}_{(3)} \le_H \underbrace{R}_{(2)})}_{(4)}$$

$$\underbrace{(P \le_H \underbrace{Y}_{14})}_{\substack{0 \\ 0\ 0 \\ 0\ 0\ 0 \\ 0\ 0\ 0\ 1 \\ 0\ 0\ 0\ 1\ 1 \\ 0\ 0\ 1\ 1 \\ 0\ 1\ 1 \\ 1\ 1 \\ 1}} \leftrightarrow \underbrace{(\underbrace{(P \land_H \underbrace{Q}_{31})}_{\substack{31 \\ 31\ 31 \\ 31\ 31\ 21 \\ 31\ 31\ 21\ 11 \\ 30\ 31\ 21\ 11\ 01 \\ 30\ 21\ 11\ 01 \\ 20\ 11\ 01 \\ 10\ 01 \\ 00}} \le_H \underbrace{R}_{12})}_{\substack{0 \\ 0\ 0 \\ 0\ 0\ 0 \\ 0\ 0\ 0\ 1 \\ 0\ 0\ 0\ 1\ 1 \\ 0\ 0\ 1\ 1 \\ 0\ 1\ 1 \\ 1\ 1 \\ 1}}$$

8 Logic in a Heyting Algebra

In the previous sextion we saw a set of conditions — called 1 to 8' — that characterize the "Heyting-Algebra-ness" of a PC-structure. It is easy to see that Heyting-Algebra-ness, or "HA-ness", is equivalent to this set of conditions:

1	$\forall P.$	$(P \le P)$					id
	$\forall P,Q,R.$	$(P \le R)$	$\leftarrow$	$(P \le Q)$	$\land$	$(Q \le R)$	comp
2	$\forall P.$	$(P \le \top)$					$\top_1$
3	$\forall Q.$	$(\bot \le Q)$					$\bot_1$
6	$\forall P,Q,R.$	$(P \le Q \land R)$	$\to$	$(P \le Q)$			$\land_1$
	$\forall P,Q,R.$	$(P \le Q \land R)$	$\to$			$(P \le R)$	$\land_2$
	$\forall P,Q,R.$	$(P \le Q \land R)$	$\leftarrow$	$(P \le Q)$	$\land$	$(P \le R)$	$\land_3$
7	$\forall P,Q,R.$	$(P \lor Q \le R)$	$\to$	$(P \le R)$			$\lor_1$
	$\forall P,Q,R.$	$(P \lor Q \le R)$	$\to$			$(Q \le R)$	$\lor_2$
	$\forall P,Q,R.$	$(P \lor Q \le R)$	$\leftarrow$	$(P \le R)$	$\land$	$(Q \le R)$	$\lor_3$
8	$\forall P,Q,R.$	$(P \le Q{\to}R)$	$\to$		$(P \land Q \le R)$		$\to_1$
	$\forall P,Q,R.$	$(P \le Q{\to}R)$	$\leftarrow$		$(P \land Q \le R)$		$\to_2$

We omitted the conditions 4 and 5, that defined '$\leftrightarrow$' and '$\neg$' in terms of the other operators. The last column gives a name to each of these new conditions.

These new conditions let us put (some) proofs about HAs in tree form, as we shall see soon.

Let us introduce two new notations. The first one,

$$(\text{expr})\left[\begin{smallmatrix} v_1{:=}\text{repl}_1 \\ v_2{:=}\text{repl}_2 \end{smallmatrix}\right]$$

indicates simultaneous substitution of all (free) occurrences of the variables v_1 and v_2 in expr by repl_1 and repl_2. For example,

$$((x+y)\cdot z)\left[\begin{smallmatrix} x{:=}a+y \\ y{:=}b+z \\ z{:=}c+x \end{smallmatrix}\right] \quad = \quad ((a+y)+(b+z))\cdot(c+x).$$

The second is a way to write '$\to$'s as horizontal bars. In

$$\dfrac{A \quad B \quad C}{D}\,\alpha \qquad \dfrac{E \quad F}{G}\,\beta \qquad \dfrac{H}{I}\,\gamma \qquad \dfrac{}{J}\,\delta \qquad \dfrac{\dfrac{}{K}\,\epsilon \quad \dfrac{L \quad M}{N}\,\zeta \quad O}{P}\,\eta$$

the trees mean:

- if A, B, C are true then D is true (by α),
- if E, F, are true then G is true (by β),
- if H is true then I is true (by γ),
- J is true (by δ, with no hypotheses),
- K is true (by ϵ); if L and M then N (by ζ); if K, N, O, then P (by η); combining all this we get a way to prove that if L, M, O, then P,

where $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta$ are usually names of rules.

The implications in the table in the beginning of this section can be rewritten as "tree rules" as:

$$\dfrac{}{P \le P}\,\text{id} \qquad \dfrac{P \le Q \quad Q \le R}{P \le R}\,\text{comp} \qquad \dfrac{}{P \le \top}\,\top_1 \qquad \dfrac{}{\bot \le Q}\,\bot_1$$

$$\dfrac{P \le Q \land R}{P \le Q}\,\land_1 \qquad \dfrac{P \le Q \land R}{P \le R}\,\land_2 \qquad \dfrac{P \le Q \quad P \le R}{P \le Q \land R}\,\land_3$$

$$\dfrac{P \lor Q \le R}{P \le R}\,\lor_1 \qquad \dfrac{P \lor Q \le R}{Q \le R}\,\lor_2 \qquad \dfrac{P \le R \quad Q \le R}{P \lor Q \le R}\,\lor_3$$

$$\dfrac{P \le Q{\to}R}{P \land Q \le R}\,{\to}_1 \qquad \dfrac{P \land Q \le R}{P \le Q{\to}R}\,{\to}_2$$

Note that the '$\forall P, Q, R \in \Omega$'s are left implicit in the tree rules, which means that *every substitution instance of the tree rules hold*; sometimes — but rarely — we will indicate the substitution explicitly, like this,

$$\left(\dfrac{P \land Q \le R}{P \le Q{\to}R}\;{\to_2}\right)\left[\begin{smallmatrix}Q:=P\to\bot\\ R:=\bot\end{smallmatrix}\right] \quad\rightsquigarrow\quad \dfrac{P \land (P{\to}\bot) \le \bot}{P \le ((P{\to}\bot){\to}\bot)}\;{\to_2}$$

$$({\to_2})\left[\begin{smallmatrix}Q:=P\to\bot\\ R:=\bot\end{smallmatrix}\right] \quad\rightsquigarrow\quad \dfrac{P \land (P{\to}\bot) \le \bot}{P \le ((P{\to}\bot){\to}\bot)}\;{\to_2}\left[\begin{smallmatrix}Q:=P\to\bot\\ R:=\bot\end{smallmatrix}\right]$$

Usually we will only say '$\to_2$' instead of '$\to_2 \left[\begin{smallmatrix}Q:=P\to\bot\\ R:=\bot\end{smallmatrix}\right]$' at the right of a bar, and the task of discovering which substitution has been used is left to the reader.

The tree rules can be composed in a nice visual way. For example, this,

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \land Q \le P \land Q}\;\mathrm{id}}{P \land Q \le P}\;\land_1 \qquad P \le R}{P \land Q \le R}\;\mathrm{comp} \qquad \dfrac{\dfrac{\dfrac{}{P \land Q \le P \land Q}\;\mathrm{id}}{P \land Q \le Q}\;\land_2 \qquad Q \le S}{P \land Q \le S}\;\mathrm{comp}}{P \land Q \le R \land S}\;\land_3$$

"is" a proof for:

$$\forall P, Q, R, S \in \Omega.\ (P \le R) \land (Q \le S) \to ((P \land Q) \le (R \land S)).$$

8.1 Derived rules

Note: in this section we will ignore the operators '$\leftrightarrow$' and '$\neg$' in PC-structures; we will think that every '$P \leftrightarrow Q$' is as abbreviation for '$(P{\to}Q) \land (Q{\to}P)$' and every '$\neg P$' is an abbreviation for '$P{\to}\top$'.

We'll write $[\top_1], \ldots, [{\to_2}]$ for the "linear" versions of the rules in last section — for example, $[{\to_2}]$ is $(\forall P, Q, R \in \Omega.\ (P \land Q \le R) \to (P \le Q{\to}R))$ — and if $S = (r_1, \ldots, r_n)$ is a set of rules, each in tree form, then $[S] = [r_1] \land \ldots \land [r_n]$, and an "$S$-tree" is a proof in tree form that only uses rules that are in the set S.

Let $\mathsf{HA\text{-}ness}_1$, $\mathsf{HA\text{-}ness}_2$, $\mathsf{HA\text{-}ness}_3$, be these sets, with the rules from sec.8:

$$\begin{array}{rcl}
\mathsf{HA\text{-}ness}_1 &=& \{\mathrm{id}, \mathrm{comp}, \top_1, \bot_1, \land_3, \lor_3, {\to_2}\}, \\
\mathsf{HA\text{-}ness}_2 &=& \{\land_1, \land_2, \lor_1, \lor_2, {\to_1}\}, \\
\mathsf{HA\text{-}ness}_3 &=& \mathsf{HA\text{-}ness}_1 \cup \mathsf{HA\text{-}ness}_2
\end{array}$$

and let $\mathsf{HA\text{-}ness}_4$, $\mathsf{HA\text{-}ness}_5$ and $\mathsf{HA\text{-}ness}_7$ be these ones, where the new rules are the ones at the left column of fig.1:

$$\begin{array}{rcl}
\mathsf{HA\text{-}ness}_4 &=& \{\land_4, \land_5, \lor_4, \lor_5, \mathsf{MP}_0, \mathsf{MP}\} \\
\mathsf{HA\text{-}ness}_5 &=& \mathsf{HA\text{-}ness}_1 \cup \mathsf{HA\text{-}ness}_4 \\
\mathsf{HA\text{-}ness}_7 &=& \mathsf{HA\text{-}ness}_1 \cup \mathsf{HA\text{-}ness}_2 \cup \mathsf{HA\text{-}ness}_4
\end{array}$$

$$\frac{}{Q \wedge R \leq Q}\,\wedge_4 \quad := \quad \dfrac{\dfrac{}{Q \wedge R \leq Q \wedge R}\,\mathrm{id}\,[\,P{:=}Q{\wedge}R\,]}{Q \wedge R \leq Q}\,\wedge_1\,[\,P{:=}Q{\wedge}R\,]$$

$$\frac{}{Q \wedge R \leq R}\,\wedge_5 \quad := \quad \dfrac{\dfrac{}{Q \wedge R \leq Q \wedge R}\,\mathrm{id}\,[\,P{:=}Q{\wedge}R\,]}{Q \wedge R \leq R}\,\wedge_2\,[\,P{:=}Q{\wedge}R\,]$$

$$\frac{}{P \leq P \vee Q}\,\vee_4 \quad := \quad \dfrac{\dfrac{}{P \vee Q \leq P \vee Q}\,\mathrm{id}\,[\,P{:=}P{\vee}Q\,]}{P \leq P \vee Q}\,\vee_1\,[\,R{:=}P{\vee}Q\,]$$

$$\frac{}{Q \leq P \vee Q}\,\vee_5 \quad := \quad \dfrac{\dfrac{}{P \vee Q \leq P \vee Q}\,\mathrm{id}\,[\,P{:=}P{\vee}Q\,]}{Q \leq P \vee Q}\,\vee_2\,[\,R{:=}P{\vee}Q\,]$$

$$\frac{}{Q \wedge (Q \to R) \leq R}\,\mathsf{MP}_0 \quad := \quad \dfrac{\dfrac{}{Q \to R \leq Q \to R}\,\mathrm{id}}{(Q \to R) \wedge Q \leq R}\,\to_1$$

$$\dfrac{P \leq Q \quad P \leq Q \to R}{P \leq R}\,\mathsf{MP} \quad := \quad \dfrac{\dfrac{P \leq Q \quad P \leq Q \to R}{P \leq Q \wedge (Q \to R)} \qquad \dfrac{}{Q \wedge (Q \to R) \leq R}\,\mathsf{MP}_0}{P \leq R}\,\mathrm{comp}$$

Figure 1: Derived rules

$$\dfrac{P \leq Q \wedge R}{P \leq Q}\ \wedge_1 \quad := \quad \dfrac{P \leq Q \wedge R \quad \overline{Q \wedge R \leq Q}\ \wedge_4}{P \leq Q}\ \text{comp}$$

$$\dfrac{P \leq Q \wedge R}{P \leq R}\ \wedge_2 \quad := \quad \dfrac{P \leq Q \wedge R \quad \overline{Q \wedge R \leq R}\ \wedge_5}{P \leq R}\ \text{comp}$$

$$\dfrac{P \vee Q \leq R}{P \leq R}\ \vee_1 \quad := \quad \dfrac{\overline{P \leq P \vee Q}\ \vee_4 \quad P \vee Q \leq R}{P \leq R}\ \text{comp}$$

$$\dfrac{P \vee Q \leq R}{Q \leq R}\ \vee_2 \quad := \quad \dfrac{\overline{Q \leq P \vee Q}\ \vee_5 \quad P \vee Q \leq R}{Q \leq R}\ \text{comp}$$

$$\dfrac{P \leq Q \to R}{P \wedge Q \leq R}\ \to_1 \quad :=$$

$$\dfrac{\dfrac{\overline{P \wedge Q \leq Q}\ \wedge_5 \quad \dfrac{\overline{P \wedge Q \leq P}\ \wedge_4 \quad P \leq Q \to R}{P \wedge Q \leq Q \to R}\ \text{comp}}{P \wedge Q \leq Q \wedge (Q \to R)}\ \wedge_3 \quad \overline{Q \wedge (Q \to R) \leq R}\ \mathsf{MP}_0}{P \wedge Q \leq R}\ \text{comp}$$

Figure 2: Derived rules (2)

Note that the trees in the right of fig.1 are $\mathsf{HA\text{-}ness}_3$-trees.

Fig.1 can be interpreted in two ways. The first one is that it shows that

$$\begin{array}{rcl}
[\mathsf{HA\text{-}ness}_3] & \to & [\wedge_4], \\
[\mathsf{HA\text{-}ness}_3] & \to & [\wedge_5], \\
[\mathsf{HA\text{-}ness}_3] & \to & [\vee_4], \\
[\mathsf{HA\text{-}ness}_3] & \to & [\vee_5], \\
[\mathsf{HA\text{-}ness}_3] & \to & [\mathsf{MP}_0], \\
[\mathsf{HA\text{-}ness}_3] & \to & [\mathsf{MP}], \\
[\mathsf{HA\text{-}ness}_3] & \to & [\mathsf{HA\text{-}ness}_4], \\
[\mathsf{HA\text{-}ness}_3] & \to & [\mathsf{HA\text{-}ness}_7];
\end{array}$$

the second one is that it shows a way to replace occurrences of $\wedge_4$, $\wedge_5$, $\vee_4$, $\vee_5$, MP_0, MP. Take an $\mathsf{HA\text{-}ness}_7$-tree, T. Call it hypotheses $H_1, \ldots, H_n$, and its conclusion C, Replace each occurrence of $\wedge_4$, $\wedge_5$, $\vee_4$, $\vee_5$, MP_0, MP in T by the corresponding tree in the right side of fig.1. The result is a new tree, T', which is "equivalent" to T in the sense of having the same hypotheses and conclusion as T. So,

- every $\mathsf{HA\text{-}ness}_3$-tree is an $\mathsf{HA\text{-}ness}_7$-tree,
- every $\mathsf{HA\text{-}ness}_7$-tree is "equivalent" to an $\mathsf{HA\text{-}ness}_3$-tree.

We call this trick "derived rules" — the rules in $\mathsf{HA\text{-}ness}_4$ are "derived" from $\mathsf{HA\text{-}ness}_3$, and $\mathsf{HA\text{-}ness}_3$ and $\mathsf{HA\text{-}ness}_7$ are "equivalent" in the sense that they "prove the same things".

Now look at fig.2. It has the rules in $\mathsf{HA\text{-}ness}_2$ at the left, and $\mathsf{HA\text{-}ness}_5$-trees at the right; it shows that

$$\begin{array}{rcl}
[\mathsf{HA\text{-}ness}_5] & \to & [\wedge_1], \\
[\mathsf{HA\text{-}ness}_5] & \to & [\wedge_2], \\
[\mathsf{HA\text{-}ness}_5] & \to & [\vee_1], \\
[\mathsf{HA\text{-}ness}_5] & \to & [\vee_2], \\
[\mathsf{HA\text{-}ness}_5] & \to & [\to_2], \\
[\mathsf{HA\text{-}ness}_5] & \to & [\mathsf{HA\text{-}ness}_2], \\
[\mathsf{HA\text{-}ness}_5] & \to & [\mathsf{HA\text{-}ness}_7],
\end{array}$$

and it also shows how to take an $\mathsf{HA\text{-}ness}_7$-tree T and replace every occurrence of an $\mathsf{HA\text{-}ness}_4$-rule in it by an $\mathsf{HA\text{-}ness}_3$-tree, producing an $\mathsf{HA\text{-}ness}_3$-tree T' which is "equivalent" to T. This means that:

- every $\mathsf{HA\text{-}ness}_5$-tree is an $\mathsf{HA\text{-}ness}_7$-tree,
- every $\mathsf{HA\text{-}ness}_7$-tree is "equivalent" to an $\mathsf{HA\text{-}ness}_5$-tree,

and that $\mathsf{HA\text{-}ness}_3$, $\mathsf{HA\text{-}ness}_7$ and $\mathsf{HA\text{-}ness}_5$ are all "equivalent".

9 Topologies

The best way to connect ZHAs to several standard concepts is by seeing that ZHAs are topologies on certain finite sets — actually on 2-column acyclical graphs. This will be done here and in the next few sections.

A *topology* on a set X is a subset $\mathcal{S}$ of $\mathcal{P}(X)$ such that:
1) $\mathcal{S}$ contains X and $\varnothing$,
2) if $P, Q \in \mathcal{S}$ then $\mathcal{S}$ contains $P \cap Q$ and $P \cup Q$,
3) if $\mathcal{S}' \subset \mathcal{S}$ then $\mathcal{S}$ contains $\bigcup \mathcal{S}'$.

A *topological space* is a pair $(X, \mathcal{S})$ where X is a set and $\mathcal{S}$ is a topology on X.

When $(X, \mathcal{S})$ is a topological space and $U \in \mathcal{S}$ we say that U is *open* in $(X, \mathcal{S})$.

For example, let X be the ZSet $\begin{smallmatrix}\bullet&&\bullet\\&\bullet&\\\bullet&&\bullet\end{smallmatrix}$, and let's use the characteristic function notation from sec.1 to denote its subsets — we write $X = \begin{smallmatrix}1&&1\\&1&\\1&&1\end{smallmatrix}$ and $\varnothing = \begin{smallmatrix}0&&0\\&0&\\0&&0\end{smallmatrix}$ instead of $X = \begin{smallmatrix}\bullet&&\bullet\\&\bullet&\\\bullet&&\bullet\end{smallmatrix}$ and $\varnothing = \begin{smallmatrix}\cdot&&\cdot\\&\cdot&\\\cdot&&\cdot\end{smallmatrix}$.

If $\mathcal{S} = \left\{ \begin{smallmatrix}1&&0\\&0&\\0&&0\end{smallmatrix}, \begin{smallmatrix}0&&1\\&0&\\0&&0\end{smallmatrix}, \begin{smallmatrix}0&&0\\&1&\\0&&0\end{smallmatrix}, \begin{smallmatrix}0&&0\\&0&\\1&&0\end{smallmatrix}, \begin{smallmatrix}0&&0\\&0&\\0&&1\end{smallmatrix} \right\}$ then $\mathcal{S} \subset \mathcal{P}(X)$ but $\mathcal{S}$ fails all the conditions in 1, 2, 3 above:
1) $X = \begin{smallmatrix}1&&1\\&1&\\1&&1\end{smallmatrix} \notin \mathcal{S}$ and $\varnothing = \begin{smallmatrix}0&&0\\&0&\\0&&0\end{smallmatrix} \notin \mathcal{S}$
2) Let $P = \begin{smallmatrix}1&&0\\&0&\\0&&0\end{smallmatrix} \in \mathcal{S}$ and $Q = \begin{smallmatrix}0&&1\\&0&\\0&&0\end{smallmatrix} \in \mathcal{S}$. Then $P \cap Q = \begin{smallmatrix}0&&0\\&0&\\0&&0\end{smallmatrix} \notin \mathcal{S}$ and $P \cup Q = \begin{smallmatrix}1&&1\\&0&\\0&&0\end{smallmatrix} \notin \mathcal{S}$.
3) Let $\mathcal{S}' = \left\{ \begin{smallmatrix}0&&1\\&0&\\0&&0\end{smallmatrix}, \begin{smallmatrix}0&&0\\&1&\\0&&0\end{smallmatrix}, \begin{smallmatrix}0&&0\\&0&\\1&&0\end{smallmatrix} \right\} \subset \mathcal{S}$. Then $\bigcup \mathcal{S}' = \begin{smallmatrix}0&&1\\&0&\\0&&0\end{smallmatrix} \cup \begin{smallmatrix}0&&0\\&1&\\0&&0\end{smallmatrix} \cup \begin{smallmatrix}0&&0\\&0&\\1&&0\end{smallmatrix} = \begin{smallmatrix}0&&1\\&1&\\1&&0\end{smallmatrix} \notin \mathcal{S}$.

Now let $K = \begin{smallmatrix}&\bullet&\\\bullet&&\bullet\\&\bullet&\\&\bullet&\end{smallmatrix}$ and $\mathcal{S} = \left\{ \begin{smallmatrix}&0&\\0&&0\\&0&\\&0&\end{smallmatrix}, \begin{smallmatrix}&0&\\0&&0\\&0&\\&1&\end{smallmatrix}, \begin{smallmatrix}&0&\\0&&0\\&1&\\&1&\end{smallmatrix}, \begin{smallmatrix}&0&\\0&&1\\&1&\\&1&\end{smallmatrix}, \begin{smallmatrix}&0&\\1&&0\\&1&\\&1&\end{smallmatrix}, \begin{smallmatrix}&0&\\1&&1\\&1&\\&1&\end{smallmatrix}, \begin{smallmatrix}&1&\\1&&1\\&1&\\&1&\end{smallmatrix} \right\}$. In this case $(K, \mathcal{S})$ is a topological space.

Some sets have "default" topologies on them, denoted with '$\mathcal{O}$'. For example, $\mathbb{R}$ is often used to mean the topological space $(\mathbb{R}, \mathcal{O}(\mathbb{R}))$, where:

$$\mathcal{O}(\mathbb{R}) = \{ U \subset \mathbb{R} \mid U \text{ is a union of open intervals} \}.$$

We say that a subset $U \subset \mathbb{R}$ is "open in $\mathbb{R}$" ("in the default sense"; note that now we are saying just "open in $\mathbb{R}$", not "open in $(\mathbb{R}, \mathcal{O}(\mathbb{R}))$") when U is a union of open intervals, i.e., when $U \in \mathcal{O}(\mathbb{R})$; but note that $\mathcal{P}(\mathbb{R})$ and $\{\varnothing, \mathbb{R}\}$ are also topologies on $\mathbb{R}$, and:

$$\begin{array}{lll}
\{2,3,4\} \in \mathcal{P}(\mathbb{R}), & \text{so} & \{2,3,4\} \text{ is open in } (\mathbb{R}, \mathcal{P}(\mathbb{R})), \\
\{2,3,4\} \notin \mathcal{O}(\mathbb{R}), & \text{so} & \{2,3,4\} \text{ is not open in } (\mathbb{R}, \mathcal{O}(\mathbb{R})), \\
\{2,3,4\} \notin \{\varnothing, \mathbb{R}\}, & \text{so} & \{2,3,4\} \text{ is not open in } (\mathbb{R}, \{\varnothing, \mathbb{R}\});
\end{array}$$

when we say just "U is open in X", this means that:
1) $\mathcal{O}(X)$ is clear from the context, and
2) $U \in \mathcal{O}(X)$.

10 The default topology on a ZSet

Let's define a default topology $\mathcal{O}(D)$ for each ZSet D.

For each ZSet D we define $\mathcal{O}(D)$ as:

$$\mathcal{O}(D) := \{\, U \subset D \mid \forall((x,y),(x',y'))\in\mathsf{BPM}(D).\ ((x,y)\in U \to (x',y')\in U)\,\}$$

whose visual meaning is this. Turn D into a ZDAG by adding arrows for the black pawns moves (sec.2), and regard each subset $U \subset D$ as a board configuration in which the black pieces may move down to empty positions through the arrows. A subset U is "stable" when no moves are possible because all points of U "ahead" of a black piece are already occupied by black pieces; a subset U is "non-stable" when there is at least one arrow $((x,y),(x',y')) \in \mathsf{BPM}(D)$ in which (x,y) had a black piece and (x',y') is an empty position.

In our two notations for subsets (sec.1) a subset $U \subset D$ is unstable when it has an arrow like '$\bullet \to \cdot$' or '$1 \to 0$'; remember that black pawn moves arrows go down. A subset $U \subset D$ is stable when none of its '$\bullet$'s or '1's can move down to empty positions.

"Open" is the same as "stable". $\mathcal{O}(D)$ is the set of stable subsets of D.

Some examples:
$\begin{smallmatrix} &0& \\ 0&&1 \\ &0& \\ &0& \end{smallmatrix}$ is not open because it has a 1 above a 0,
$\mathcal{O}\left(\begin{smallmatrix} &\bullet& \\ \bullet&&\bullet \\ &\bullet& \\ &\bullet& \end{smallmatrix}\right) = \left\{ \begin{smallmatrix} &0& \\ 0&&0 \\ &0& \\ &0& \end{smallmatrix}, \begin{smallmatrix} &0& \\ 0&&0 \\ &0& \\ &1& \end{smallmatrix}, \begin{smallmatrix} &0& \\ 0&&0 \\ &1& \\ &1& \end{smallmatrix}, \begin{smallmatrix} &0& \\ 0&&1 \\ &1& \\ &1& \end{smallmatrix}, \begin{smallmatrix} &0& \\ 1&&0 \\ &1& \\ &1& \end{smallmatrix}, \begin{smallmatrix} &0& \\ 1&&1 \\ &1& \\ &1& \end{smallmatrix}, \begin{smallmatrix} &1& \\ 1&&1 \\ &1& \\ &1& \end{smallmatrix} \right\}$,
$\mathcal{O}\left(\begin{smallmatrix} &\bullet& \\ \bullet&&\bullet \\ \bullet&&\bullet \end{smallmatrix}\right) = \left\{ \begin{smallmatrix} &0& \\ 0&&0 \\ 0&&0 \end{smallmatrix}, \begin{smallmatrix} &0& \\ 0&&0 \\ 0&&1 \end{smallmatrix}, \begin{smallmatrix} &0& \\ 0&&0 \\ 1&&0 \end{smallmatrix}, \begin{smallmatrix} &0& \\ 0&&0 \\ 1&&1 \end{smallmatrix}, \begin{smallmatrix} &0& \\ 0&&1 \\ 0&&1 \end{smallmatrix}, \begin{smallmatrix} &0& \\ 0&&1 \\ 1&&1 \end{smallmatrix}, \begin{smallmatrix} &0& \\ 1&&0 \\ 1&&0 \end{smallmatrix}, \begin{smallmatrix} &0& \\ 1&&0 \\ 1&&1 \end{smallmatrix}, \begin{smallmatrix} &0& \\ 1&&1 \\ 1&&1 \end{smallmatrix}, \begin{smallmatrix} &1& \\ 1&&1 \\ 1&&1 \end{smallmatrix} \right\}$.

The definition of $\mathcal{O}(D)$ above can be generalized to any directed graph. If (A,R) is a directed graph, then $(A,\mathcal{O}_R(A))$ is a topological space if we define:

$$\mathcal{O}_R(A) := \{\, U \subseteq A \mid \forall(a,b) \in R.\ (a \in U \to b \in U)\,\}$$

The two definitions are related as this: $\mathcal{O}(D) = \mathcal{O}_{\mathsf{BPM}(D)}(D)$.

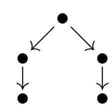
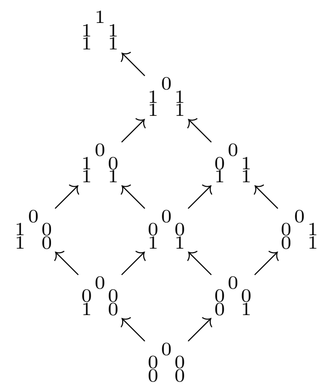
Note that we can see the arrows in $\mathsf{BPM}(D)$ or in R as *obligations* that open sets must obey; each arrow $a \to b$ says that every open set that contains a is forced to contain b too.

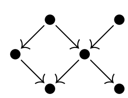
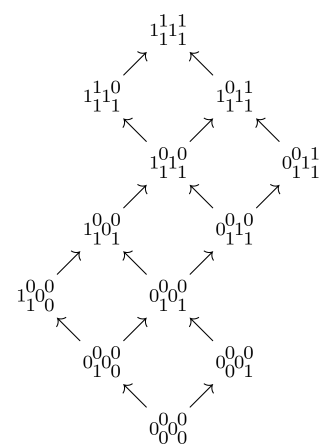
11 Topologies as partial orders

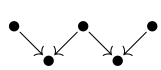
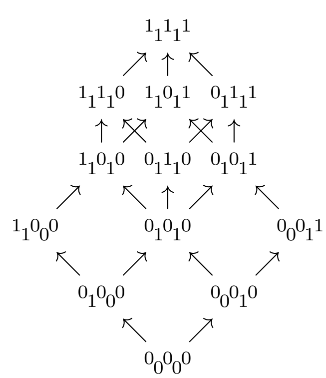
For any topological space $(X,\mathcal{O}(X))$ we can regard $\mathcal{O}(X)$ as a partial order, ordered by inclusion, with $\varnothing$ as its minimal element and X as its maximal element; we denote that partial order by $(\mathcal{O}(X),\subseteq)$.

Take any ZSet D. The partial order $(\mathcal{O}(D),\subseteq)$ will *sometimes* be a ZHA when we draw it with $\varnothing$ at the bottom, D at the top, and inclusions pointing up, as can be seen in the three figures below; when $D = \begin{smallmatrix} &\bullet& \\ \bullet&&\bullet \\ \bullet&&\bullet \end{smallmatrix}$ or $D = \begin{smallmatrix} &\bullet&&\bullet \\ \bullet&&\bullet& \\ &\bullet&&\bullet \end{smallmatrix}$ the result is a ZHA, but when $D = \begin{smallmatrix} \bullet&&\bullet&&\bullet \\ &\bullet&&\bullet& \end{smallmatrix}$ it not.

Let's write "$V \subset_1 U$" for "$V \subseteq U$ and V and U differ in exactly one point". When D is a ZSet the relation $\subseteq$ on $\mathcal{O}(D)$ is the transitive-reflexive closure of $\subset_1$, and $(\mathcal{O}(D), \subset_1)$ is easier to draw than $(\mathcal{O}(D), \subseteq)$.

$(H, \mathsf{BPM}(H)) =$  $(\mathcal{O}(H), \subset_1) =$ 

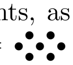
$(G, \mathsf{BPM}(G)) =$  $(\mathcal{O}(G), \subset_1) =$ 

$(W, \mathsf{BPM}(W)) =$  $(\mathcal{O}(W), \subset_1) =$ 

We can formalize a "way to draw $\mathcal{O}(D)$ as a ZHA" (or "...as a ZDAG") as a bijective function f from a ZHA (or from a ZSet) S to $\mathcal{O}(D)$ that creates a perfect correspondence between the white moves in S and the "$V \subset_1 U$-arrows"; more precisely, an f such that this holds: if $a, b \in S$ then $(a, b) \in \mathsf{WPM}(S)$ iff $f(a) \subset_1 f(b)$.

Note that the *number of elements* in an open set corresponds to the *height* where it is drawn; if $f : S \to \mathcal{O}(D)$ is a way to draw $\mathcal{O}(D)$ as a ZHA or a ZDAG then f takes points of the form $(\_\_, y)$ to open sets with y elements, and if $f : S \to \mathcal{O}(D)$ is a way to draw $\mathcal{O}(D)$ as a ZHA (not a ZDAG!) then we also have that $f((0, 0)) = \varnothing \in \mathcal{O}(D)$.

The diagram for $(\mathcal{O}(H), \subset_1)$ above is a way to draw $\mathcal{O}(H)$ as a ZHA.
The diagram for $(\mathcal{O}(G), \subset_1)$ above is a way to draw $\mathcal{O}(H)$ as a ZHA.
The diagram for $(\mathcal{O}(W), \subset_1)$ above is *not* a way to draw $\mathcal{O}(W)$ as a ZSet. Look at ${}^0{}_1{}^1{}_1{}^0$ and ${}^1{}_1{}^0{}_1{}^1$ in the middle of the cube formed by all open sets of the form ${}^a{}_1{}^b{}_1{}^c$. We don't have ${}^0{}_1{}^1{}_1{}^0 \subset_1 {}^1{}_1{}^0{}_1{}^1$, but we do have a white pawn move (not draw in the diagram!) from $f^{-1}({}^0{}_1{}^1{}_1{}^0)$ to $f^{-1}({}^1{}_1{}^0{}_1{}^1)$. We say that a ZSet is *thin* when it doesn't have three independent points.

Every time that a ZSet D has three independent points, as in W, we will have a situation like in $(\mathcal{O}(W), \subset_1)$; for example, if $B =$  then the open sets of B of the form $a{}^0_1 b{}^0_1 c$ form a cube.

12 2-Column Graphs

Note: in this section we will manipulate objects with names like $1\_, 2\_, 3\_, \ldots,$ $\_1, \_2, \_3, \ldots$; here are two good ways to formalize them:

$$\begin{array}{cc} \vdots & \vdots \\ 4\_ = (0,4) & \_4 = (1,4) \\ 3\_ = (0,3) & \_3 = (1,3) \\ 2\_ = (0,2) & \_2 = (1,2) \\ 1\_ = (0,1) & \_1 = (1,1) \end{array} \quad \text{or} \quad \begin{array}{cc} \vdots & \vdots \\ 4\_ = \texttt{"4\_"} & \_4 = \texttt{"\_4"} \\ 3\_ = \texttt{"3\_"} & \_3 = \texttt{"\_3"} \\ 2\_ = \texttt{"2\_"} & \_2 = \texttt{"\_2"} \\ 1\_ = \texttt{"1\_"} & \_1 = \texttt{"\_1"} \end{array},$$

where `"1_"`, `"_2"`, `""`, `"Hello!"`, etc are strings.

We define:

$$\begin{array}{rcl} LC(l) & := & \{1\_, 2\_, \ldots, l\_\} \\ RC(l) & := & \{\_1, \_2, \ldots, \_r\}, \end{array}$$

which generate a "left column" of height l and a "right column" of height r.

A *description for a 2-column graph* (a "D2CG") is a 4-tuple (l, r, R, L), where $l, r \in \mathbb{N}$, $R \subset \mathsf{LC}(l) \times \mathsf{RC}(r)$, $L \subset \mathsf{RC}(r) \times \mathsf{LC}(l)$; l is the height of the left column, r is the height of the right column, and R and L are set of intercolumn arrows (going right and left respectively).

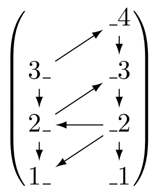
The operation $\mathsf{2CG}$ (in a sans-serif font) generates a directed graph from a D2CG:

$$\mathsf{2CG}(l,r,R,L) \quad := \quad \left(\mathsf{LC}(l)\cup\mathsf{RC}(r), \left\{ \begin{array}{c} \{l\_{\to}(l{-}1)\_,\ \ldots,\ 2\_{\to}1\_\}\cup \\ \{\_r{\to}\_(r{-}1),\ \ldots,\ \_2{\to}\_1\}\cup \\ R\cup L \end{array} \right\}\right)$$

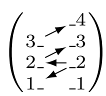
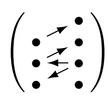
For example,

$$\mathsf{2CG}(3,4,\left\{ \begin{array}{c} 3\_{\to}\_4, \\ 2\_{\to}\_3 \end{array} \right\}, \left\{ \begin{array}{c} 2\_{\leftarrow}\_2, \\ 1\_{\leftarrow}\_2 \end{array} \right\}) \quad := \quad \left(\left\{ \begin{array}{c} 3\_,\ 2\_,\ 1\_, \\ \_4,\ \_3,\ \_2,\ \_1 \end{array} \right\}, \left\{ \begin{array}{c} 3\_{\to}2\_,\ 2\_{\to}1\_, \\ \_4{\to}\_3,\ \_3{\to}\_2,\ \_2{\to}\_1, \\ 3\_{\to}\_4,\ 2\_{\to}\_3, \\ 2\_{\leftarrow}\_2,\ 1\_{\leftarrow}\_2 \end{array} \right\} \right)$$

which is:

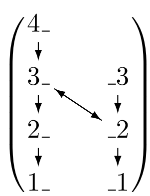
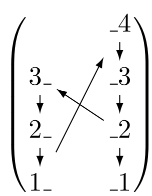


we will usually draw that more compactly, by omitting the intracolumn (i.e., vertical) arrows:

 or .

A *2-column graph* (a "2CG") is a directed graph that is of the form $\mathsf{2CG}(l,r,R,L)$. We will often say $(P,A) = \mathsf{2CG}(l,r,R,L)$, where the P stand for "points" and A for "arrows".

A *2-column acyclical graph* (a "2CAG") is a 2CG that doesn't have cycles. If L has an arrow that is the opposite of an arrow in R, this generates a cycle of length 2; if R has an arrow $l\_{\to}\_r'$ and L has an arrow $l'\_{\leftarrow}\_r$, where $l \le l'$ and $r \le r'$, this generates a cycle that can have a more complex shape — a triangle or a bowtie. For example,

 and .

13 Topologies on 2CGs

In this section we will see that ZHAs are topologies on 2CAGs.

Let $(P,A) = \mathsf{2CG}(l,r,R,L)$ be a 2-column graph.
What happens if we look at the open sets of (P,A), i.e., at $\mathcal{O}_A(P)$? Two things:

1) every open set $U \in \mathcal{O}_A(P)$ is of the form $\mathsf{LC}(a)\cup\mathsf{LC}(b)$,

2) arrows in R and L forbids some '$\mathsf{LC}(a)\cup\mathsf{LC}(b)$'s from being open sets.

In order to understand that we need to introduce some notations for "piles".

The function

$$\mathsf{pile}(\langle a,b\rangle) := \mathsf{LC}(a) \cup \mathsf{LC}(b)$$

converts an element $\langle a,b\rangle \in \mathbb{LR}$ into a pile of elements in the left column of height a and a pile of elements in the right column of height b. We will write subsets of the points of a 2CG using a positional notation with arrows. So, for example, if $(P,A) = \mathsf{2CG}(3,4,\{2\_{\to}\_3\},\{2\_{\leftarrow}\_2\})$ then

$$(P,A) = \begin{pmatrix} & & \_4 \\ 3\_ & \nearrow & \_3 \\ 2\_ & \leftarrow & \_2 \\ 1\_ & & \_1 \end{pmatrix} \qquad \text{and} \qquad \mathsf{pile}(21) = \begin{pmatrix} & & 0 \\ 0 & \nearrow & 0 \\ 1 & \leftarrow & 0 \\ 1 & & 1 \end{pmatrix} \quad \text{(as a subset of } P\text{)}.$$

Note that $\mathsf{pile}(21)$ is not open in $(P, \mathcal{O}_A(P))$, as it has an arrow '$1 \to 0$'. In fact, the presence of the arrow $\{2\_{\to}\_3\}$ in A means that all piles of the form

$$\begin{pmatrix} & & 0 \\ ? & \nearrow & 0 \\ 1 & \leftarrow & ? \\ 1 & & ? \end{pmatrix}$$

are not open, the presence of the arrow $\{2\_{\leftarrow}\_2\}$ means that the piles of the form

$$\begin{pmatrix} & & ? \\ 0 & \nearrow & ? \\ 0 & \leftarrow & 1 \\ ? & & 1 \end{pmatrix}$$

are not open sets.

The effect of these prohibitions can be expressed nicely with implications. If

$$(P,A) = \mathsf{2CG}(l, r, \left\{ \begin{smallmatrix} c\_{\to}\_d, \\ e\_{\to}\_f \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} g\_{\leftarrow}\_h, \\ i\_{\leftarrow}\_j \end{smallmatrix} \right\})$$

then

$$\mathcal{O}_A(P) = \{\, \mathsf{pile}(ab) \mid a \in \{0,\dots,l\}, b \in \{0,\dots,r\}, \begin{pmatrix} a{\geq}c{\to}b{\geq}d \,\land \\ a{\geq}e{\to}b{\geq}f \,\land \\ a{\geq}g{\leftarrow}b{\geq}h \,\land \\ a{\geq}i{\leftarrow}b{\geq}j \end{pmatrix} \}$$

Let's use a shorter notation for comparing 2CGs and their topologies:

$$\mathcal{O}\begin{pmatrix} & \_5 \\ & \downarrow \\ 4\_ & \_4 \\ \downarrow & \downarrow \\ 3\_ & \_3 \\ \downarrow & \downarrow \\ 2\_ & \_2 \\ \downarrow & \downarrow \\ 1\_ & \_1 \end{pmatrix} = \begin{array}{ccccccc} & & & 45 & & & \\ & & 44 & & 35 & & \\ & 43 & & 34 & & 25 & \\ 42 & & 33 & & 24 & & \\ & 32 & & 23 & & 14 & \\ & & 22 & & 13 & & \\ & 21 & & 12 & & 03 & \\ 20 & & 11 & & 02 & & \\ & 10 & & 01 & & & \\ & & 00 & & & & \end{array}$$

the arrows in R and L and the values of l and r are easy to read from the 2CG at the left, and we omit the 'pile's at the right.

In a situation like the above we say that the 2CG in the '$\mathcal{O}(\dots)$' *generates* the ZHA at the right. There is an easy way to draw the ZHA generated by a

2CG, and a simple way to find the 2CG that generates a given ZHA. To describe them we need two new concepts.

If (A,R) is a directed graph and $S \subset A$ then $\downarrow S$ is the smallest open set in $\mathcal{O}_R(A)$ that contains S. If (A,R) is a ZDAG with black pawns moves as its arrows, think that the '1's in S are painted with a black paint that is very wet, and that that paint flows into the '0's below; the result of $\downarrow S$ is what we get when all the '0's below '1's get painted black. For example: $\downarrow 0^{0}_{0}0^{1}_{0} = 0^{0}_{1}1^{1}_{1}$. When (P,A) is a 2CG and $S \subseteq P$, we have to think that the paint flows along the arrows, even if some of the intercolumn arrows point upward. For example:

$$\downarrow \begin{pmatrix} & & 0 \\ 0 & \nearrow & 0 \\ 0 & \leftarrow & 1 \\ 1 & & 0 \end{pmatrix} = \begin{pmatrix} & & 0 \\ 0 & \nearrow & 1 \\ 1 & \leftarrow & 1 \\ 1 & & 1 \end{pmatrix}$$

and if S consists of a single point, $S = \{s\}$, then we may write $\downarrow s$ instead of $\downarrow\{s\} = \downarrow S$. In the 2CG above, we have (omitting the 'pile's):

$$\downarrow \_2 \;=\; \downarrow\{\_2\} = \downarrow \begin{pmatrix} & & 0 \\ 0 & \nearrow & 0 \\ 0 & \leftarrow & 1 \\ 0 & & 0 \end{pmatrix} = \begin{pmatrix} & & 0 \\ 0 & \nearrow & 1 \\ 1 & \leftarrow & 1 \\ 1 & & 1 \end{pmatrix} = 23, \quad \text{and} \quad \begin{array}{ll} & \downarrow\_4{=}24, \\ \downarrow 3\_{=}33, & \downarrow\_3{=}23, \\ \downarrow 2\_{=}23, & \downarrow\_2{=}23, \\ \downarrow 1\_{=}10, & \downarrow\_1{=}01, \end{array}$$

The second concept is this: the "generators" of a ZDAG D with white pawns moves as its arrows — or of a ZHA D — are the points of D that have exactly one white pawn move pointing *to* them (not *going out of* them).

If (P,A) is a 2CAG, then $\mathcal{O}_A(P)$ is a ZHA, and '$\downarrow$' is a bijection from P to the generators of $\mathcal{O}_A(P)$; for example:

$$\mathcal{O}\begin{pmatrix} & \_5 \\ & \downarrow \\ 4\_ & \_4 \\ \downarrow & \downarrow \\ 3\_ & \_3 \\ \downarrow & \downarrow \\ 2\_ & \_2 \\ \downarrow & \downarrow \\ 1\_ & \_1 \end{pmatrix} = \begin{array}{ccccccc} &&& 45 &&& \\ && 44 && 35 && \\ & 43 && 34 && 25 & \\ 42 && 33 && 24 && \\ & 32 && 23 && 14 & \\ && 22 && 13 && \\ & 21 && 12 && 03 & \\ 20 && 11 && 02 && \\ & 10 && 01 &&& \\ && 00 &&&& \end{array} \qquad \begin{array}{ccccccc} &&& \cdot &&& \\ && \cdot && \cdot && \\ & \cdot && \cdot && \_5 & \\ 4\_ && \cdot && \cdot && \\ & 3\_ && \cdot && \_4 & \\ && \cdot && \cdot && \\ & \cdot && \cdot && \_3 & \\ 2\_ && \cdot && \_2 && \\ & 1\_ && \_1 &&& \\ && \cdot &&&& \end{array}$$

but if (P,A) is a 2CG with cycles, then $\mathcal{O}_A(P)$ is not a ZHA because each cycle generates a "gap" that disconnects the points of $\mathcal{O}_A(P)$. We just saw an example of a 2CG with a cycle in which $\downarrow 2\_ = 23 = \downarrow\_3 = \downarrow\_2$; look at its topology:

$$\mathcal{O}\begin{pmatrix} & & \_4 \\ & & \downarrow \\ 3\_ & & \_3 \\ \downarrow & \nearrow & \downarrow \\ 2\_ & \leftarrow & \_2 \\ \downarrow & & \downarrow \\ 1\_ & & \_1 \end{pmatrix} = \begin{array}{ccccc} && 34 && \\ & 33 && 24 & \\ && 23 && \\ &&&& \\ && 11 && \\ & 10 && 01 & \\ && 00 && \end{array}$$

14 Converting bewteen ZHAs and 2CAGs

Let's now see how to start from a 2CAG and produce its topology (a ZHA) quickly, and how to find quickly the 2CAG that generates a given ZHA.

From 2CAGs to ZHAs. Let $(P, A) = \mathsf{2CG}(l, r, R, L)$ be a 2CAG, and call the ZHA generated by it H. Then the top point of H is lr, its bottom point is 00. Let $C := \{00, \downarrow 1_{\_}, \downarrow 2_{\_}, \ldots, \downarrow l_{\_}, lr\}$; then C has some of the points of the left wall (sec.4) of H, but usually not all. To "complete" C, apply this operation repeatedly: if $ab \in C$ and $ab \neq lr$, then test if either $(a+1)b$ or $a(b+1)$ are in C; if none of them are, add $a(b+1)$, which is northeast of ab. When there is nothing else to add, then C is the whole of the left wall of H. For the right wall, start with $D := \{00, \downarrow {}_{\_}1, \downarrow {}_{\_}2, \ldots, \downarrow {}_{\_}r, lr\}$, and for each $ab \in C$ with $ab \neq lr$, test if either $(a+1)b$ or $a(b+1)$ are in D; if none of them are, add $(a+1)b$, which is northwest of ab. When there is nothing else to add, then D is the whole of the right wall of H.

In the acyclic example of the last section this yields:

$$
\begin{array}{rcl}
C & = & \{00, \downarrow 1_{\_}, \downarrow 2_{\_}, \downarrow 3_{\_}, \downarrow 4_{\_}, lr\} \\
 & = & \{00, 10, 20, 32, 42, 45\} \\
 & \rightsquigarrow & \{00, 10, 20, 21, 22, 32, 42, 43, 44, 45\}, \\
D & = & \{00, \downarrow {}_{\_}1, \downarrow {}_{\_}2, \downarrow {}_{\_}3, \downarrow {}_{\_}4, \downarrow {}_{\_}5, lr\} \\
 & = & \{00, 01, 02, 03, 14, 25, 45\} \\
 & \rightsquigarrow & \{00, 01, 02, 03, 13, 14, 24, 25, 35, 45\}.
\end{array}
$$

and the ZHA is everything between the "left wall" C and the "right wall" D.

From ZHAs to 2CAGs. Let H be a ZHA and let lr be its top point. Form the sequence of its left wall generators (the generators of H in which the arrow pointing to them points northwest) and the sequence of its right wall generators (the generators of H in which the arrow pointing to them points northeast). Look at where there are "gaps" in these sequences; each gap in the left wall generators becomes an intercolumn arrow going right, and each gap in the right wall generators becomes an intercolun arrow going left. In the acyclic example of the last section, this yields:

$$
\begin{array}{ll}
 & {}_{\_}5 = 25 \\
 & \quad \text{(gap becomes } 2_{\_} \leftarrow {}_{\_}5\text{)} \\
4_{\_} = 42 & {}_{\_}4 = 14 \\
\quad \text{(no gap)} & \quad \text{(gap becomes } 1_{\_} \leftarrow {}_{\_}4\text{)} \\
3_{\_} = 32 & {}_{\_}3 = 03 \\
\quad \text{(gap becomes } 3_{\_} \rightarrow {}_{\_}2\text{)} & \quad \text{(no gap)} \\
2_{\_} = 20 & {}_{\_}2 = 02 \\
\quad \text{(no gap)} & \quad \text{(no gap)} \\
1_{\_} = 10 & {}_{\_}1 = 01
\end{array}
$$

We know l and r from the top point of the ZHA, and from the gaps we get R

and L; the 2CAG that generates this ZHA is:

$$(4, 5, \left\{3_\text{-}{\rightarrow}_\text{-}2\right\}, \left\{\begin{matrix} 2_\text{-}{\leftarrow}_\text{-}5, \\ 1_\text{-}{\leftarrow}_\text{-}4 \end{matrix}\right\}).$$

15 Piccs and slashings

A picc ("partition into contiguous classes") of an interval $I = \{0, \dots, n\}$ is a partition P of I that obeys this condition ("picc-ness"):

$$\forall a, b, c \in \{0, \dots, n\}.\ (a < b < c\ \ \&\ \ a \sim_P c) \to (a \sim_P b \sim_P c).$$

So $P = \{\{0\}, \{1, 2, 3\}, \{4, 5\}\}$ is a picc of $\{0, \dots, 5\}$, and $P' = \{\{0\}, \{1, 2, 4, 5\}, \{3\}\}$ is a partition of $\{0, \dots, 5\}$ that is not a picc.

A short notation for piccs is this:

$$0|123|45 \equiv \{\{0\}, \{1, 2, 3\}, \{4, 5\}\}$$

we list all digits in the "interval" in order, and we put bars to indicate where we change from one equivalence class to another.

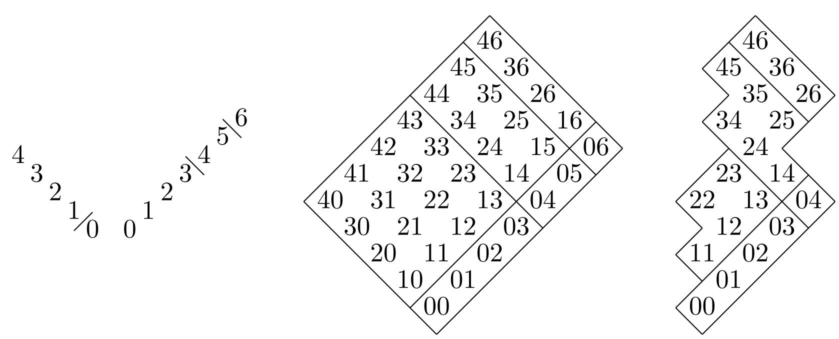
Let's define a notation for "intervals" in $\mathbb{LR}$,

$$[ab, cd] := [\langle a, b\rangle, \langle c, d\rangle] := \{\, \langle l, r\rangle \in \mathbb{LR} \mid a \le l \le c \& b \le r \le d \,\},$$

Note that it can be adapted to define "intervals" in a ZHAs H:

$$\begin{array}{rcl} [ab, cd] \cap H & := & \{\, \langle l, r\rangle \in \mathbb{LR} \mid a \le l \le c \& b \le r \le d \,\} \cap H \\ & = & \{\, \langle l, r\rangle \in H \mid a \le l \le c \& b \le r \le d \,\}. \end{array}$$

A *slashing* S on a ZHA H with top element ab is a pair of piccs, $S = (L, R)$, where L is a picc on $\{0, \dots, a\}$ and R is a picc on $\{0, \dots, b\}$; for example, $S = (4321/0,\ 0123\backslash 45\backslash 6)$ is a slashing on $[00, 46]$. We write the bars in L as '/'s and the bars in R as '\' as a reminder that they are to be interpreted as northeast and northwest "cuts" respectively; $S = (4321/0,\ 0123\backslash 45\backslash 6)$ is interpreted as the diagram at the left below, and it "slashes" $[00, 46]$ and the ZHA at the right below as:



A slashing $S = (L, R)$ on a ZHA H with top element ab induces an equivalence relation '$\sim_S$' on H that works like this: $\langle c, d\rangle \sim_S \langle e, f\rangle$ iff $c \sim_L e$ and $d \sim_R f$. We write

$$\begin{array}{rcl} [c]_L & := & \{\, e \in \{0, \dots, a\} \mid c \sim_L a \,\} \\ [d]_R & := & \{\, f \in \{0, \dots, b\} \mid d \sim_L f \,\} \\ [cd]_S & := & \{\, ef \in H \mid cd \sim_S ef \,\} \end{array}$$

for the equivalence classes, and note that

$$\begin{array}{lrcl} \text{if} & [c]_L & = & \{c', \ldots, c''\} \\ \text{and} & [d]_L & = & \{d', \ldots, d''\} \\ \text{then} & [cd]_S & = & [c'd', c''d''] \cap H; \end{array}$$

for example, in the ZHA at the right at the example above we have:

$$\begin{array}{rcl} [1]_L & = & \{1,2,3,4\}, \\ [2]_R & = & \{0,1,2,3\}, \\ [12]_S & = & [10,43] \cap H \;=\; \{11,12,13,22,23\}. \end{array}$$

We say that a slashing S on a ZHA H partitions H into *slash-regions*; later (sec.21) we will see that a J-operator J also partitions H, and we will refer to its equivalence classes as *J-regions*.

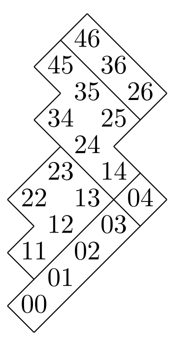
Slash-regions are intervals, but note that neither 10 or 43 belong to the slash-region $[12]_S = [10,43] \cap H$ above.

A *slash-partition* is a partition on a ZHA induced by a slashing, and a *slash-equivalence* is an equivalence relation on a ZHA induced by a slashing. Formally, a slash-partition on H is a set of subsets of H, and a slash-equivalence is subset of $H \times H$, but it is so easy to convert between partitions and equivalence relations that we will often use both terms interchangeably. Our visual representation for slash-partitions and slash-equivalences on a ZHA H will be the same: H slashed by diagonal cuts.

16 From slash-partitions back to slashings

We saw how to go from a slashing $S = (L, R)$ on H to an equivalence relation $\sim_S$ on H; let's see now how to recover L and R from $\sim_S$.

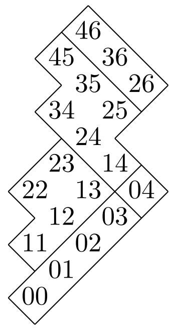
Let LW_H be the left wall of H, and RW_H the right wall of H. For example,

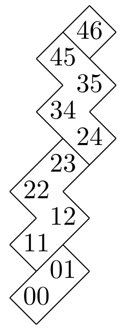
$H =$ 

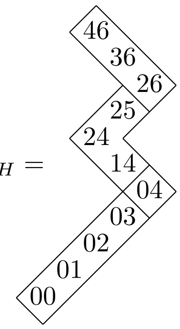
$$\begin{array}{l} LW_H = \{00,01,11,12,22,23,24,34,35,45,46\} \\ RW_H = \{00,01,02,03,04,14,24,25,26,36,46\} \end{array}$$

To recover the picc L — which is a picc on $\{0,1,2,3,4\}$ — we need to find where we change from an L-equivalence class to another when we go from one digit to the next; and to recover the picc R — which is a picc on $\{0,1,2,3,4,5,6\}$ — we need to find where we change from an R-equivalence class to another when we go from one digit to the next.

We can recover L and R by walking LW_H (or RW_H) from bottom to top in a series of white pawns moves, and checking when we change from one S-equivalence class to another. Northwest moves give information about L, and northeast moves give information about R. Look at the example below, in which we walk on RW_H:

$H =$ 

$LW_H =$ 

$RW_H =$ 

$$
\begin{array}{ll}
 & {}_{25}\nearrow^{26} : 25 \not\sim_S 26 \;\Rightarrow\; 5 \not\sim_R 6 \;\Rightarrow\; 5\backslash 6 \\
 & {}_{24}\nearrow^{25} : 24 \sim_S 25 \;\Rightarrow\; 4 \sim_R 5 \;\Rightarrow\; 45 \\
{}^{46}\nwarrow_{36} : 36 \sim_S 46 \;\Rightarrow\; 3 \sim_L 4 \;\Rightarrow\; 43 & {}_{03}\nearrow^{04} : 03 \not\sim_S 04 \;\Rightarrow\; 3 \not\sim_R 4 \;\Rightarrow\; 3\backslash 4 \\
{}^{36}\nwarrow_{26} : 26 \sim_S 36 \;\Rightarrow\; 2 \sim_L 3 \;\Rightarrow\; 32 & {}_{02}\nearrow^{03} : 02 \sim_S 03 \;\Rightarrow\; 2 \sim_R 3 \;\Rightarrow\; 23 \\
{}^{24}\nwarrow_{14} : 14 \sim_S 24 \;\Rightarrow\; 1 \sim_L 2 \;\Rightarrow\; 21 & {}_{01}\nearrow^{02} : 01 \sim_S 02 \;\Rightarrow\; 1 \sim_R 2 \;\Rightarrow\; 12 \\
{}^{14}\nwarrow_{04} : 04 \not\sim_S 14 \;\Rightarrow\; 0 \not\sim_L 1 \;\Rightarrow\; 1/0 & {}_{00}\nearrow^{01} : 00 \sim_S 01 \;\Rightarrow\; 0 \sim_R 1 \;\Rightarrow\; 01
\end{array}
$$

$$(L, R) \;=\; (4321/0, 0123\backslash 45\backslash 6)$$

17 Slash-regions have maximal elements

...here is how our argument will work, in a particular case:

$$
\begin{array}{rcl}
[1]_L &=& \{1, 2, 3, 4\}, \\
[2]_R &=& \{0, 1, 2, 3\}, \\
I &=& [10, 43], \\
[12]_S &=& I \cap H \;=\; \{11, 12, 13, 22, 23\}.
\end{array}
$$

$$
\underbrace{\underbrace{\underbrace{(((\underbrace{11 \vee 12}_{=12\in I}) \vee 13)}_{=13\in I} \vee 22)}_{=23\in I} \vee 23}_{=23\in I}
\qquad
\underbrace{\underbrace{\underbrace{(((\underbrace{11 \vee 12}_{=12\in H}) \vee 13)}_{=13\in H} \vee 22)}_{=23\in H} \vee 23}_{=23\in H}
$$

$$\bigvee [12]_S = \bigvee \{11, 12, 13, 22, 23\} = 11 \vee 12 \vee 13 \vee 22 \vee 23 \in I \cap H$$

$$11 \leq \bigvee[12]_S, \ 12 \leq \bigvee[12]_S, \ \ldots, \ 23 \leq \bigvee[12]_S$$

We have $[12]_S = I \cap H$, and $\bigvee[12]_S$ belongs to $I \cap H$ and is greter-or-equal than all elements of $I \cap H$, so $\bigvee[12]_S$ is the maximal element of $[12]_S$.

Here is how we can do that in the general case. Let $S = (L, R)$ be a slashing on a ZHA H. Let P be a point of H. The equivalence class $[P]_S$ is a finite set $\{P_1, \ldots, P_n\}$, and we know that $[P]_S = H \cap I$ for some interval I. Look at the elements P_1, $P_1 \vee P_2$, $(P_1 \vee P_2) \vee P_3$, ..., $((P_1 \vee P_2) \vee \ldots) \vee P_n$ We can see that all of them belong to both H and I, so we conclude that $\bigvee[P]_S = ((P_1 \vee P_2) \vee \ldots) \vee P_n$ belongs to $H \cap I$, and it is easy to see that it is greater-or-equal that all elements in $H \cap I$, so it is the maximal element of $H \cap I$.

A similar argument shows that $\bigwedge[P]_S = ((P_1 \wedge P_2) \wedge \ldots) \wedge P_n$ is the smallest element of $[P]_S$.

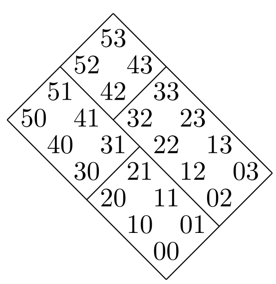
The same argument shows that if C is any non-empty set of the form $I \cap H$, where I is an interval, then $\bigvee C \in C$, $\bigwedge C \in C$, $[\bigwedge C, \bigvee C] \cap H = C$.

Remember that an *interval* in a ZHA H is any set of the form $[P, Q] \cap H$. Let's introduce a new definition: a *closed interval* in a ZHA H is a non-empty set $C \subset H$, with $\bigvee C \in C$, $\bigwedge C \in C$, $[\bigwedge C, \bigvee C] \cap H = C$; informally, a closed interval in a ZHA has a lowest and highest element, and it "is" everything between them.

18 Cuts stopping midway

We saw in the last section that every slash-region is a closed interval. A *partition into closed intervals* of a ZHA H is, as its name says, a partition of H whose equivalence classes are all closed intervals in H.

Some partitions into closed intervals of a ZHA are not slashings — for example, take the partition P with these equivalence classes:



Here is an easy way to prove formally that the partition above does not come from a slashing $S = (L, R)$. We will adapt the idea from sec.16, where we recovered L and R from northwest and northeast steps.

$$\begin{array}{ccccc}
\underbrace{21 \sim_P 31}_{\text{false}} & \leftrightarrow & \underbrace{2 \sim_L 3}_{=(} & \leftrightarrow & \underbrace{22 \sim_P 32}_{\text{true}} \\
\underbrace{31 \sim_P 41}_{\text{true}} & \leftrightarrow & \underbrace{3 \sim_L 4}_{=(} & \leftrightarrow & \underbrace{32 \sim_P 42}_{\text{false}}
\end{array}$$

The problem is that the figure above has "cuts stopping midway"... if its cuts all crossed the ZHA all the way through, we would have this for L and northeast cuts,

$$\begin{array}{ccccccccc}
0 \sim_L 1 & \leftrightarrow & 00 \sim_P 10 & \leftrightarrow & 01 \sim_P 11 & \leftrightarrow & 02 \sim_P 12 & \leftrightarrow & 03 \sim_P 13 \\
1 \sim_L 2 & \leftrightarrow & 10 \sim_P 20 & \leftrightarrow & 11 \sim_P 21 & \leftrightarrow & 12 \sim_P 22 & \leftrightarrow & 13 \sim_P 23 \\
2 \sim_L 3 & \leftrightarrow & 20 \sim_P 30 & \leftrightarrow & 21 \sim_P 31 & \leftrightarrow & 22 \sim_P 32 & \leftrightarrow & 23 \sim_P 33 \\
3 \sim_L 4 & \leftrightarrow & 30 \sim_P 40 & \leftrightarrow & 31 \sim_P 41 & \leftrightarrow & 32 \sim_P 42 & \leftrightarrow & 33 \sim_P 43 \\
4 \sim_L 5 & \leftrightarrow & 40 \sim_P 50 & \leftrightarrow & 41 \sim_P 51 & \leftrightarrow & 42 \sim_P 52 & \leftrightarrow & 43 \sim_P 53 \\
5 \sim_L 6 & \leftrightarrow & 50 \sim_P 60 & \leftrightarrow & 51 \sim_P 61 & \leftrightarrow & 52 \sim_P 62 & \leftrightarrow & 53 \sim_P 63
\end{array}$$

and something similar for R and northwest cuts.

Formally, a partition P on H has an "L-cut between c and c^+ stopping midway" if $cd \sim_P c^+d \not\leftrightarrow cd \sim_P c^+d$ for some d, and it has an "R-cut between d and d^+ stopping midway" if $cd \sim_P cd^+ \not\leftrightarrow c^+d \sim_P c^+d^+$ for some c; here we are writing x^+ for $x+1$.

Theorem: a partition of H into closed intervals is a slash-partition if and only if it doesn't have any cuts stopping midway. Proof: use the ideas above to recover L and R from $\sim_P$, and then check that $S = (L, R)$ induces an equivalence relation $\sim_S$ that coincides with $\sim_P$.

19 Slash-operators

We can define operations that take each each $P \in H$ to the maximal and to the minimal element of its S-equivalent class, now that we know that these maximal and minimal elements exist:

$$\begin{array}{rcll}
P^S & := & \bigvee [P]_S & \text{(maximal element)}, \\
P^{\mathrm{co}S} & := & \bigwedge [P]_S & \text{(minimal element)}.
\end{array}$$

Note that $[P]_S = [P^{\mathrm{co}S}, P^S] \cap H$.

We will use the operation $\cdot^S$ a lot and $\cdot^{\mathrm{co}S}$ very little. The 'co' in 'co$S$' means that $\cdot^{\mathrm{co}S}$ is dual to $\cdot^S$, in a sense that will be made precise later.

A *slash-operator* on a ZHA H is a function $\cdot^S : H \to H$ induced by a slashing $S = (L, R)$ on H. It is easy to see that $P \leq P^S$ ("$\cdot^S$ is non-decreasing") and that $P^S = (P^S)^S$ ("$\cdot^S$ is idempotent").

Any idempotent function $\cdot^F : H \to H$ induces an equivalence relation on H: $P \sim_F Q$ iff $P^F = Q^F$. We can use that to test if a given $\cdot^F : H \to H$ is a slash-operator: $\cdot^F$ is a slash-operator iff it obeys all this:
1) $\cdot^F$ is idempotent,
2) $\cdot^F$ is non-decreasing,
3) $\sim_F$ partitions H into closed intervals,
4) $\sim_F$ doesn't have cuts stopping midway.

20 Slash-operators: a property

Slash-operators obey a certain property that will be very important later. Let's state that property in five equivalent ways:

1) If $cd \sim_S c'd'$ and $ef \sim_S e'f'$ then $cd \land ef \sim_S c'd' \land e'f'$.
2) If $P \sim_S P'$ and $Q \sim_S Q'$ then $P \land Q \sim_S P' \land Q'$.
3) If $P \sim_S P'$ and $Q \sim_S Q'$ then $(P \land Q)^S = (P' \land Q')^S$.
4) If $P \sim_S P'$ and $Q \sim_S Q'$ then

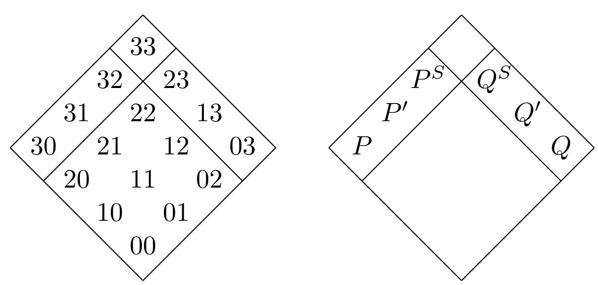
$$\begin{array}{rcll} (P \land Q)^S & = & (P^S \land Q^S)^S & \text{(a)} \\ & = & ((P')^S \land (Q')^S)^S & \text{(b)} \\ & = & (P' \land Q')^S & \text{(c)} \end{array}$$

5) $(P \land Q)^S = (P^S \land Q^S)^S$.

Here's a proof of $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4 \leftrightarrow 5$.

$1 \leftrightarrow 2$: we just changed notation,
$2 \leftrightarrow 3$: because $A \sim_S B$ iff $A^S = B^S$,
$3 \to 5$: make the substitution $\begin{bmatrix} P' := P^S \\ Q' := Q^S \end{bmatrix}$ in 3,
$5 \to 4$: 4a is just a copy of 5, and 4c is a copy of 5 with $\begin{bmatrix} P := P' \\ Q := Q' \end{bmatrix}$. For 4b, note that $P \sim_P P'$ implies $P^S = (P')^S$ and $Q \sim_P Q'$ implies $Q^S = (Q')^S$,
$4 \to 3$: 4 is an equality between more expressions than 3,

...and here is a way to visualize what is going on:



$$\underbrace{(\underbrace{\underbrace{P}_{30} \land \underbrace{Q}_{03}}_{00})^S}_{22} = \underbrace{(\underbrace{\underbrace{\underbrace{P}_{30}{}^S}_{32} \land \underbrace{\underbrace{Q}_{03}{}^S}_{23}}_{22})^S}_{22} = \underbrace{(\underbrace{\underbrace{\underbrace{P'}_{31}{}^S}_{32} \land \underbrace{\underbrace{Q'}_{13}{}^S}_{23}}_{22})^S}_{22} = \underbrace{(\underbrace{\underbrace{P'}_{31} \land \underbrace{Q'}_{13}}_{11})^S}_{22}$$

Note that all subexpressions belong to three S-regions: a region with P, P', $P^S = P'^S$, another with Q, Q', $Q^S = Q'^S$, and one with all the '$\land$'s. If we had cuts stopping midway then some of the '$\land$'s could be in different regions.

I think that the clearest way to show (1) is by putting its proof in tree form:

$$\dfrac{\dfrac{\dfrac{\dfrac{cd \sim_S c'd'}{c \sim_L c'} \qquad \dfrac{ef \sim_S e'f'}{e \sim_L e'}}{\min(c,e) \sim_L \min(c',e')} \qquad \dfrac{\dfrac{cd \sim_S c'd'}{d \sim_R d'} \qquad \dfrac{ef \sim_S e'f'}{f \sim_R f'}}{\min(d,f) \sim_L \min(d',f')}}{\min(c,e)\min(d,f) \sim_S \min(c',e')\min(d',f')}}{cd \wedge ef \sim_S c'd' \wedge e'f'}$$

21 J-operators and J-regions

A *J-operator* on a Heyting Algebra $H = (\Omega, \leq, \top, \bot, \land, \lor, \to, \leftrightarrow, \neg)$ is a function $J : \Omega \to \Omega$ that obeys the axioms $\mathsf{J1}$, $\mathsf{J2}$, $\mathsf{J3}$ below; we usually write J as $\cdot^* : \Omega \to \Omega$, and write the axioms as rules.

$$\frac{}{P \leq P^*}\,\mathsf{J1} \qquad \frac{}{P^* = P^{**}}\,\mathsf{J2} \qquad \frac{}{(P\&Q)^* = P^*\&Q^*}\,\mathsf{J3}$$

$\mathsf{J1}$ says that the operation $\cdot^*$ is non-decreasing.
$\mathsf{J2}$ says that the operation $\cdot^*$ is idempotent.
$\mathsf{J3}$ is a bit mysterious but will have interesting consequences.

Note that when H is a ZHA then any slash-operator on H is a J-operator on it; see secs.19 and 20.

A J-operator induces an equivalence relation and equivalence classes on Ω, like slashings do:

$$\begin{array}{rcl} P \sim_J Q & \text{iff} & P^* = Q^* \\ {[P]}^J & := & \{\, Q \in \Omega \mid P^* = Q^* \,\} \end{array}$$

The axioms $\mathsf{J1}$, $\mathsf{J2}$, $\mathsf{J3}$ have many consequences. The first ones are listed in Figure 3 as derived rules, whose names mean:
Mop (monotonicity for products): a lemma used to prove Mo,
Mo (monotonicity): $P \leq Q$ implies $P^* \leq Q^*$,
Sand (sandwiching): all truth values between P and P^* are equivalent,
$\mathsf{EC\&}$: equivalence classes are closed by '&',
$\mathsf{EC\lor}$: equivalence classes are closed by '$\lor$',
ECS: equivalence classes are closed by sandwiching,

Take a J-equivalence class, $[P]^J$, and list its elements: $[P]^J = \{P_1, \ldots, P_n\}$. Let $P_\land := ((P_1 \land P_2) \land \ldots) \land P_n$ and Let $P_\lor := ((P_1 \lor P_2) \lor \ldots) \lor P_n$. It turns out that $[P]^J = [P_\land, P_\lor] \cap \Omega$; let's prove that by doing '$\subseteq$' first, then '$\supseteq$'.
Using $\mathsf{EC\&}$ and $\mathsf{EC\lor}$ several times we see that

$$\begin{array}{rcr} P_1 \land P_2 \sim_J P & \qquad & P_1 \lor P_2 \sim_J P \\ (P_1 \land P_2) \land P_3 \sim_J P & & (P_1 \lor P_2) \lor P_3 \sim_J P \\ \vdots & & \vdots \\ ((P_1 \land P_2) \land \ldots) \land P_n \sim_J P & & ((P_1 \lor P_2) \lor \ldots) \lor P_n \sim_J P \end{array}$$

so $P_\land \sim_J P_\lor \sim_J P$, and by the sandwich lemma $([P_\land, P_\lor] \cap \Omega) \subseteq [P]^J$.
For any $P_i \in [P]^J$ we have $P_\land \leq P_i \leq P_\lor$, which means that:

$$\begin{array}{rcl} {[P]}^J & = & \{P_1, \ldots, P_n\} \\ & \subseteq & \{\, Q \in \Omega \mid P_\land \leq Q \leq P_\lor \,\} \\ & = & [P_\land, P_\lor] \cap \Omega, \end{array}$$

so $[P]^J \subseteq [P_\land, P_\lor] \cap \Omega$.

$$\dfrac{}{(P\&Q)^* \le Q^*}\ \mathsf{Mop} \quad := \quad \dfrac{\dfrac{}{(P\&Q)^* = P^*\&Q^*}\ \mathsf{J3} \quad \dfrac{}{P^*\&Q^* \le Q^*}}{(P\&Q)^* \le Q^*}$$

$$\dfrac{P \le Q}{P^* \le Q^*}\ \mathsf{Mo} \quad := \quad \dfrac{\dfrac{\dfrac{P \le Q}{P = P\&Q}}{P^* = (P\&Q)^*} \quad \dfrac{}{(P\&Q)^* \le Q^*}\ \mathsf{Mop}}{P^* \le Q^*}$$

$$\dfrac{P \le Q \le P^*}{P^* = Q^*}\ \mathsf{Sand} \quad := \quad \dfrac{\dfrac{P \le Q}{P^* \le Q^*}\ Mo \quad \dfrac{\dfrac{Q \le P^*}{Q^* \le P^{**}}\ \mathsf{Mo} \quad \dfrac{}{P^{**} = P^*}\ \mathsf{J2}}{Q^* \le P^*}}{P^* = Q^*}$$

$$\dfrac{P^* = Q^*}{P^* = Q^* = (P\&Q)^*}\ \mathsf{EC\&} \quad := \quad \dfrac{\dfrac{P^* = Q^*}{P^* = Q^* = P^*\&Q^*} \quad \dfrac{}{P^*\&Q^* = (P\&Q)^*}\ \mathsf{J3}}{P^* = Q^* = (P\&Q)^*}$$

$$\dfrac{P^* = Q^*}{P^* = Q^* = (P \vee Q)^*}\ \mathsf{EC\vee} \quad := \quad \dfrac{P^* = Q^* \quad \dfrac{\dfrac{\dfrac{}{P \le P \vee Q} \quad \dfrac{\dfrac{}{P \le P^*}\ \mathsf{J1} \quad \dfrac{\dfrac{}{Q \le Q^*}\ \mathsf{J1} \quad \dfrac{P^* = Q^*}{Q^* = P^*}}{Q \le P^*}}{P \vee Q \le P^*}}{P \le P \vee Q \le P^*}}{P^* = (P \vee Q)^*}\ \mathsf{Sand}}{P^* = Q^* = (P \vee Q)^*}$$

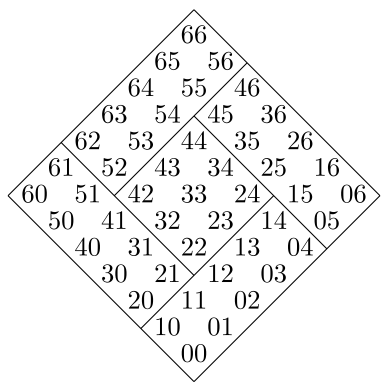
$$\dfrac{P \le Q \le R \quad P^* = R^*}{P^* = Q^* = R^*}\ \mathsf{ECS} \quad := \quad \dfrac{\dfrac{\dfrac{P \le Q \le R \quad \dfrac{}{R \le R^*}\ \mathsf{J1} \quad \dfrac{P^* = R^*}{R^* = P^*}}{P \le Q \le P^*}}{P^* = Q^*}\ \mathsf{Sand} \qquad P^* = R^*}{P^* = Q^* = R^*}$$

Figure 3: J-operators: basic derived rules

As the operation '$\cdot^*$' is increasing and idempotent, each equivalence class $[P]^J$ has exactly one maximal element, which is P^*; but $P_\vee$ is also the maximal element of $[P]^J$, so $P_\vee = P^*$, and we can interpret the operation '$\cdot*$' as "take each P to the top element in its equivalence class", which is similar to how we defined an(other) operation '$\cdot^*$' on slashings in the previous section.

The operation "take each P to the bottom element in its equivalence class" will be useful in a few occasions; we will call it '$\cdot^{\mathrm{co}*}$' to indicate that it is dual to '$\cdot^*$' in some sense. Note that $P^{\mathrm{co}*} = P_\wedge$.

Look at the figure below, that shows a partition of a ZHA $A = [00, 66]$ into five regions, each region being an interval; this partition does not come from a slashing, as it has cuts that stop midway. Define an operation '$\cdot^*$' on A, that works by taking each truth-value P in it to the top element of its region; for example, $30^* = 61$.



It is easy to see that '$\cdot^*$' obeys J1 and J2; however, it does *not* obey J3 — we will prove that in sec.23. As we will see, *the partitons of a ZHA into intervals that obey* J1, J2, J3 *ae exactly the slashings;* or, in other words, *every J-operator comes from a slashing.*

22 How J-operators interact with connectives

23 J-regions come from slashings

24 Appendix: on “children”

...from the slides of my minicourse in the UniLog 2016 (Istanbul):

- Why study Category Theory *now*?

  Public education in Brazil is being dismantled — maybe we should be doing better things than studying very technical & inaccessible subjects with no research grants —

  (Here I showed a photo called “The New Girl From Ipanema” — a girl walking on the Ipanema beach at night with a gas mask, with a huge cloud of tear gas behind her)

- Category theory should be more accessible

  Most texts about CT are for specialists in research universities... *Category theory should be more accessible..*

  To whom?...

  - Non-specialists (in research universities)
  - Grad students (in research universities)
  - Undergrads (in research universities)
  - Non-specialists (in conferences - where we have to be quick)
  - Undergrads (e.g., in CompSci - in teaching colleges) - (“Children”)

- What do we mean by ”accessible”?

  - Done on very firm grounds: mathematical objects made from numbers, sets and tuples; FINITE, SMALL mathematical objects whenever possible. Avoid references to non-mathematical things like windows, cars and pizzas (like the object-orientation people do); avoid reference to Physics; avoid Quantum Mecanics at all costs; time is difficult to draw, prefer *static* rather than *changing*
  - People have very short attention spans nowadays
  - Self-contained, but not *isolated* or *isolating*; our material should make the literature more accessible
  - We learn better by doing. Our material should have lots of space for exercises.
  - Most people with whom I interact are not from Maths/CS/etc
  - *Proving* general cases is relatively hard. *Checking* and *calculating* is much easier. People can believe that something can be generalized after seeing a convincing particular case. (Sometimes leave them to look for the right generalization by themselves)

I've been using "for children" in titles for a while. This is a bit of a marketing strategy, of course, but the term "children" here has a precise, though unusual, meaning: it means "people with very little mathematical maturity", where I am taking these as the main aspects of "mathematical maturity": the ability to *work on very abstract settings*, to *generalize*, to *particularize*, and to *use infinite objects*.

Writing things "for children" in this sense results in material that [is accessible] [exercises, not included here] [visual, easy to check] [who I've tested this with]

A note for "adults". In [Ochs2013] I sketched a method for working in a general case and in a particular case (an "archetypal case") in parallel, and also a way to prove things in the archetypal case and then "lift" the proofs to the general case. This paper is an offspring of that one; I believe that planar Heyting Algebras presented here (ZHAs, sec.4) are archetypal Heyting Algebras, and when we add "closure operators" to ZHAs (as in the seminar notes `http://angg.twu.net/math-b.html#zhas-for-children`, pp.13–30; they are called "J-operators" there) we get something that is archetypal for studying toposes and sheaves; that will be the subject of a sequel of this paper.

[Topos theory books are too hard for me] [a bridge between philosophers and toposophers]

25 Appendix: notations for set comprehension

This is section is just to clarify the exact meaning of the "$\{\ldots \mid \ldots\}$-expressions" in the previous sections.

We'll use three notations for set comprehensions: a "low-level" one, with generators and filters separated by commas, then a semicolon and then the result expression, and two higher-level notations using a '|', that are closer to the standard ones.

Here are some examples of the low-level notation,

$$
\begin{array}{rcl}
\{\underbrace{a \in \{1,2,3,4\}}_{\text{gen}}; \underbrace{10a}_{\text{expr}}\} & = & \{10,20,30,40\} \\
\{\underbrace{a \in \{1,2,3,4\}}_{\text{gen}}; \underbrace{\;a\;}_{\text{expr}}\} & = & \{1,2,3,4\} \\
\{\underbrace{a \in \{1,2,3,4\}}_{\text{gen}}, \underbrace{a \geq 3}_{\text{filt}}; \underbrace{\;a\;}_{\text{expr}}\} & = & \{3,4\} \\
\{\underbrace{a \in \{1,2,3,4\}}_{\text{gen}}, \underbrace{a \geq 3}_{\text{filt}}; \underbrace{10a}_{\text{expr}}\} & = & \{30,40\} \\
\{\underbrace{a \in \{10,20\}}_{\text{gen}}, \underbrace{b \in \{3,4\}}_{\text{gen}}; \underbrace{a+b}_{\text{expr}}\} & = & \{13,14,23,24\} \\
\{\underbrace{a \in \{1,2\}}_{\text{gen}}, \underbrace{b \in \{3,4\}}_{\text{gen}}; \underbrace{(a,b)}_{\text{expr}}\} & = & \{(1,3),(1,4),(2,3),(2,4)\}
\end{array}
$$

Here is how to calculate the results of some low-level comprehensions using tables; note that when a filter yields "false" we stop — this is indicated by a vertical bar — and we don't calculate the rest of the line. The result of the comprehension is the set of the results in the lines where all filters yielded "true".

$$\{\underbrace{x \in \{1,2,3\}}_{\text{gen}}, \underbrace{y \in \{3,4\}}_{\text{gen}}, \underbrace{x+y<6}_{\text{filt}}; \underbrace{(x,y)}_{\text{expr}}\} = \{(1,3),(1,4),(2,3)\}$$

x	y	$x{+}y{<}6$	(x,y)
1	3	**T**	(1,3)
1	4	**T**	(1,4)
2	3	**T**	(2,3)
2	4	**F**	|
3	3	**F**	|
3	4	**F**	|

$$\{\underbrace{(x,y) \in \{1,2,3\}^2}_{\text{gen}}, \underbrace{x>y}_{\text{filt}}; \underbrace{(x,y)}_{\text{expr}}\} = \{(2,1),(3,1),(3,2)\}$$

(x,y)	x	y	$x{>}y$	(x,y)
(1,1)	1	1	**F** |	
(1,2)	1	2	**F** |	
(1,3)	1	3	**F** |	
(2,1)	2	1	**T**	(2,1)
(2,2)	2	2	**F** |	
(2,3)	2	3	**F** |	
(3,1)	3	1	**T**	(3,1)
(3,2)	3	2	**T**	(3,2)
(3,3)	3	3	**F** |	

Here are some examples of the higher-level, standard-ish notations for set comprehensions, and how they can be translated into the low-level notation:

$$\begin{array}{lcl}
\text{(standard)} & & \text{(low-level)} \\
\{\, \underbrace{10a}_{\text{expr}} \mid \underbrace{a \in \{1,2,3,4\}}_{\text{gen}} \,\} & = & \{\underbrace{a \in \{1,2,3,4\}}_{\text{gen}}; \underbrace{10a}_{\text{expr}}\} \\
\{\, \underbrace{a}_{\text{expr}} \mid \underbrace{a \in \{1,2,3,4\}}_{\text{gen}} \,\} & = & \{\underbrace{a \in \{1,2,3,4\}}_{\text{gen}}; \underbrace{a}_{\text{expr}}\} \\
\{\, \underbrace{10a}_{\text{expr}} \mid \underbrace{a \in \{1,2,3,4\}}_{\text{gen}}, \underbrace{a \geq 3}_{\text{filt}} \,\} & = & \{\underbrace{a \in \{1,2,3,4\}}_{\text{gen}}, \underbrace{a \geq 3}_{\text{filt}}; \underbrace{10a}_{\text{expr}}\} \\
\{\, \underbrace{a}_{\text{expr}} \mid \underbrace{a \in \{1,2,3,4\}}_{\text{gen}}, \underbrace{a \geq 3}_{\text{filt}} \,\} & = & \{\underbrace{a \in \{1,2,3,4\}}_{\text{gen}}, \underbrace{a \geq 3}_{\text{filt}}; \underbrace{a}_{\text{expr}}\} \\
\{\, \underbrace{a \in \{1,2,3,4\}}_{\text{gen}} \mid \underbrace{a \geq 3}_{\text{filt}} \,\} & = & \{\underbrace{a \in \{1,2,3,4\}}_{\text{gen}}, \underbrace{a \geq 3}_{\text{filt}}; \underbrace{a}_{\text{expr}}\}
\end{array}$$

The first four are of the form "{ expr | generators and filters }" ("e|gf"), and the last one is of the form "{ generator | filters }" ("g|f"). In "g|f" comprehensions the final 'expr' is the variable of the generator:

$$\{\, \underbrace{\underbrace{a}_{\text{var}} \in \{1,2,3,4\}}_{\text{gen}} \mid \underbrace{a \geq 3}_{\text{filt}} \,\} = \{\underbrace{a \in \{1,2,3,4\}}_{\text{gen}}, \underbrace{a \geq 3}_{\text{filt}}; \underbrace{a}_{\text{expr}}\}$$