Cálculo 2
PURO-UFF - 2017.1
P1 - 17/jul/2017 - Eduardo Ochs
Respostas sem justificativas não serão aceitas.
Proibido usar quaisquer aparelhos eletrônicos.

1) **(Total: 1.0)** Demonstre que $\frac{d}{dx} \arctan x = \frac{1}{x^2+1}$. Você vai precisar do truque pra derivar funções inversas — se $f(g(x)) = x$ então $\frac{d}{dx}(f(g(x))) = \frac{d}{dx}x = 1$ — e de um outro truque extra... dica: na demonstração de que $\frac{d}{dx} \operatorname{arcsen} x = \frac{1}{\sqrt{1-x^2}}$ o truque extra é $\cos^2 x + \operatorname{sen}^2 x = 1$.

2) **(Total: 1.0)** Sejam $f(x) = x^{-4}$ e $F(x) = \frac{x^{-3}}{-3}$.
a) **(0.1 pts)** Mostre que a área $\int_{x=-1}^{x=1} f(x)\,dx$ é positiva.
b) **(0.2 pts)** Calcule $F(x)\big|_{x=-1}^{x=1}$.
c) **(0.3 pts)** Calcule $G(\epsilon) = \int_{x=-1}^{x=-\epsilon} f(x)\,dx + \int_{x=\epsilon}^{x=1} f(x)\,dx$. Obs: $\epsilon > 0$.
d) **(0.4 pts)** Calcule $\lim_{\epsilon\to 0^+} G(\epsilon)$.

3) **(Total: 2.0)** Calcule

$$\int_{x=-1}^{x=2} \frac{1}{3+|x|}\,dx.$$

4) **(Total: 2.0)** Calcule

$$\int \frac{x^2}{x^2+4x-5}\,dx.$$

5) **(Total: 2.0)** Calcule

$$\int \frac{x}{1+x^2}\,dx.$$

6) **(Total: 2.0)** Calcule

$$\int (\cos x)^4\,dx.$$

Dicas: $\begin{bmatrix} s=\operatorname{sen}\theta \\ \sqrt{1-s^2}=\cos\theta \\ ds=\cos\theta\,d\theta \\ \theta=\operatorname{arcsen} s \end{bmatrix}$, $\begin{bmatrix} t=\tan\theta \\ \sqrt{1+t^2}=\sec\theta=z \\ dt=z^2\,d\theta \\ \theta=\arctan t \end{bmatrix}$, $\begin{bmatrix} z=\sec\theta \\ \sqrt{z^2-1}=\tan\theta=t \\ dz=zt\,d\theta \\ \theta=\operatorname{arcsec} z \end{bmatrix}$,
$E = e^{i\theta} = \cos\theta + i\operatorname{sen}\theta = c + is$,
$c = (E + E^{-1})/2$,
$s = (E - E^{-1})/2i$.

Gabarito parcial:

1) $\frac{d}{dx}\arctan(\tan x) = \frac{d}{dx}x = 1$
$\frac{d}{dx}\arctan(\tan x) = \arctan'(\tan x)\tan' x$
$\arctan'(\tan x) = \frac{1}{\tan' x}$
$\frac{d}{d\theta}\tan\theta = \frac{d}{d\theta}\frac{s}{c} = \frac{s'c - sc'}{c^2} = \frac{c^2+s^2}{c^2} = \frac{c^2}{c^2} + \frac{s^2}{c^2} = 1 + t^2$
$\frac{d}{dx}\tan x = (\tan x)^2 + 1$
$\arctan'(\tan x) = \frac{1}{(\tan x)^2+1}$
$\arctan'(t) = \frac{1}{t^2+1}$
$\arctan'(x) = \frac{1}{x^2+1}$

2a) (um desenho)

2b) $F(x)|_{x=-1}^{x=1} = F(1) - F(-1) = \frac{1^{-3}}{-3} - \frac{(-1)^{-3}}{-3} = \frac{1}{-3} - \frac{-1}{-3} = \frac{2}{-3} = -\frac{2}{3}$

2c)
$$\begin{aligned} G(\epsilon) &= F(x)|_{x=-1}^{x=-\epsilon} + F(x)|_{x=\epsilon}^{x=1} \\ &= F(-\epsilon) - F(-1) + F(1) - F(\epsilon) \\ &= \frac{(-\epsilon)^{-3}}{-3} - \frac{(-1)^{-3}}{-3} + \frac{1^{-3}}{-3} - \frac{\epsilon^{-3}}{-3} \\ &= \frac{(-\epsilon)^{-3}+1+1-\epsilon^{-3}}{-3} \\ &= \frac{2-2(\epsilon^{-3})}{-3} \\ &= \frac{2(\epsilon^{-3})-2}{3} \\ &= \frac{2}{3}(\epsilon^{-3} - 1) \end{aligned}$$

2d) $\lim_{\epsilon\to 0^+} G(\epsilon) = \frac{2}{3}(\lim_{\epsilon\to 0^+}\epsilon^{-3} - \lim_{\epsilon\to 0^+} 1) = \frac{2}{3}(+\infty - 1) = +\infty$

3) Seja $f(x) = \frac{1}{3+|x|}$. Então f é contínua e:

$$f(x) = \begin{cases} \frac{1}{3+|x|} & \text{quando } x < 0 \\ \frac{1}{3+|x|} & \text{quando } x \geq 0 \end{cases} = \begin{cases} \frac{1}{3-x} & \text{quando } x < 0 \\ \frac{1}{3+x} & \text{quando } x \geq 0 \end{cases} = \begin{cases} -\frac{1}{x-3} & \text{quando } x < 0 \\ \frac{1}{x+3} & \text{quando } x \geq 0 \end{cases}$$

$$\begin{aligned} \int_{x=-1}^{x=2} f(x)\,dx &= \int_{x=-1}^{x=0} -\frac{1}{x-3}\,dx + \int_{x=0}^{x=2}\frac{1}{x+3}\,dx \\ &= (-\ln|x-3|)|_{x=-1}^{x=0} + \ln|x+3||_{x=0}^{x=2} \\ &= (-\ln|-1-3|) - (-\ln|0-3|) + (-\ln|2+3|) - (-\ln|0+3|) \\ &= (-\ln 4) - (-\ln 3) + (-\ln 5) - (-\ln 3) \\ &= 2\ln 3 - \ln 4 - \ln 5 \end{aligned}$$

4) $\int \frac{x^2}{x^2+4x-5}\,dx = \int 1 + \frac{1}{6}\frac{1}{x-1} - \frac{25}{6}\frac{1}{x+5}\,dx = x + \frac{1}{6}\ln|x-1| - \frac{25}{6}\ln|x+5|$