Geometria Analítica
PURO-UFF - 2017.1
VR - 18/jul/2017 - Eduardo Ochs
Respostas sem justificativas não serão aceitas.
Proibido usar quaisquer aparelhos eletrônicos.

1) **(Total: 2.5)** Sejam $A = (-5, 0)$, $B = (5, 0)$, $\vec{v}_m = \overrightarrow{(1, m)}$, $\vec{w}_m = \overrightarrow{(1, -1/m)}$, $r_m = \{ A + t\vec{v}_m \mid t \in \mathbb{R} \}$, $s_m = \{ A + t\vec{w}_m \mid t \in \mathbb{R} \}$, $P_m = r_m \cap s_m$. $C_\alpha = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = \alpha \}$.
a) **(0.2 pts)** Represente graficamente A, B, r_1, s_1, P_1.
b) **(0.3 pts)** Represente graficamente A, B, r_2, s_2, P_2.
c) **(1.0 pts)** Dê as coordenadas de P_m para $m = 1, 2, 3, -1, -2, -3, \frac{1}{2}, \frac{1}{3}, -\frac{1}{2}, -\frac{1}{3}$.
d) **(1.0 pts)** Para que valor de α o círculo C_α fica tangente a r_2?

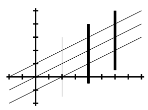
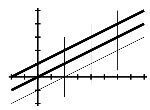
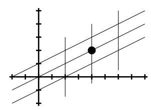
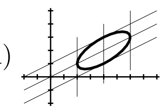
2) **(Total: 3.0)** Faça esboços das cônicas com as equações abaixo. Algumas delas são degeneradas. Em todos os itens abaixo considere que $u = (x-4)/2$ e $v = y - x/2$ — ou que u e v são *abreviações* para $(x-4)/2$ e $y - x/2$.

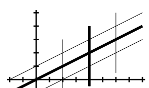
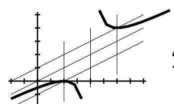
a) (0.5 pts)	$u(u-1) = 0$	e) (0.2 pts)	$uv = 0$
b) (0.5 pts)	$v(v-1) = 0$	f) (0.4 pts)	$uv = 1$
c) (0.2 pts)	$u^2 + v^2 = 0$	g) (0.4 pts)	$uv = -1$
d) (0.3 pts)	$u^2 + v^2 = 1$	h) (0.5 pts)	$u^2 = v$
		i) (0.5 pts)	$u = v^2$

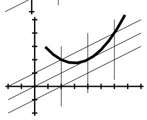
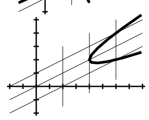
3) **(Total: 5.5)** Sejam $\pi = \{ (x, y, z) \in \mathbb{R}^3 \mid 2x + 2y + z = 4 \}$, $O = (0, 0, 0)$ e $r = \{ (1, 2, 3) + t\overrightarrow{(2, 1, 0)} \mid t \in \mathbb{R} \}$.
a) **(0.2 pts)** Dê as coordenadas do ponto $F \in \pi \cap r$.
b) **(1.0 pts)** Encontre o ponto $G \in r$ mais próximo do ponto $(0, 0, 0)$.
c) **(1.0 pts)** Encontre o ponto $H \in \pi$ mais próximo do ponto $(0, 0, 0)$.
d) **(1.0 pts)** Calcule $d(\pi, (0, 0, 0))$ usando '$\times$'.
e) **(0.3 pts)** Verifique se os resultados dos seus itens (b) e (c) são coerentes.
f) **(1.0 pts)** Dê a equação de um plano π' paralelo a π e tal que $d(\pi, \pi') = 1$.
g) **(0.5 pts)** Encontre um ponto $J \in r$ tal que $d(\pi, J) = 1$.
h) **(0.5 pts)** Encontre um ponto $K \in r$ tal que $d(\pi, K) = 2$.

Gabarito: (não revisado)

1a) (desenho)
1b) (desenho)
1c) $P_1 = (0,5)$, $P_2 = (3,4)$, $P_3 = (4,3)$,
$P_{-1} = (0,-5)$, $P_{-2} = (3,-4)$, $P_{-3} = (4,-3)$,
$P_{1/2} = (-3,4)$, $P_{1/3} = (-4,3)$,
$P_{-1/2} = (-3,-4)$, $P_{-1/3} = (-4,-3)$.
1d) $\alpha = 20$

2a)  2b)  2c)  2d) 

2e)  2f)  2g) 

2h)  2i) 

3) Sejam $\vec{n} = \overrightarrow{(2,2,1)}$, $\vec{n}' = \overrightarrow{(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})}$, $A = (1,2,3)$, $\vec{v} = \overrightarrow{(2,1,0)}$.

3a) $F = (1+2t, 2+t, 3)$,
$F \in \pi \quad \Rightarrow \quad 2(1+2t) + 2(2+t) + 3 = 4 \quad \Rightarrow \quad 9 + 6t = 4$
$\Rightarrow \quad t = -\frac{5}{6} \quad \Rightarrow \quad F = (1 - 2\frac{5}{6}, 2 - \frac{5}{6}, 3) = (-\frac{4}{6}, \frac{7}{6}, 3)$

3b)
$$
\begin{aligned}
\mathrm{Pr}_{\vec{v}}\overrightarrow{AO} &= \mathrm{Pr}_{\overrightarrow{(2,1,0)}}\overrightarrow{(-1,-2,-3)} \\
&= \frac{\overrightarrow{(2,1,0)}\cdot\overrightarrow{(-1,-2,-3)}}{\overrightarrow{(2,1,0)}\cdot\overrightarrow{(2,1,0)}}\overrightarrow{(2,1,0)} \\
&= -\frac{4}{5}\overrightarrow{(2,1,0)} \\
&= \overrightarrow{(-\frac{8}{5}, -\frac{4}{5}, 0)} \\
G &= A + \mathrm{Pr}_{\vec{v}}\overrightarrow{AO} \\
&= (1,2,3) + \overrightarrow{(-\frac{8}{5}, -\frac{4}{5}, 0)} \\
&= (-\frac{3}{5}, \frac{6}{5}, 3)
\end{aligned}
$$

Repare que $\overrightarrow{GO} = \overrightarrow{(\frac{3}{5}, -\frac{6}{5}, -3)}$ é ortogonal a $\vec{v}$.

3c) Seja $s = \{\, O + t\vec{n} \mid t \in \mathbb{R} \,\} = \{\, (2t, 2t, t) \mid t \in \mathbb{R} \,\}$; $O \in s$ e $s \perp \pi$. Seja $H \in s \cap s$; $(2t, 2t, t) \in \pi$ quando $9t = 4 \ \Rightarrow\ t = \frac{4}{9} \ \Rightarrow\ H = (\frac{8}{9}, \frac{8}{9}, \frac{4}{9})$.

3d) Sejam $A = (2,0,0)$, $B = (0,2,0)$, $C = (0,0,4)$; $A, B, C \in \pi$. Sejam $\vec{u} = \overrightarrow{AB} = \overrightarrow{(-2,2,0)}$, $\vec{v} = \overrightarrow{AC} = \overrightarrow{(-2,0,4)}$, $\vec{w} = \overrightarrow{AO} = \overrightarrow{(-2,0,0)}$. Então $\vec{u} \times \vec{v} = \overrightarrow{(8,8,4)}$, $\mathsf{área}(\vec{u}, \vec{v}) = ||\vec{u} \times \vec{v}|| = 12$, $|[\vec{u}, \vec{v}, \vec{w}]| = \begin{vmatrix} -2 & 2 & 0 \\ -2 & 0 & 4 \\ -2 & 0 & 0 \end{vmatrix} = -16$, $\frac{|[\vec{u},\vec{v},\vec{w}]|}{\mathsf{área}(\vec{u},\vec{v})} = \frac{-16}{12} = -\frac{4}{3}$, $d(\pi, O) = |-\frac{4}{3}| = \frac{4}{3}$.

3e) $d(H, O) = ||\overrightarrow{(-\frac{8}{9}, -\frac{8}{9}, -\frac{4}{9})}|| = ||\frac{1}{9}\overrightarrow{(8,8,4)}|| = \frac{1}{9}||\overrightarrow{(8,8,4)}|| = \frac{12}{9} = \frac{4}{3}$; ou seja, $d(H, O) = d(\pi, O)$.

3f) O modo mais rápido é aproveitar que $\vec{n}' = \overrightarrow{(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})}$ é um vetor normal a π com $||\vec{n}'|| = 1$. Como $A = (2,0,0) \in \pi$, $d(A + \vec{n}', \pi) = d(A - \vec{n}', \pi) = 1$; generalizando: $d(A + k\vec{n}', \pi) = ||k\vec{n}'|| = |k|$.
Vou fazer umas contas que vão resolver este item e os próximos dois de uma vez só.
Seja $A_k = A + k\vec{n}' = (2,0,0) + k\overrightarrow{(\frac{2}{3}, \frac{2}{3}, \frac{k}{3})} = (\frac{6+2k}{3}, \frac{2k}{3}, \frac{k}{3})$. As soluções deste item são o plano paralelo a π que contêm A_1 e o plano paralelo a π que contém A_{-1}. Os planos paralelos a π são da forma $\{ (x,y,z) \in \mathbb{R}^3 \mid 2x+2y+z = \alpha \}$, então vamos definir $\pi_\alpha = \{ (x,y,z) \in \mathbb{R}^3 \mid 2x+2y+z = \alpha \}$; repare que $\pi = \pi_4$, $A = A_0$, $A_0 \in \pi_4$.

$$\begin{array}{rcl}
(x,y,z) \in \pi_\alpha & \Rightarrow & \alpha = 2x+2y+z \\
A_k \in \pi_\alpha & \Rightarrow & \alpha = 2\frac{6+2k}{3} + 2\frac{2k}{3} + \frac{k}{3} \\
& \Rightarrow & \alpha = \frac{12+4k+4k+k}{3} = \frac{12+9k}{3} = 4+3k \\
& \Rightarrow & \pi_\alpha = \{ (x,y,z) \in \mathbb{R}^3 \mid 2x+2y+z = 4+3k \}, \\
A_1 \in \pi_\alpha & \Rightarrow & \pi_\alpha = \{ (x,y,z) \in \mathbb{R}^3 \mid 2x+2y+z = 7 \}, \\
A_{-1} \in \pi_\alpha & \Rightarrow & \pi_\alpha = \{ (x,y,z) \in \mathbb{R}^3 \mid 2x+2y+z = 1 \}.
\end{array}$$

3g) Seja $R_t = (1,2,3) + t\overrightarrow{(2,1,0)} = (1+2t, 2+t, 3)$. Temos:

$$\begin{array}{rcl}
R_t \in \pi_\alpha & \Rightarrow & (1+2t, 2+t, 3) \in \pi_\alpha \\
& \Rightarrow & \alpha = 2(1+2t) + 2(2+t) + 3 = 2 + 4t + 4 + 2t + 3 = 9 + 6t \\
& \Rightarrow & t = \frac{\alpha-9}{6} \\
& \Rightarrow & R_t = (1 + 2\frac{\alpha-9}{6}, 2 + \frac{\alpha-9}{6}, 3) = (\frac{6+2\alpha-9}{6}, \frac{12+\alpha-9}{6}, 3) \\
& \Rightarrow & r \cap \pi_\alpha = R_t = (\frac{2\alpha-3}{6}, \frac{\alpha+3}{6}, 3)
\end{array}$$

As soluções são

$$\begin{array}{rcl}
r \cap \pi_7 & = & (\frac{2\cdot 7-3}{6}, \frac{7+3}{6}, 3) = (\frac{11}{6}, \frac{10}{6}, 3), \\
r \cap \pi_1 & = & (\frac{2\cdot 1-3}{6}, \frac{1+3}{6}, 3) = (\frac{-1}{6}, \frac{4}{6}, 3).
\end{array}$$

3f) As soluções são

$$\begin{array}{rcl}
r \cap \pi_{10} & = & (\frac{2\cdot 10-3}{6}, \frac{10+3}{6}, 3) = (\frac{17}{6}, \frac{13}{6}, 3), \\
r \cap \pi_{-2} & = & (\frac{2\cdot(-2)-3}{6}, \frac{-2+3}{6}, 3) = (\frac{-7}{6}, \frac{1}{6}, 3).
\end{array}$$

Obs: se $\pi = \{ A + t\vec{u} + t'\vec{v} \mid t, t' \in \mathbb{R} \}$ então:
$d(\pi, A + \vec{w}) = \left| \frac{|[\vec{u}, \vec{v}, \vec{w}]|}{||\vec{u} \times \vec{v}||} \right| = \left| \frac{(\vec{u} \times \vec{v}) \cdot \vec{w}}{||\vec{u} \times \vec{v}||} \right| = \frac{|(\vec{u} \times \vec{v}) \cdot \vec{w}|}{||\vec{u} \times \vec{v}||}$.
Usando o $\vec{u}$ e o $\vec{v}$ do item 3d, temos $\vec{u} \times \vec{v} = \overrightarrow{(8,8,4)}$ e
$d(\pi, A + \vec{w}) = \frac{|\overrightarrow{(8,8,4)} \cdot \vec{w}|}{12} = \frac{|\overrightarrow{(2,2,1)} \cdot \vec{w}|}{3}$.
Isto também vale para planos paralelos a π:
se $\pi_\alpha = \{ (x,y,z) \in \mathbb{R}^3 \mid 2x+2y+z = \alpha \}$ e $(x_0, y_0, z_0) \in \pi_\alpha$ então
$d(\pi_\alpha, A_\alpha + \vec{w}) = \frac{|\overrightarrow{(2,2,1)} \cdot \vec{w}|}{3}$,
$d(\pi_\alpha, (x,y,z)) = \frac{|\overrightarrow{(2,2,1)} \cdot \overrightarrow{(x-x_0, y-y_0, z-z_0)}|}{3}$.