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II. Constructions on Categories

p.45: 6. Comma Categories

Fix C , $b \in C$. The category $(b \downarrow C)$ of *objects under b* has objects like $\langle f, c \rangle$, where $c \in C$ and $f : b \rightarrow c$.

Fix C , $a \in C$. The category $(C \downarrow a)$ of *objects over a* has objects like $\langle c, f \rangle$, where $c \in C$ and $f : c \rightarrow a$.

Fix C , D , $b \in C$. $S : D \rightarrow C$. The category $(b \downarrow S)$ of *objects S -under b* has objects like $\langle f, d \rangle$, where $d \in D$ and $f : b \rightarrow Sd$.

Fix C , E , $a \in C$. $T : E \rightarrow C$. The category $(T \downarrow a)$ of *objects T -over a* has objects like $\langle f, d \rangle$, where $d \in D$ and $f : b \rightarrow Sd$.

Fix C , D , E , and S , T with $E \xrightarrow{T} C \xleftarrow{S} D$. The *comma category* (T, S) has objects like $\langle e, d, f \rangle$, where $d \in D$, $e \in E$ and $f : Te \rightarrow Sd$.

An object $b \in C$ may be regarded as a functor $b : 1 \rightarrow C$ with image b .

A category C may be regarded as the identity functor $C \rightarrow C$.

We have:

$(b \downarrow C)$ has objects like $\langle *, c, f \rangle$, where $c \in C$ and $f : b \rightarrow c$.

$(C \downarrow a)$ has objects like $\langle c, *, f \rangle$, where $c \in C$ and $f : c \rightarrow a$.

$(b \downarrow S)$ has objects like $\langle *, d, f \rangle$, where $d \in D$ and $f : b \rightarrow Sd$.

$(T \downarrow a)$ has objects like

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III. Universals and Limits

p.55: Definition: universal arrow from c to S Fix $D, C, S : D \rightarrow C, r \in D, c \in C$.

Then we have functors

 $D(r, -) : D \rightarrow \mathbf{Set}$ and $C(c, S-) : D \rightarrow \mathbf{Set}$.

Every $u : c \rightarrow Sr$ induces a NT

$$\begin{array}{rcl} (S - \circ u) & : & D(r, -) \rightarrow C(c, S-), \\ (S - \circ u)d & : & D(r, d) \rightarrow C(c, Sd) \\ & & f' \mapsto Sf' \circ u. \end{array}$$
We say that a pair $\langle r, u \rangle$ is a *universal arrow from c to S* when $(S - \circ u)$ (i.e., $\lambda d. \lambda f'. Sf' \circ u$) is a natural isomorphism, i.e., when every $(S - \circ u)d$ (i.e., $\lambda f'. Sf' \circ u$) is a bijection.

$$D \xrightarrow{S} C$$

$$\begin{array}{ccc} & & c \\ & & \downarrow u \\ r & \xrightarrow{S} & Sr \\ f' \downarrow & & \downarrow Sf' \\ d & \xrightarrow{S} & Sd \\ & & \downarrow Sf' \circ u \end{array}$$

$$D(r, -) \xrightarrow[\cong]{(S - \circ u)} C(c, S-)$$

$$D(r, d) \xrightarrow[\cong]{(S - \circ u)d} C(c, Sd)$$

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III. Universals and Limits

p.58: Definition: universal arrow from S to c Fix $D, C, S : D \rightarrow C, r \in C, c \in C$.

Then we have functors

 $D(-, r) : D^{\text{op}} \rightarrow \mathbf{Set}$ and $C(S-, c) : D^{\text{op}} \rightarrow \mathbf{Set}$.

Every $v : Sr \rightarrow c$ induces a NT

$$\begin{array}{rcl} (v \circ S-) & : & D(-, r) \xrightarrow{\quad} C(S-, c), \\ (v \circ S-)d & : & D(d, r) \xrightarrow{\quad} C(Sd, c) \\ & & f' \mapsto v \circ Sf'. \end{array}$$
We say that a pair $\langle r, v \rangle$ is a *universal arrow from S to c* when $(v \circ S-)$ (i.e., $\lambda d. \lambda f'. v \circ Sf'$) is a natural isomorphism, i.e., when every $(v \circ S-)d$ (i.e., $\lambda f'. v \circ Sf'$) is a bijection.

$$\begin{array}{ccc} C & \xleftarrow{S} & D \\ \\ \begin{array}{ccc} \mathbf{S}d & \xleftarrow{S} & d \\ \downarrow Sf' & & \downarrow f' \\ \mathbf{S}r & \xleftarrow{S} & r \\ \downarrow v & & \\ c & & \end{array} & & \end{array}$$

$v \circ Sf'$ \downarrow

$$C(S-, c) \xleftarrow[\cong]{(S- \circ u)} D(-, r)$$

$$C(Sd, c) \xleftarrow[\cong]{(S- \circ u)d} D(d, r)$$

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III. Universals and Limits

p.57: universal element

Fix $D, H : D \rightarrow \mathbf{Set}$.A *universal element* of H is a pair $\langle r, e \rangle$

Then we have a functor

 $D(r, -) : D \rightarrow \mathbf{Set}$.Every $e \in Hr$, which can be seen as an arrow $e : * \rightarrow Hr$,...induces a NT $((H-)e) : D(r, -) \dot{\rightarrow} C(c, S-)?$,

$$\begin{aligned} ((H-)e)d & : D(r, d) \dot{\rightarrow} Hd \\ f & \mapsto (Hf)e? \end{aligned}$$

We say that a pair $\langle r, u \rangle$ is a *universal arrow from c to S* when $(S - \circ u)$ (i.e., $\lambda d. \lambda f'. Sf' \circ u$) is a natural isomorphism, i.e., when every $(S - \circ u)d$ (i.e., $\lambda f'. Sf' \circ u$) is a bijection.

$$D \xrightarrow{H} \mathbf{Set}$$

$$\begin{array}{ccc} & & * \\ & & \downarrow u \\ r & \xrightarrow{S} & Hr \\ f \downarrow & & \downarrow Hf \\ d & \xrightarrow{S} & Hd \end{array} \quad \begin{array}{c} \downarrow \\ \text{---} \\ \downarrow \\ \text{---} \\ \downarrow \end{array} \begin{array}{c} (Hf)e \\ \text{---} \\ x \end{array}$$

$$D(r, -) \xrightarrow[\cong]{(S - \circ u)} H$$

$$D(r, d) \xrightarrow[\cong]{(S - \circ u)d} Hd$$

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IV. Adjoints

p.79: Adjunctions

Fix $X, A, F : X \rightarrow A, G : A \rightarrow X$.

Then we have functors

 $A(F-, -) : X^{\text{op}} \times A \rightarrow \mathbf{Set}$ and $X(-, G-) : X^{\text{op}} \times A \rightarrow \mathbf{Set}$.An *adjunction from X to A* is a triple $\langle F, G, \varphi \rangle : X \rightarrow A$ where $\varphi : A(F-, -) \rightarrow X(-, G-)$ is a natural iso, i.e.,for all $x \in X, a \in A$ this is a bijection: $\varphi_{x,a} : A(Fx, a) \rightarrow X(x, Ga)$ and φ is natural in the sense that...

$$\begin{array}{c}
\langle F, G, \varphi \rangle : X \rightarrow A \\
\\
A \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{G} \end{array} X \\
\\
\begin{array}{ccc}
\begin{array}{ccc}
F x' & \xleftarrow{F} & x' \\
\downarrow Fh & & \downarrow h \\
F x & \xleftarrow{F} & x \\
\downarrow f & & \downarrow g \\
a & \xrightarrow{\varphi} & G a
\end{array} & \xrightarrow{h^* g := g \circ h} & \begin{array}{ccc}
A(F x', a) & \xrightarrow{\varphi_{x', a}} & X(x', G a) \\
\uparrow (Fh)^* & & \uparrow h^* \\
A(F x, a) & \xrightarrow{\varphi_{x, a}} & X(x, G a)
\end{array} \\
\downarrow (Fh)^* f := f \circ Fh & & \downarrow (Fh)^*
\end{array} & & \begin{array}{ccc}
(Fh)^* f & \xrightarrow{\varphi_{x', a}} & \varphi(f \circ Fh) \\
= f \circ Fh & & = \\
\uparrow (Fh)^* & & \uparrow h^* \\
f & \xrightarrow{\varphi_{x, a}} & \varphi f \\
& & \uparrow h^*
\end{array} \\
\\
\begin{array}{ccc}
\begin{array}{ccc}
F x & \xleftarrow{F} & x \\
\downarrow f & & \downarrow g \\
a & \xrightarrow{\varphi} & G a \\
\downarrow k & & \downarrow Gk \\
a' & \xrightarrow{G} & G a'
\end{array} & \xrightarrow{(Gk)_* g := Gk \circ g} & \begin{array}{ccc}
A(F x, a) & \xrightarrow{\varphi_{x, a}} & X(x, G a) \\
\downarrow k_* & & \downarrow (Gk)_* \\
A(F x, a') & \xrightarrow{\varphi_{x, a'}} & X(x, G a')
\end{array} \\
\downarrow k_* f := k \circ f & & \downarrow (Gk)_*
\end{array} & & \begin{array}{ccc}
f & \xrightarrow{\varphi_{x, a}} & \varphi f \\
\downarrow k_* & & \downarrow (Gk)_* \\
k_* f & \xrightarrow{\varphi_{x, a'}} & \varphi(k \circ f) \\
= k \circ f & & =
\end{array}
\end{array}$$

$$\begin{array}{c}
A(F-, -) \begin{array}{c} \xleftarrow{\varphi^{-1}} \\ \xrightarrow{\varphi} \end{array} X(-, G-) \\
\\
A(Fx, a) \begin{array}{c} \xleftarrow{\varphi_{x, a}^{-1}} \\ \xrightarrow{\varphi_{x, a}} \end{array} X(x, Ga)
\end{array}$$

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IV. Adjoints

p.79: Adjunctions - the naturality of φ Fix $X, A, F : X \rightarrow A, G : A \rightarrow X, \langle F, G, \varphi \rangle : X \dashv A$.

Remember that we have functors

 $A(F-, -) : X^{\text{op}} \times A \rightarrow \mathbf{Set}$ and $X(-, G-) : X^{\text{op}} \times A \rightarrow \mathbf{Set}$ and $\varphi : A(F-, -) \rightarrow X(-, G-)$ is a natural transformation (a natural iso)...Let $\langle h, k \rangle : \langle x, a \rangle \rightarrow \langle x', a' \rangle$ be a morphism in $X^{\text{op}} \times A$.The naturality of φ is easier to see in this diagram:

$$\begin{array}{c}
\langle F, G, \varphi \rangle : X \dashv A \\
\\
A \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{G} \end{array} X \\
\\
\begin{array}{ccc}
Fx' & \xleftarrow{F} & x' \\
Fh \downarrow & & \downarrow h \\
Fx & \xleftarrow{F} & x \\
f \downarrow & \xrightarrow{\varphi} & \downarrow \varphi f \\
a & \xrightarrow{G} & Ga \\
k \downarrow & & \downarrow Gk \\
a' & \xrightarrow{G} & Ga'
\end{array}
\end{array}
\begin{array}{c}
A(F-, -)\langle x, a \rangle \xrightarrow{\varphi_{\langle x, a \rangle}} X(-, G-)\langle x, a \rangle \\
A(F-, -)\langle x', a' \rangle \xrightarrow{\varphi_{\langle x', a' \rangle}} X(-, G-)\langle x', a' \rangle \\
\\
A(Fx, a) \xrightarrow{\varphi_{x, a}} X(x, Ga) \\
A(Fx', a') \xrightarrow{\varphi_{x', a'}} X(x', Ga') \\
\\
\begin{array}{ccc}
f & \xrightarrow{\quad} & \varphi f \\
\downarrow & & \downarrow \\
k \circ f \circ Fh & \xrightarrow{\quad} & \varphi(k \circ f \circ Fh)
\end{array}
\end{array}$$

$A(F-, -) \begin{array}{c} \xleftarrow{\varphi_{-1}} \\ \xrightarrow{\varphi_{-1}} \end{array} X(-, G-)$
 $A(Fx, a) \begin{array}{c} \xleftarrow{\varphi_{x, a}^{-1}} \\ \xrightarrow{\varphi_{x, a}} \end{array} X(x, Ga)$

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VI. Monads and Algebras

p.137: Monads

Fix $X, T : X \rightarrow X, \mu : T^2 \rightarrow T, \eta : I_X \rightarrow T$.

Then we can make a diagram:

$$\begin{array}{ccc}
 I & \xrightarrow{\eta} & T \xleftarrow{\mu} T^2 & & x & \xrightarrow{\eta x} & Tx \xleftarrow{\mu x} T^2 x \\
 \\
 T & \xrightarrow{T\eta} & T^2 \xleftarrow{T\mu} T^3 & & Tx & \xrightarrow{T(\eta x)} & T^2 x \xleftarrow{T(\mu x)} T^3 x \\
 \eta T \downarrow & \searrow \text{id} & \downarrow \mu & & \eta(Tx) \downarrow & \searrow \text{id} & \downarrow \mu x \\
 T^2 & \xrightarrow{\mu} & T \xleftarrow{\mu} T^2 & & T^2 x & \xrightarrow{\mu x} & Tx \xleftarrow{\mu x} T^2 x \\
 & & \downarrow \mu T & & & & \downarrow \mu(Tx)
 \end{array}$$

A monad $T = \langle T, \eta, \mu \rangle$ in a category X is a triple as above that obeys $\mu \circ \eta T = I_X = \mu \circ T\eta$ and $\mu \circ T\mu = \mu \circ \mu T$.

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VI. Monads and Algebras

2. Algebras for a monad

p.140: T -algebrasFix X and a monad $T = \langle T, \eta, \mu \rangle$ in X .A T -algebra is a pair $\langle x, h \rangle$ with $x \in X$ and $h : Tx \rightarrow x$ that obeys $\text{id}_x = h \circ \eta_x$, $h \circ \mu_x = h \circ Th$:

$$\begin{array}{ccc}
 x & \xrightarrow{\eta_x} & Tx & \xleftarrow{\mu_x} & T^2x \\
 & \searrow \text{id} & \downarrow h & & \downarrow Th \\
 & & x & \xleftarrow{h} & Tx
 \end{array}
 \quad \text{or} \quad
 \begin{array}{ccc}
 x & \xrightarrow{\eta_x} & Tx & \xleftarrow{\mu_x} & T^2x \\
 \downarrow \text{id} & \nearrow h & & \nearrow Th & \\
 x & \xleftarrow{h} & Tx & &
 \end{array}$$

A morphism $f : \langle x, h \rangle \rightarrow \langle x', h' \rangle$ (in the category X^T of T -algebras) is a morphism $f : x \rightarrow x'$ obeying $f \circ h = h' \circ Tf$.

$$\begin{array}{ccc}
 x & \xleftarrow{h} & Tx \\
 f \downarrow & & \downarrow Tf \\
 x' & \xleftarrow{h'} & Tx'
 \end{array}$$

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VI. Monads and Algebras

First examples

Let M be a monoid.We will call its identity e and its elements a, b, c , etc.Multiplication in M will be written as ab .Let Q, R, S be (arbitrary) sets.Then $T = (\times M) : \mathbf{Set} \rightarrow \mathbf{Set}$ and $\langle \times M, \eta, \mu \rangle$ is a monad on \mathbf{Set} , where:

$$\eta S : S \rightarrow S \times M \quad \text{and} \quad \mu S : (S \times M) \times M \rightarrow S \times M$$

$$s \mapsto \langle s, e \rangle \quad \text{and} \quad \langle \langle s, a \rangle, b \rangle \mapsto \langle s, ab \rangle.$$

In λ -notation they are $\eta = \lambda S. \lambda s. \langle s, e \rangle$ and $\mu = \lambda S. \lambda \langle \langle s, a \rangle, b \rangle. \langle s, ab \rangle$.Note that the conditions on η and μ , that we gave abstractly as:

$$\begin{array}{ccccc} x & \xrightarrow{\eta x} & Tx & \xleftarrow{\mu x} & T^2x \\ & & \downarrow \eta(Tx) & \searrow \text{id} & \downarrow \mu x \\ Tx & \xrightarrow{T(\eta x)} & T^2x & \xleftarrow{T(\mu x)} & T^3x \\ & & \downarrow \eta(T^2x) & \searrow \text{id} & \downarrow \mu(T^2x) \\ T^2x & \xrightarrow{\mu x} & Tx & \xleftarrow{\mu x} & T^2x \end{array}$$

become:

$$\begin{array}{ccccc} q \vdash & \longrightarrow & \langle q, e \rangle & \quad \langle q, ab \rangle \longleftarrow & \vdash \langle \langle q, a \rangle, b \rangle \\ & & \downarrow & \searrow & \downarrow \\ \langle q, a \rangle \vdash & \longrightarrow & \langle \langle q, e \rangle, a \rangle & \quad \langle \langle q, ab \rangle, c \rangle \longleftarrow & \vdash \langle \langle \langle q, a \rangle, b \rangle, c \rangle \\ & & \downarrow & \searrow & \downarrow \\ \langle \langle q, a \rangle, e \rangle \vdash & \longrightarrow & \langle q, ae \rangle & \quad \langle q, (ab)c \rangle \longleftarrow & \vdash \langle q, a(bc) \rangle \\ & & \downarrow & \searrow & \downarrow \\ \langle \langle q, a \rangle, e \rangle \vdash & \longrightarrow & \langle q, ea \rangle & \quad \langle q, a(bc) \rangle \longleftarrow & \vdash \langle \langle q, a \rangle, bc \rangle \end{array}$$

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VI. Monads and Algebras

First examples (2)

Fix a monoid M and sets Q, R .

An *action of M on a set Q* is a map $h : Q \times M \rightarrow Q$
 $\langle q, a \rangle \mapsto qa$

obeying $q(ab) = (qa)b$ and $qe = q$.

An *action of M on a set R* is a map $h' : R \times M \rightarrow R$
 $\langle r, a \rangle \mapsto ra$

obeying $r(ab) = (ra)b$ and $re = r$.

Note that we don't write h or h' .