CWM2
II. Constructions on Categories
p.45: 6. Comma Categories

Fix C, $b \in C$. The category $(b \downarrow C)$ of *objects under* b has objects like $\langle f, c\rangle$, where $c \in C$ and $f : b \to c$.
Fix C, $a \in C$. The category $(C \downarrow a)$ of *objects over* a has objects like $\langle c, f\rangle$, where $c \in C$ and $f : c \to a$.
Fix C, D, $b \in C$. $S : D \to C$. The category $(b \downarrow S)$ of *objects* S*-under* b has objects like $\langle f, d\rangle$, where $d \in D$ and $f : b \to Sd$.
Fix C, E, $a \in C$. $T : E \to C$. The category $(T \downarrow a)$ of *objects* T*-over* a has objects like $\langle f, d\rangle$, where $d \in D$ and $f : b \to Sd$.
Fix C, D, E, and S, T with $E \stackrel{T}{\to} C \stackrel{S}{\leftarrow} D$. The *comma category* (T, S) has objects like $\langle e, d, f\rangle$, where $d \in D$, $e \in E$ and $f : Te \to Sd$.

An object $b \in C$ may be regarded as a functor $b : 1 \to C$ with image b.
A category C may be regarded as the identity functor $C \to C$.
We have:
$(b \downarrow C)$ has objects like $\langle *, c, f\rangle$, where $c \in C$ and $f : b \to c$.
$(C \downarrow a)$ has objects like $\langle c, *, f\rangle$, where $c \in C$ and $f : c \to a$.
$(b \downarrow S)$ has objects like $\langle *, d, f\rangle$, where $d \in D$ and $f : b \to Sd$.
$(T \downarrow a)$ has objects like

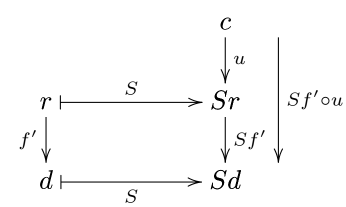
CWM2
III. Universals and Limits
p.55: Definition: universal arrow from c to S

Fix D, C, $S : D \to C$, $r \in D$, $c \in C$.
Then we have functors
$D(r, -) : D \to \mathbf{Set}$ and
$C(c, S-) : D \to \mathbf{Set}$.
Every $u : c \to Sr$ induces a NT

$$\begin{array}{rcccl} (S - \circ u) & : & D(r,-) & \dot{\to} & C(c, S-), \\ (S - \circ u)d & : & D(r,d) & \dot{\to} & C(c, Sd) \\ & & f' & \mapsto & Sf' \circ u. \end{array}$$

We say that a pair $\langle r, u \rangle$ is a *universal arrow from c to S*
when $(S - \circ u)$ (i.e., $\lambda d.\lambda f'.Sf' \circ u$) is a natural isomorphism, i.e., when every $(S - \circ u)d$ (i.e., $\lambda f'.Sf' \circ u$) is a bijection.

$$D \xrightarrow{S} C$$



$$D(r,-) \xrightarrow[(\cong)]{(S - \circ u)} C(c, S-)$$

$$D(r,d) \xrightarrow[(\cong)]{(S - \circ u)d} C(c, Sd)$$

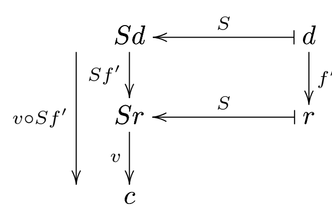
CWM2
III. Universals and Limits
p.58: Definition: universal arrow from S to c

Fix D, C, $S : D \to C$, $r \in C$, $c \in C$.
Then we have functors
$D(-, r) : D^{\mathrm{op}} \to \mathbf{Set}$ and
$C(S-, c,) : D^{\mathrm{op}} \to \mathbf{Set}$.
Every $v : Sr \to c$ induces a NT

$$\begin{aligned} (v \circ S-) &: D(-, r) \dot{\to} C(S-, c), \\ (v \circ S-)d &: D(d, r) \dot{\to} C(Sd, c) \\ f' &\mapsto v \circ Sf'. \end{aligned}$$

We say that a pair $\langle r, v \rangle$ is a *universal arrow from S to c*
when $(v \circ S-)$ (i.e., $\lambda d. \lambda f'. v \circ Sf'$) is a natural isomorphism, i.e., when every $(v \circ S-)d$ (i.e., $\lambda f'. v \circ Sf'$) is a bijection.

$$C \xleftarrow{S} D$$



$$C(S-, c) \xleftarrow[(\cong)]{(S-\circ u)} D(-, r)$$

$$C(Sd, c) \xleftarrow[(\cong)]{(S-\circ u)d} D(d, r)$$

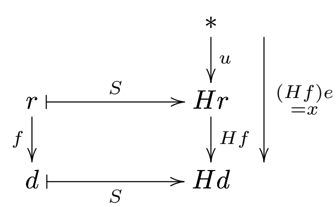
CWM2
III. Universals and Limits
p.57: universal element

Fix D, $H : D \to \mathbf{Set}$.
A *universal element of* H is a pair $\langle r, e \rangle$
Then we have a functor
$D(r, -) : D \to \mathbf{Set}$.
Every $e \in Hr$, which can be seen as an arrow $e : * \to Hr$,
...induces a NT

$$\begin{array}{rcrcl} ((H-)e) & : & D(r,-)? & \dot{\to} & C(c, S-)?, \\ ((H-)e)d & : & D(r,d) & \dot{\to} & Hd \\ & & f & \mapsto & (Hf)e?. \end{array}$$

We say that a pair $\langle r, u \rangle$ is a *universal arrow from* c *to* S
when $(S - \circ u)$ (i.e., $\lambda d.\lambda f'.Sf' \circ u$) is a natural isomorphism, i.e., when every $(S - \circ u)d$ (i.e., $\lambda f'.Sf' \circ u$) is a bijection.

$$D \xrightarrow{H} \mathbf{Set}$$



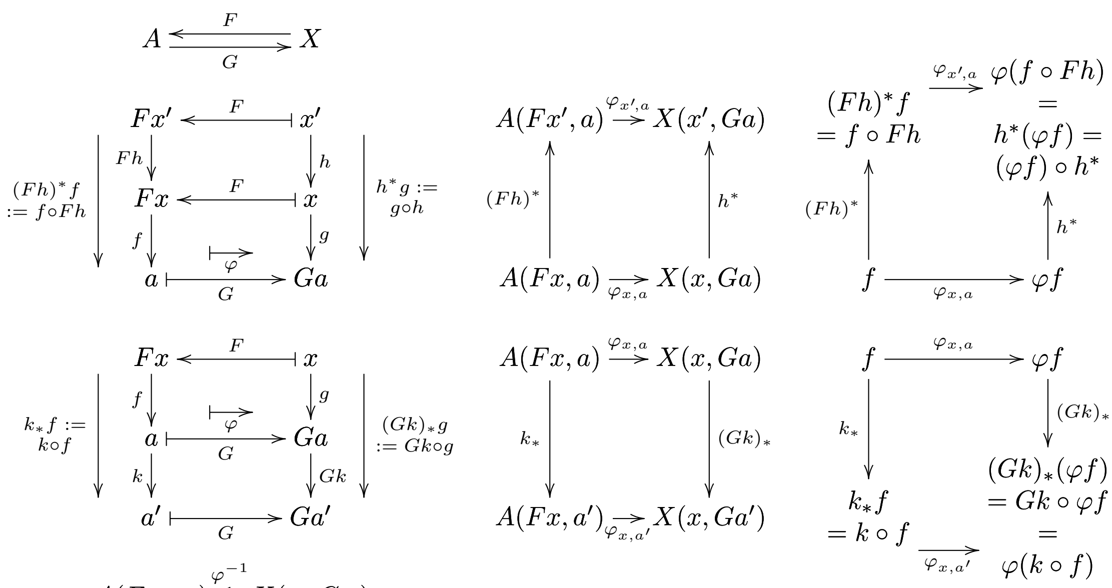
$$D(r,-) \xrightarrow[(\cong)]{(S-\circ u)} H$$

$$D(r,d) \xrightarrow[(\cong)]{(S-\circ u)d} Hd$$

CWM2
IV. Adjoints
p.79: Adjunctions

Fix X, A, $F : X \to A$, $G : A \to X$.
Then we have functors
$A(F-,-) : X^{\mathrm{op}} \times A \to \mathbf{Set}$ and
$X(-,G-) : X^{\mathrm{op}} \times A \to \mathbf{Set}$.
An *adjunction from X to A* is a triple $\langle F, G, \varphi \rangle : X \rightharpoonup A$
where $\varphi : A(F-,-) \to X(-,G-)$ is a natural iso, i.e.,
for all $x \in X$, $a \in A$ this is a bijection: $\varphi_{x,a} : A(Fx,a) \to X(x,Ga)$
and φ is natural in the sense that...

$\langle F, G, \varphi \rangle : X \rightharpoonup A$



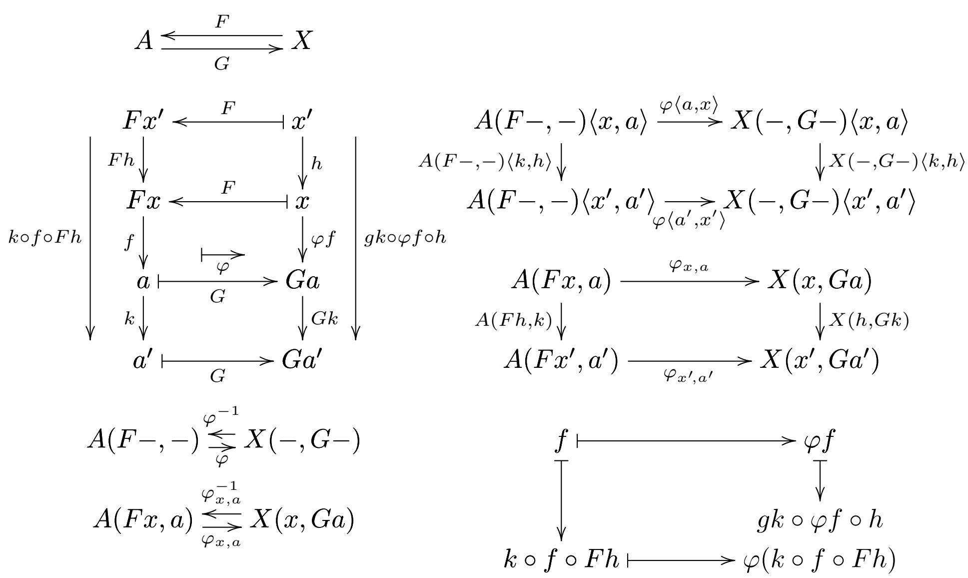
$$A(F-,-) \underset{\varphi}{\overset{\varphi^{-1}}{\rightleftarrows}} X(-,G-)$$

$$A(Fx,a) \underset{\varphi_{x,a}}{\overset{\varphi_{x,a}^{-1}}{\rightleftarrows}} X(x,Ga)$$

CWM2
IV. Adjoints
p.79: Adjunctions - the naturality of φ

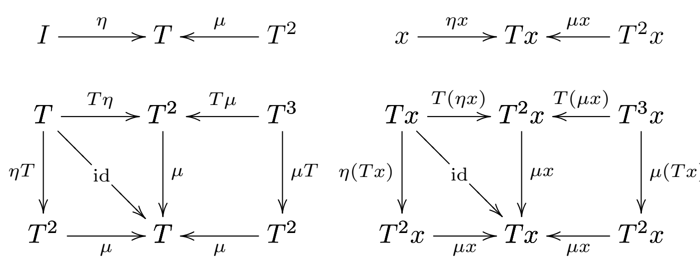
Fix X, A, $F : X \to A$, $G : A \to X$, $\langle F, G, \varphi\rangle : X \rightharpoonup A$.
Remember that we have functors
$A(F-,-) : X^{\text{op}} \times A \to \mathbf{Set}$ and
$X(-,G-) : X^{\text{op}} \times A \to \mathbf{Set}$
and $\varphi : A(F-,-) \to X(-,G-)$ is a natural transformation (a natural iso)...
Let $\langle h, k\rangle : \langle x, a\rangle \to \langle x', a'\rangle$ be a morphism in $X^{\text{op}} \times A$.
The naturality of φ is easier to see in this diagram:

$\langle F, G, \varphi\rangle : X \rightharpoonup A$



CWM2
VI. Monads and Algebras
p.137: Monads

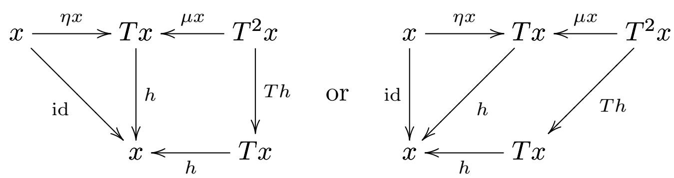
Fix X, $T : X \to X$, $\mu : T^2 \dot{\to} T$, $\eta : I_X \dot{\to} T$.
Then we can make a diagram:



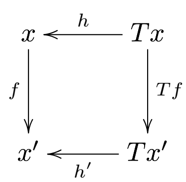
A *monad* $T = \langle T, \eta, \mu \rangle$ *in a category* X is a triple as above that obeys $\mu \circ \eta T = I_X = \mu \circ T\eta$ and $\mu \circ T\mu = \mu \circ \mu T$.

CWM2
VI. Monads and Algebras
2. Algebras for a monad
p.140: T-algebras

Fix X and a monad $T = \langle T, \eta, \mu \rangle$ in X.
A T-*algebra* is a pair $\langle x, h \rangle$ with $x \in X$ and $h : Tx \to x$
that obeys $\mathrm{id}_x = h \circ \eta x$, $h \circ \mu x = h \circ Th$:



A morphism $f : \langle x, h \rangle \to \langle x', h' \rangle$ (in the category X^T of T-algebras)
is a morphism $f : x \to x'$ obeying $f \circ h = h' \circ Tf$.

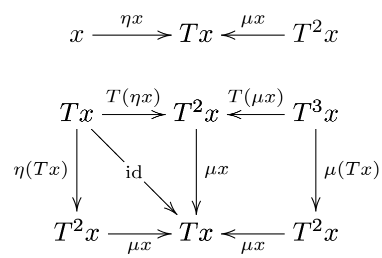


CWM2
VI. Monads and Algebras
First examples

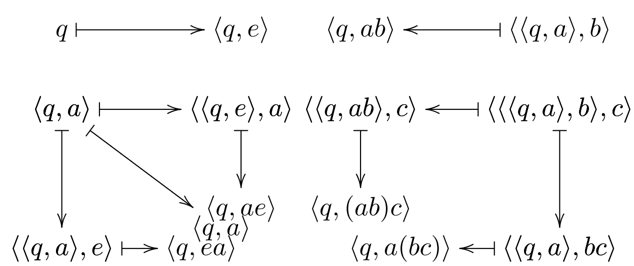
Let M be a monoid.
We will call its identity e and its elements a, b, c, etc.
Multiplication in M will be written as ab.
Let Q, R, S be (arbitrary) sets.
Then $T = (\times M) : \mathbf{Set} \to \mathbf{Set}$ and $\langle \times M, \eta, \mu \rangle$ is a monad on **Set**, where:

$$\begin{array}{rcccl} \eta S & : & S & \to & S \times M \\ & & s & \mapsto & \langle s, e \rangle \end{array} \quad \text{and} \quad \begin{array}{rcccl} \mu S & : & (S \times M) \times M & \to & S \times M \\ & & \langle \langle s, a \rangle, b \rangle & \mapsto & \langle s, ab \rangle. \end{array}$$

In λ-notation they are $\eta = \lambda S.\lambda s.\langle s, e \rangle$ and $\mu = \lambda S.\lambda \langle \langle s, a \rangle, b \rangle.\langle s, ab \rangle$.
Note that the conditions on η and μ, that we gave abstractly as:



become:



CWM2
VI. Monads and Algebras
First examples (2)

Fix a monoid M and sets Q, R.
An *action of* M *on a set* Q is a map $\begin{array}{rcrcl} h & : & Q \times M & \to & Q \\ & & \langle q, a \rangle & \mapsto & qa \end{array}$
obeying $q(ab) = (qa)b$ and $qe = q$.
An *action of* M *on a set* R is a map $\begin{array}{rcrcl} h' & : & R \times M & \to & R \\ & & \langle r, a \rangle & \mapsto & qa \end{array}$
obeying $r(ab) = (ra)b$ and $re = r$.
Note that we don't *write* h or h'.