Notes on notation: CWM

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http://angg.twu.net/LATEX/2017cwm.pdf
http://angg.twu.net/math-b.html#notes-on-notation

From http://angg.twu.net/math-b.html#idarct:

> Different people have different measures for "mental space"; someone with a good algebraic memory may feel that an expression like $\mathsf{Frob}^\natural : \Sigma_f(P \land f^*Q) \overset{\cong}{\longrightarrow} \Sigma_f P \land Q$ is easy to remember, while I always think diagramatically, and so what I do is that I remember this diagram [...] and I reconstruct the formula from it.

These are very informal notes showing my favourite ways to draw the "missing diagrams" in MacLane's *Categories for the Working Mathematician*, and my favourite choices of letters for them. Work in progress changing often, contributions and chats very welcome, etc. I am also doing something similar for parts of *Sketches of an Elephant* — see the link "#notes-on-notation" above.

The good parts are the one on Yoneda (pp.7–10), the "interdefinabilities" for some components of adjunctions (pp.16–17), and the part on monads (pp.18–21). The rest is a mess.

CWM2
II. Constructions on Categories
p.45: 6. Comma Categories (in my notation)

The most general case is with functors $\mathbf{A} \xrightarrow{F} \mathbf{B} \xleftarrow{G} \mathbf{C}$.
The comma category $(F \downarrow G)$ is

$$\begin{tikzcd}
A \arrow[r, mapsto] \arrow[d, "\alpha"'] & FA \arrow[r, "h"] \arrow[d, "F\alpha"'] & GC \arrow[d, "F\gamma"] & C \arrow[l, mapsto] \arrow[d, "\gamma"] & (A,h,C) \arrow[d, "{(\alpha,\gamma)}"] \\
A' \arrow[r, mapsto] & FA' \arrow[r, "h'"] & GC' & C' \arrow[l, mapsto] & (A',h',C') \\
\mathbf{A} \arrow[r, "F"] & \mathbf{B} \arrow[r, equal] & \mathbf{B} & \mathbf{C} \arrow[l, "G"'] & (F \downarrow G)
\end{tikzcd}$$

The obtain the other 8 cases I replace the functors F and G by $\mathrm{Id}_{\mathbf{B}}$ or S_B, where $\mathrm{S}_B : 1 \to \mathbf{B}$ is the functor that "selects" the object B — it takes the only object $\bullet \in 1$ to B. For example, $(\mathrm{S}_B, \mathrm{Id}_{\mathbf{B}})$ is:

$$\begin{tikzcd}
\bullet \arrow[r, mapsto] \arrow[d, equal] & B \arrow[r, "h"] \arrow[d, equal] & B' \arrow[d, "\beta"] & B' \arrow[l, mapsto] \arrow[d, "\beta"] & (\bullet,h,B') \arrow[d, "{(\mathrm{id}_\bullet,\beta)}"] \\
\bullet \arrow[r, mapsto] & B \arrow[r, "h'"] & B'' & B'' \arrow[l, mapsto] & (\bullet,h',B'') \\
\mathbf{A} \arrow[r, "\mathrm{S}_B"] & \mathbf{B} \arrow[r, equal] & \mathbf{B} & \mathbf{B} \arrow[l, "\mathrm{Id}_{\mathbf{B}}"'] & (\mathrm{S}_B \downarrow \mathrm{Id}_{\mathbf{B}})
\end{tikzcd}$$

Shorthands:
1) Use '_' in the pairs and triples in the positions where the information there is trivial — $(\mathrm{id}_\bullet, \beta) : (\bullet, h, B') \to (\bullet, h', B'')$ becomes $(\_, \beta) : (\_, h, B') \to (\_, h', B'')$.
2) Use B instead of S_B.
3) Use $\mathbf{B}$ instead of Id_B.

The correspondence with the names in CWM is:
The comma category $(F \downarrow G)$
The category $(B \downarrow \mathbf{B})$ of objects under B
The category $(\mathbf{B} \downarrow B)$ of objects over B
The category $(B \downarrow G)$ of objects G-under B
The category $(F \downarrow B)$ of objects F-over B

The nine cases:

$$\begin{array}{ccc}
(F \downarrow G) & (F \downarrow \mathbf{B}) & (F \downarrow B) \\
(\mathbf{B} \downarrow G) & (\mathbf{B} \downarrow \mathbf{B}) & (\mathbf{B} \downarrow B) \\
(B \downarrow G) & (B \downarrow \mathbf{B}) & (B \downarrow B')
\end{array}
\quad \Rightarrow \quad
\begin{array}{ccc}
(F \downarrow G) & (F \downarrow \mathrm{Id}_{\mathbf{B}}) & (F \downarrow \mathrm{S}_B) \\
(\mathrm{Id}_{\mathbf{B}} \downarrow G) & (\mathrm{Id}_{\mathbf{B}} \downarrow \mathrm{Id}_{\mathbf{B}}) & (\mathrm{Id}_{\mathbf{B}} \downarrow B) \\
(\mathrm{S}_B \downarrow G) & (\mathrm{S}_B \downarrow \mathrm{Id}_{\mathbf{B}}) & (\mathrm{S}_B \downarrow \mathrm{S}_{B'})
\end{array}$$

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II. Constructions on Categories
p.45: 6. Comma Categories

MacLane uses a notation with lots of names and shorthands.

Fix C, $b \in C$. The category $(b \downarrow C)$ of *objects under* b has
objects like $\langle f, c\rangle$, where $c \in C$ and $f : b \to c$.
Fix C, $a \in C$. The category $(C \downarrow a)$ of *objects over* a has
objects like $\langle c, f\rangle$, where $c \in C$ and $f : c \to a$.
Fix C, D, $b \in C$. $S : D \to C$. The category $(b \downarrow S)$ of *objects* S*-under* b has
objects like $\langle f, d\rangle$, where $d \in D$ and $f : b \to Sd$.
Fix C, E, $a \in C$. $T : E \to C$. The category $(T \downarrow a)$ of *objects* T*-over* a has
objects like $\langle f, d\rangle$, where $d \in D$ and $f : b \to Sd$.

Fix C, D, E, and S, T with $E \overset{T}{\to} C \overset{S}{\leftarrow} D$. The *comma category* (T, S) has
objects like $\langle e, d, f\rangle$, where $d \in D$, $e \in E$ and $f : Te \to Sd$.

An object $b \in C$ may be regarded as a functor $b : 1 \to C$ with image b.
A category C may be regarded as the identity functor $C \to C$.
We have:
$(b \downarrow C)$ has objects like $\langle *, c, f\rangle$, where $c \in C$ and $f : b \to c$.
$(C \downarrow a)$ has objects like $\langle c, *, f\rangle$, where $c \in C$ and $f : c \to a$.
$(b \downarrow S)$ has objects like $\langle *, d, f\rangle$, where $d \in D$ and $f : b \to Sd$.
$(T \downarrow a)$ has objects like

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III. Universals and Limits
p.55: Definition: universal arrow from c to S

Fix D, C, $S : D \to C$, $r \in D$, $c \in C$.
Then we have functors
$D(r, -) : D \to \mathbf{Set}$ and
$C(c, S-) : D \to \mathbf{Set}$.
Every $u : c \to Sr$ induces a NT

$$\begin{array}{rcrcl} (S - \circ u) & : & D(r,-) & \dot{\to} & C(c, S-), \\ (S - \circ u)d & : & D(r,d) & \dot{\to} & C(c, Sd) \\ & & f' & \mapsto & Sf' \circ u. \end{array}$$

We say that a pair $\langle r, u\rangle$ is a *universal arrow from c to S*
when $(S - \circ u)$ (i.e., $\lambda d.\lambda f'.Sf' \circ u$) is a natural isomorphism, i.e., when every $(S - \circ u)d$ (i.e., $\lambda f'.Sf' \circ u$) is a bijection.
In MacLane's and in my notation:

$$\begin{tikzcd}
 & c \arrow[d, "u"] \arrow[dd, bend left, "{f= Sf' \circ u}"] \\
r \arrow[r, mapsto, "S"] \arrow[d, "f'"'] & Sr \arrow[d, "Sf'"] \\
d \arrow[r, mapsto, "S"'] & Sd
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[d, "u"] \arrow[dd, bend left, "{g= u;R\beta}"] \\
B \arrow[r, mapsto] \arrow[d, "\beta"'] & RB \arrow[d, "R\beta"] \\
B' \arrow[r, mapsto] & RB'
\end{tikzcd}$$

$$D \xrightarrow{S} C \qquad \mathbf{B} \xrightarrow{R} \mathbf{A}$$

$$D(r,-) \xrightarrow[(\cong)]{(S-\circ u)} C(c, S-) \qquad (B,-) \longrightarrow (A, R-)$$

$$D(r,d) \xrightarrow[(\cong)]{(S-\circ u)d} C(c, Sd) \qquad (B,B') \longrightarrow (A, RB')$$

As comma categories (universal arrows are initial in comma categories):

$$\begin{tikzcd}
* \arrow[r, mapsto] \arrow[d, equal] & c \arrow[r, "u"] \arrow[d, equal] & Sr \arrow[d, "Sf'"] & r \arrow[l, mapsto] \arrow[d, "f'"] & \langle r, u\rangle \arrow[d, "f'"] \\
* \arrow[r, mapsto] & c \arrow[r, "f"] & Sd & d \arrow[l, mapsto] & \langle d, f\rangle \\
1 \arrow[r, "c"] & C \arrow[r, equal] & C & D \arrow[l, "S"'] & (c \downarrow S)
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, mapsto] \arrow[d, equal] & A \arrow[r, "u"] \arrow[d, equal] & RB \arrow[d, "R\beta"] & B \arrow[l, mapsto] \arrow[d, "\beta"] & (\_, u, B) \arrow[d, "(\_,\beta)"] \\
\bullet \arrow[r, mapsto] & A \arrow[r, "g"] & RB' & B' \arrow[l, mapsto] & (\_, g, B') \\
1 \arrow[r, "\mathrm{S}_A"] & \mathbf{A} \arrow[r, equal] & \mathbf{A} & \mathbf{B} \arrow[l, "R"'] & (\mathrm{S}_A \downarrow R)
\end{tikzcd}$$

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III. Universals and Limits
p.58: Definition: universal arrow from S to c

Fix D, C, $S : D \to C$, $r \in C$, $c \in C$.
Then we have functors
$D(-, r) : D^{\mathrm{op}} \to \mathbf{Set}$ and
$C(S-, c,) : D^{\mathrm{op}} \to \mathbf{Set}$.
Every $v : Sr \to c$ induces a NT

$$\begin{array}{rcrcl} (v \circ S-) & : & D(-, r) & \dot{\to} & C(S-, c), \\ (v \circ S-)d & : & D(d, r) & \dot{\to} & C(Sd, c) \\ & & f' & \mapsto & v \circ Sf'. \end{array}$$

We say that a pair $\langle r, v \rangle$ is a *universal arrow from S to c*
when $(v \circ S-)$ (i.e., $\lambda d. \lambda f'. v \circ Sf'$) is a natural isomorphism, i.e., when every $(v \circ S-)d$ (i.e., $\lambda f'. v \circ Sf'$) is a bijection.

$$\begin{tikzcd} C & & D \arrow[ll, "S"'] \end{tikzcd}$$

$$\begin{tikzcd} Sd \arrow[d, "Sf'"'] \arrow[dd, bend right=60, "v\circ Sf'"'] & & d \arrow[ll, mapsto, "S"'] \arrow[d, "f'"] \\ Sr \arrow[d, "v"'] & & r \arrow[ll, mapsto, "S"'] \\ c & & \end{tikzcd}$$

$$\begin{tikzcd} C(S-, c) & D(-, r) \arrow[l, "(S-\circ u)", "(\cong)"'] \end{tikzcd}$$

$$\begin{tikzcd} C(Sd, c) & D(d, r) \arrow[l, "(S-\circ u)d", "(\cong)"'] \end{tikzcd}$$

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III. Universals and Limits
p.57: universal element

Fix D, $H : D \to \mathbf{Set}$.
A *universal element of H* is a pair $\langle r, e\rangle$
Then we have a functor
$D(r, -) : D \to \mathbf{Set}$.
Every $e \in Hr$, which can be seen as an arrow $e : * \to Hr$,
...induces a NT

$$\begin{array}{rcrcl} ((H-)e) & : & D(r,-)? & \dot{\to} & C(c, S-)?, \\ ((H-)e)d & : & D(r,d) & \dot{\to} & Hd \\ & & f & \mapsto & (Hf)e?. \end{array}$$

We say that a pair $\langle r, u\rangle$ is a *universal arrow from c to S*
when $(S - \circ u)$ (i.e., $\lambda d.\lambda f'.Sf' \circ u$) is a natural isomorphism, i.e., when every $(S - \circ u)d$ (i.e., $\lambda f'.Sf' \circ u$) is a bijection.

$$\begin{tikzcd} D \arrow[r, "H"] & \mathbf{Set} \end{tikzcd}$$

$$\begin{tikzcd}
 & * \arrow[d, "u"] \arrow[dd, bend left, "\substack{(Hf)e\\=x}"] \\
r \arrow[r, mapsto, "S"] \arrow[d, "f"'] & Hr \arrow[d, "Hf"] \\
d \arrow[r, mapsto, "S"'] & Hd
\end{tikzcd}$$

$$\begin{tikzcd} D(r,-) \arrow[r, "(S-\circ u)", "(\cong)"'] & H \end{tikzcd}$$

$$\begin{tikzcd} D(r,d) \arrow[r, "(S-\circ u)d", "(\cong)"'] & Hd \end{tikzcd}$$

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III. Universals and Limits
p.59: 2. The Yoneda Lemma

The lemma behind Yoneda, in my notation

$$\begin{tikzcd} & A \arrow[d, "\eta"] \\ B \arrow[r, mapsto] & RB \end{tikzcd}$$

$$D\big\downarrow\big\uparrow U$$

$$(B,-) \xrightarrow[S]{} (A,R-)$$

There is a bijection between morphisms $\eta : A \to RB$
and natural transformations $S : (B,-) \to (A,R-)$.
D is $SCf := \eta; Rf$, i.e., $S := \lambda C.\lambda f.(\eta; Rf)$ and $D := \lambda\eta.\lambda C.\lambda f.(\eta; Rf)$.
U is $\epsilon := SB\mathrm{id}_B$, i.e., $U := \lambda S.SB\mathrm{id}_B$.
We want to check that $U(D\eta) = \eta$ and $D(US) = S$.
Using just (untyped) λ-calculus we can prove $U(D\eta) = \eta$ easily,
but the proof of $D(US) = S$ stops halfway...

$$\begin{aligned} U(D\eta) &= (\lambda S.SB(\mathrm{id}_B))((\lambda\eta.\lambda C.\lambda f.(\eta; Rf))(\eta)) \\ &= (\lambda S.SB(\mathrm{id}_B))(\lambda C.\lambda f.(\eta; Rf)) \\ &= (\lambda C.\lambda f.(\eta; Rf))B(\mathrm{id}_B) \\ &= (\lambda f.(\eta; Rf))(\mathrm{id}_B) \\ &= \eta; R(\mathrm{id}_B) \\ &= \eta; \mathrm{id}_{RB} \\ &= \eta \end{aligned}$$

$$\begin{aligned} D(US) &= (\lambda\eta.\lambda C.\lambda f.(\eta; Rf))(SB(\mathrm{id}_B)) \\ &= \lambda C.\lambda f.((SB(\mathrm{id}_B)); Rf) \end{aligned}$$

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III. Universals and Limits
p.59: 2. The Yoneda Lemma

The lemma behind Yoneda, in my notation (2)

We need the naturality (a.k.a. the "condition on squares") of S:

$$\begin{tikzcd}
& A \arrow[d, "\eta"] \arrow[dd, bend left=60, "h"] \\
B \arrow[r, mapsto] \arrow[d, "f"'] & RB \arrow[d, "Rf"] \\
C \arrow[r, mapsto] \arrow[d, "g"'] & RC \arrow[d, "Rg"] \\
D \arrow[r, mapsto] & RD
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[d, "g"'] & (B,C) \arrow[r, "SC"] \arrow[d, mapsto, "\lambda g.(f;g)"] & (A,RC) \arrow[d, mapsto, "\lambda h.(h;Rg)"] \\
D & (B,D) \arrow[r, "SD"] & (A,RD) \\
& (B,-) \arrow[r, "S"] & (A,R-)
\end{tikzcd}$$

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[dd, mapsto] & SCf \arrow[d, mapsto] \\
& (SCf);Rg \\
f;g \arrow[r, mapsto] & SD(f;g)
\end{tikzcd}$$

which yields $(SCf);Rg = SD(f;g)$. Substituting $\begin{bmatrix} C{:=}B \\ D{:=}C \\ f{:=}\mathrm{id}_B \\ g{:=}f \end{bmatrix}$ in that we get $(SB(\mathrm{id}_B));Rf = SC(\mathrm{id}_B;f)$, and so:

$$\begin{array}{rcl}
D(US) & = & (\lambda\eta.\lambda C.\lambda f.(\eta;Rf))(SB(\mathrm{id}_B)) \\
& = & \lambda C.\lambda f.((SB(\mathrm{id}_B));Rf) \\
& = & \lambda C.\lambda f.SC(\mathrm{id}_B;f) \\
& = & \lambda C.\lambda f.SCf \\
& = & S.
\end{array}$$

The last step can be explained as:

$$\begin{array}{rcl}
D(US)Cf & = & (\lambda C.\lambda f.SCf)Cf \\
& = & (\lambda f.SCf)f \\
& = & SCf
\end{array}$$

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III. Universals and Limits
p.59: 2. The Yoneda Lemma
p.61: Lemma (Yoneda).
Yoneda in my notation:

$$\begin{tikzcd}
 & A \arrow[d, "\alpha"] & & & 1 \arrow[d, "r"] & & & 1 \arrow[d, "f"] \\
C \arrow[r, mapsto] & RC & & C \arrow[r, mapsto] & RC & & C \arrow[r, mapsto] & (B,C) \\
\downarrow\uparrow & & \longrightarrow & & \downarrow\uparrow y & \longrightarrow & Y\downarrow\uparrow & \\
(C,-) \arrow[r, "T"'] & (A,R-) & & (C,-) \arrow[r] \arrow[dr, "T'"'] & (1,R-) \arrow[d, leftrightarrow] & & (C,-) \arrow[r] \arrow[dr, "f^*"'] & (1,(B,-)) \arrow[d, leftrightarrow] \\
 & & & & R & & & (B,-)
\end{tikzcd}$$

Left part:
Fix categories $\mathbf{A}$ and $\mathbf{C}$, a functor $R : \mathbf{C} \to \mathbf{A}$, and objects $A \in \mathbf{A}$, $C \in \mathbf{C}$.
We have functors $(C,-) : \mathbf{C} \to \mathbf{Set}$ and $(A, R-) : \mathbf{C} \to \mathbf{Set}$.
Each map $\alpha : A \to RC$ induces a NT $T : (C,-) \to (A, R-)$ and vice-versa.
The formulas are $T := \lambda D : \mathbf{C}.\lambda f : (C,D).(a; Rf)$ and $\alpha = T_C(\mathrm{id}_C)$,
and the '$\downarrow\uparrow$' is a bijection.

Middle part:
We take the left part and substitute $\mathbf{A} := \mathbf{Set}$ and $A := 1$.
The functor R becomes a functor from $\mathbf{C}$ to $\mathbf{Set}$.
There is a natural iso ('$\updownarrow$', unnamed) between the functors $(1, R-)$ and R.
We have a bijection between arrows $r : 1 \to RC$ (or elements of RC)
and natural transformations $T' : (C,-) \to R$.
The *Yoneda map* 'y' in '$\downarrow\uparrow y$' is a bijection $y : \mathrm{Nat}((C,-), R) \cong RC$.

Right part:
Choose an object $B \in \mathbf{C}$. Take the middle part and substitute $R := (B,-)$.
We get a bijection $Y\downarrow\uparrow$ between maps $f : B \to C$ and NTs
$f^* : (C,-) \to (B,-)$. The *Yoneda Functor* $Y : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathbf{C}}$ behaves as:

$$\begin{tikzcd}
B \arrow[d, "f"'] & B^{\mathrm{op}} \arrow[r, mapsto] & (B,-) \\
C & C^{\mathrm{op}} \arrow[r, mapsto] \arrow[u, "f^{\mathrm{op}}"] & (C,-) \arrow[u, "Yf"']
\end{tikzcd}$$

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III. Universals and Limits
p.59: 2. The Yoneda Lemma
p.61: Lemma (Yoneda).

$$\begin{tikzcd}
 & c \arrow[d, "u"] & & & * \arrow[d, "u"] & & & * \arrow[d, "f"] \\
r \arrow[r, mapsto] & Sr & & r \arrow[r, mapsto] & Kr & & r \arrow[r, mapsto] & D(s,r) \\
\downarrow\uparrow & & \longrightarrow & \downarrow\uparrow y & & \longrightarrow & Y\downarrow\uparrow & \\
D(r,-) \arrow[r, "T"'] & C(c,S-) & & D(r,-) \arrow[r] \arrow[dr, "T'"'] & \mathbf{Set}(*,K-) \arrow[d, leftrightarrow] & & D(r,-) \arrow[r] \arrow[dr, "{D(f,-)}"'] & \mathbf{Set}(*,D(s,-)) \arrow[d, leftrightarrow] \\
 & & & & K & & & D(s,-)
\end{tikzcd}$$

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III. Universals and Limits
p.59: 2. The Yoneda Lemma
Proposition 1

$$\begin{tikzcd}
D \arrow[r, "S"] & C
\end{tikzcd}$$

$$\begin{tikzcd}
 & c \arrow[d, "u"] \arrow[dd, bend left, "Sf'\circ u"] \\
r \arrow[r, mapsto, "S"] \arrow[d, "f'"'] & Sr \arrow[d, "Sf'"] \\
d \arrow[r, mapsto, "S" description] \arrow[d, "k"'] & Sd \arrow[d, "Sk"] \\
d' \arrow[r, mapsto, "S"'] & Sd'
\end{tikzcd}$$

$$\begin{tikzcd}
D(r,-) \arrow[r, "(S-\circ u)", "(\cong)"'] & C(c,S-)
\end{tikzcd}
\qquad
\begin{tikzcd}
D(r,-) \arrow[r, "\varphi", "(\cong)"'] & C(c,S-)
\end{tikzcd}$$

$$\begin{tikzcd}
D(r,d) \arrow[r, "(S-\circ u)d", "(\cong)"'] & C(c,Sd)
\end{tikzcd}$$

$$\begin{tikzcd}
D(r,r) \arrow[r, "(S-\circ u)r", "(\cong)"'] & C(c,Sr)
\end{tikzcd}
\qquad
\begin{tikzcd}
D(r,r) \arrow[r, "\varphi_r", "(\cong)"'] & C(c,Sr)
\end{tikzcd}$$

$$\begin{tikzcd}
1_r \arrow[r, mapsto, "(S-\circ u)r"] & \begin{array}{l} S1_r\circ u \\ =1_{Sr}\circ u \\ =u \end{array}
\end{tikzcd}
\qquad
\begin{tikzcd}
1_r \arrow[r, mapsto, "\varphi_r"] & \varphi_r(1_r)
\end{tikzcd}$$

$$\begin{tikzcd}
r \arrow[d, "\rho"'] & & \\
r \arrow[rr, mapsto, "D(r,-)"] \arrow[d, "f'"'] & & D(r,r) \arrow[d, "{D(r,f')=}" near start, "\lambda\rho.f'\circ\rho" near end] \\
d \arrow[rr, mapsto] \arrow[d, "k"'] & \arrow[r, phantom, "\longmapsto"', shift left=6] & D(r,d) \arrow[d, "{D(r,k)=}" near start, "\lambda f'.k\circ f'" near end] \\
d' \arrow[rr, mapsto] & \arrow[r, phantom, "\longmapsto"', shift left=6] & D(r,d')
\end{tikzcd}
\qquad
\begin{tikzcd}
r \arrow[rr, mapsto, "C(s{,}S-)"] \arrow[d, "f'"'] & & C(c,Sr) \arrow[d, "{C(c,Sf')=}" near start, "\lambda u.Sf'\circ u" near end] \\
d \arrow[rr, mapsto] \arrow[d, "k"'] & \arrow[r, phantom, "\longmapsto"', shift left=6] & C(c,Sd) \arrow[d, "{C(c,Sk)=}" near start, "\lambda g.Sk\circ g" near end] \\
d' \arrow[rr, mapsto] & \arrow[r, phantom, "\longmapsto"', shift left=6] & C(c,Sd')
\end{tikzcd}$$

$$\begin{tikzcd}
D(r,r) \arrow[r, "\varphi_r"] \arrow[d, "{D(r,f')}"'] & C(c,Sr) \arrow[d, "{C(c,Sf')}"] \\
D(r,d) \arrow[r, "\varphi_r"'] & C(c,Sd)
\end{tikzcd}$$

$$\begin{tikzcd}
1_r \arrow[r, mapsto, "\varphi_r"] \arrow[d, mapsto, "\lambda\rho.f'\circ\rho"'] & \varphi_r(1_r) \arrow[d, mapsto, "{C(c,Sf')}"] \\
f' \arrow[r, mapsto, "\varphi_r"'] & C(c,Sd)
\end{tikzcd}$$

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III. Universals and Limits
p.59: 2. The Yoneda Lemma
p.60: Proposition 2

$$\begin{tikzcd}D \arrow[r, "K"] & \mathbf{Set}\end{tikzcd}$$

$$\begin{tikzcd}
 & * \arrow[d, "u"] \\
r \arrow[r, mapsto] \arrow[d, "f'"'] & Kr \arrow[d, "Kf'"] \\
d \arrow[r, mapsto] & Kd
\end{tikzcd}$$

$$\begin{tikzcd}D(r,d) \arrow[r, "(K-\circ u)d", "(\cong)"'] & \mathbf{Set}(*,Kd) \arrow[r, "(\cong)"'] & Kd\end{tikzcd}$$

$$\begin{tikzcd}D(r,-) \arrow[r, "(K-\circ u)", "(\cong)"'] & \mathbf{Set}(*,K-) \arrow[r, "(\cong)"'] & K\end{tikzcd}$$

Yoneda: GF

$$\begin{array}{llrcllll}
f & : & A & \to & B & & & \\
 & & Nat(yB, F) & \mapsto & Nat(yA, F) & & & \\
 & & c & \mapsto & c \circ (f \circ -) & : & yA \to F & & \\
 & & & & c \circ (f \circ -)_C & : & yAC \to FC & & \\
 & & & & & : & \mathbf{C}(C, A) & \to & FC \\
 & & & & & & g & \mapsto & \mathbf{Cat}_c(f \circ g)
\end{array}$$

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IV. Adjoints
p.79: Adjunctions

Fix X, A, $F : X \to A$, $G : A \to X$.
Then we have functors
$A(F-,-) : X^{\mathrm{op}} \times A \to \mathbf{Set}$ and
$X(-,G-) : X^{\mathrm{op}} \times A \to \mathbf{Set}$.
An *adjunction from X to A* is a triple $\langle F, G, \varphi\rangle : X \rightharpoonup A$
where $\varphi : A(F-,-) \to X(-,G-)$ is a natural iso, i.e.,
for all $x \in X$, $a \in A$ this is a bijection: $\varphi_{x,a} : A(Fx, a) \to X(x, Ga)$
and φ is natural in the sense that...

$\langle F, G, \varphi\rangle : X \rightharpoonup A$

$$\begin{tikzcd}
A \arrow[r, shift right, "G"'] & X \arrow[l, shift right, "F"']
\end{tikzcd}$$

$$\begin{tikzcd}
Fx' \arrow[d, "Fh"'] \arrow[dd, bend right=60, "\substack{(Fh)^*f \\ := f\circ Fh}"'] & x' \arrow[l, mapsto, "F"'] \arrow[d, "h"] \arrow[dd, bend left=60, "\substack{h^*g := \\ g\circ h}"] \\
Fx \arrow[d, "f"'] & x \arrow[l, mapsto, "F"'] \arrow[d, "g"] \\
a \arrow[r, mapsto, "G"'] \arrow[r, phantom, shift left=3, "\overset{\mapsto}{\varphi}"] & Ga
\end{tikzcd}
\qquad
\begin{tikzcd}
A(Fx', a) \arrow[r, "\varphi_{x',a}"] & X(x', Ga) \\
A(Fx, a) \arrow[r, "\varphi_{x,a}"'] \arrow[u, "(Fh)^*"] & X(x, Ga) \arrow[u, "h^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\substack{(Fh)^*f \\ = f\circ Fh} \arrow[r, "\varphi_{x',a}"] & \substack{\varphi(f\circ Fh) \\ = \\ h^*(\varphi f) = \\ (\varphi f)\circ h^*} \\
f \arrow[r, "\varphi_{x,a}"'] \arrow[u, "(Fh)^*"] & \varphi f \arrow[u, "h^*"']
\end{tikzcd}$$

$$\begin{tikzcd}
Fx \arrow[d, "f"'] \arrow[dd, bend right=60, "\substack{k_*f := \\ k\circ f}"'] & x \arrow[l, mapsto, "F"'] \arrow[d, "g"] \arrow[dd, bend left=60, "\substack{(Gk)_*g \\ := Gk\circ g}"] \\
a \arrow[d, "k"'] \arrow[r, mapsto, "G"'] \arrow[r, phantom, shift left=3, "\overset{\mapsto}{\varphi}"] & Ga \arrow[d, "Gk"] \\
a' \arrow[r, mapsto, "G"'] & Ga'
\end{tikzcd}
\qquad
\begin{tikzcd}
A(Fx, a) \arrow[r, "\varphi_{x,a}"] \arrow[d, "k_*"'] & X(x, Ga) \arrow[d, "(Gk)_*"] \\
A(Fx, a') \arrow[r, "\varphi_{x,a'}"'] & X(x, Ga')
\end{tikzcd}
\qquad
\begin{tikzcd}
f \arrow[r, "\varphi_{x,a}"] \arrow[d, "k_*"'] & \varphi f \arrow[d, "(Gk)_*"] \\
\substack{k_*f \\ = k\circ f} \arrow[r, "\varphi_{x,a'}"'] & \substack{(Gk)_*(\varphi f) \\ = Gk\circ\varphi f \\ = \\ \varphi(k\circ f)}
\end{tikzcd}$$

$$\begin{tikzcd}
A(F-,-) \arrow[r, shift right, "\varphi"'] & X(-,G-) \arrow[l, shift right, "\varphi^{-1}"']
\end{tikzcd}$$

$$\begin{tikzcd}
A(Fx,a) \arrow[r, shift right, "\varphi_{x,a}"'] & X(x,Ga) \arrow[l, shift right, "\varphi_{x,a}^{-1}"']
\end{tikzcd}$$

CWM2
IV. Adjoints
p.79: Adjunctions - the naturality of φ

Fix X, A, $F : X \to A$, $G : A \to X$, $\langle F, G, \varphi\rangle : X \rightharpoonup A$.
Remember that we have functors
$A(F-,-) : X^{\mathrm{op}} \times A \to \mathbf{Set}$ and
$X(-,G-) : X^{\mathrm{op}} \times A \to \mathbf{Set}$,
and $\varphi : A(F-,-) \to X(-,G-)$ is a natural transformation (and a natural iso)...
Let $\langle h, k\rangle : \langle x, a\rangle \to \langle x', a'\rangle$ be a morphism in $X^{\mathrm{op}} \times A$.
The naturality of φ is easier to see in this diagram:

$\langle F, G, \varphi\rangle : X \rightharpoonup A$

$$\begin{tikzcd}
A \arrow[r, shift right, "G"'] & X \arrow[l, shift right, "F"']
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[ddd, "k\circ f\circ Fh"'] & Fx' \arrow[d, "Fh"'] & x' \arrow[l, mapsto, "F"'] \arrow[d, "h"] & {} \arrow[ddd, "gk\circ\varphi f\circ h"] \\
& Fx \arrow[d, "f"'] & x \arrow[l, mapsto, "F"'] \arrow[d, "\varphi f"] & \\
& a \arrow[r, mapsto, "\underset{\varphi}{\longmapsto}", "G"'] \arrow[d, "k"'] & Ga \arrow[d, "Gk"] & \\
{} & a' \arrow[r, mapsto, "G"'] & Ga' & {}
\end{tikzcd}$$

$$\begin{tikzcd}
A(F-,-)\langle x,a\rangle \arrow[r, "\varphi\langle a{,}x\rangle"] \arrow[d, "A(F-{,}-)\langle k{,}h\rangle"'] & X(-,G-)\langle x,a\rangle \arrow[d, "X(-{,}G-)\langle k{,}h\rangle"] \\
A(F-,-)\langle x',a'\rangle \arrow[r, "\varphi\langle a'{,}x'\rangle"'] & X(-,G-)\langle x',a'\rangle
\end{tikzcd}$$

$$\begin{tikzcd}
A(Fx,a) \arrow[r, "\varphi_{x,a}"] \arrow[d, "A(Fh{,}k)"'] & X(x,Ga) \arrow[d, "X(h{,}Gk)"] \\
A(Fx',a') \arrow[r, "\varphi_{x',a'}"'] & X(x',Ga')
\end{tikzcd}$$

$$\begin{tikzcd}
A(F-,-) \arrow[r, shift right, "\varphi"'] & X(-,G-) \arrow[l, shift right, "\varphi^{-1}"']
\end{tikzcd}$$

$$\begin{tikzcd}
A(Fx,a) \arrow[r, shift right, "\varphi_{x,a}"'] & X(x,Ga) \arrow[l, shift right, "\varphi_{x,a}^{-1}"']
\end{tikzcd}$$

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[dd, mapsto] & \varphi f \arrow[d, mapsto] \\
& gk\circ\varphi f\circ h \\
k\circ f\circ Fh \arrow[r, mapsto] & \varphi(k\circ f\circ Fh)
\end{tikzcd}$$

CWM2
IV. Adjoints
p.82: Adjunctions - interdefinabilities
(In MacLane's notation; unrevised)

$F \dashv G, \quad \langle F, G, \varphi\rangle : X \rightharpoonup A,$

$\mathcal{A} \underset{G}{\overset{F}{\leftrightarrows}} X, \quad A(F-,-) \underset{\varphi}{\overset{\varphi^{-1}}{\leftrightarrows}} X(-,G-).$

$$\begin{tikzcd}
Fx' \arrow[d, "Fh"'] & & x' \arrow[ll, mapsto] \arrow[d, "h"] \\
Fx \arrow[dd, "\substack{\varphi^{-1}g\\ f}"'] & & x \arrow[ll, mapsto] \arrow[dd, "\substack{g\\ \varphi f}"] \\
 & \substack{\varphi^{-1}\\ \longleftarrow\!\!\dashv\\ \longmapsto\\ \varphi} & \\
a \arrow[rr, mapsto] \arrow[d, "k"'] & & Ga \arrow[d, "Gk"] \\
a' \arrow[rr, mapsto] & & Ga'
\end{tikzcd}$$

$$\begin{tikzcd}
Fx \arrow[d, "\mathrm{id}_{Fx}"'] \arrow[drr, phantom, "\underset{\varphi}{\longmapsto}"] & & x \arrow[ll, mapsto] \arrow[d, "\substack{\eta x=\\ (\mathrm{id}_{Fx})^\sharp}"] \\
Fx \arrow[rr, mapsto] & & GFx
\end{tikzcd}
\qquad
\begin{tikzcd}
Fx \arrow[d, "Fg"'] \arrow[dd, bend right=60, "\substack{\varphi^{-1}g=\\ \epsilon_a\circ Fg}"'] & x \arrow[l, mapsto] \arrow[d, "g"] \\
FGa \arrow[d, "\epsilon_a"'] & Ga \arrow[l, mapsto] \\
a \arrow[ur, mapsto] &
\end{tikzcd}
\qquad
\begin{tikzcd}
Fx' \arrow[dd, "\substack{Fh=\\ \varphi^{-1}(\eta_x\circ h)}"'] & x' \arrow[l, mapsto] \arrow[d, "h"] \\
 & x \arrow[dl, mapsto] \arrow[d, "\eta_x"] \\
Fx \arrow[r, mapsto] & GFx
\end{tikzcd}$$

$$\begin{tikzcd}
FGa \arrow[d, "\substack{\epsilon_a=\\ \varphi^{-1}(\mathrm{id}_{Ga})}"'] \arrow[drr, phantom, "\substack{\varphi^{-1}\\ \longleftarrow\!\!\dashv}"] & & Ga \arrow[ll, mapsto, "\varphi^{-1}"'] \arrow[d, "\mathrm{id}_{Ga}"] \\
a \arrow[rr, mapsto] & & Ga
\end{tikzcd}
\qquad
\begin{tikzcd}
 & x \arrow[dl, mapsto] \arrow[d, "\eta_x"] \arrow[dd, bend left=60, "\substack{\varphi f=\\ Gf\circ\eta_x}"] \\
Fx \arrow[r, mapsto] \arrow[d, "f"'] & GFx \arrow[d, "Gf"] \\
a \arrow[r, mapsto] & Ga
\end{tikzcd}
\qquad
\begin{tikzcd}
FGa \arrow[r, mapsto] \arrow[d, "\epsilon_a"'] & Ga \arrow[dd, "\substack{Gk=\\ \varphi(k\circ\epsilon_a)}"] \\
a \arrow[ur, mapsto] \arrow[d, "k"'] & \\
a' \arrow[r, mapsto] & Ga'
\end{tikzcd}$$

Theorem 2. $\langle L, R, \sharp\rangle : \mathbf{A} \to \mathbf{B}$ is completely determined by:
(i) L, R, η, with each η_A universal
(ii) G, F_0 and universal arrows η_A
(iii) F, G, ϵ with each ϵ_a universal
(iv) F, G_0 and universal arrows ϵ_a
(v)

CWM2
IV. Adjoints
p.82: Adjunctions - interdefinabilities
(In my notation)

$L \dashv R, \quad \langle L, R, \sharp \rangle : \mathbf{A} \to \mathbf{B},$

$\mathbf{B} \underset{R}{\overset{L}{\leftrightarrows}} \mathbf{A}, \quad \mathbf{B}(L-,-) \underset{\sharp}{\overset{\flat}{\leftrightarrows}} \mathbf{A}(-,R-).$

$$\begin{tikzcd}
LA' \arrow[d, "L\alpha"'] & & A' \arrow[ll, mapsto] \arrow[d, "\alpha"] \\
LA \arrow[d, "\substack{g^\flat \\ f}"'] \arrow[drr, phantom, "\substack{\overset{\flat}{\longleftarrow\!\!\dashv} \\ \underset{\sharp}{\vdash\!\!\longrightarrow}}"] & & A \arrow[ll, mapsto] \arrow[d, "\substack{g \\ f^\sharp}"] \\
B \arrow[rr, mapsto] \arrow[d, "\beta"'] & & RB \arrow[d, "R\beta"] \\
B' \arrow[rr, mapsto] & & RB'
\end{tikzcd}$$

$$\begin{tikzcd}
LA \arrow[d, "\mathrm{id}_{LA}"'] \arrow[dr, phantom, "\underset{\sharp}{\vdash\!\!\longrightarrow}"] & A \arrow[l, mapsto] \arrow[d, "\substack{\eta_A= \\ (\mathrm{id}_{LA})^\sharp}"] \\
LA \arrow[r, mapsto] & RLA
\end{tikzcd}$$

$$\begin{tikzcd}
LA \arrow[d, "Lg"'] \arrow[dd, bend right=60, "\substack{g^\flat= \\ Lg;\epsilon_B}"'] & A \arrow[l, mapsto] \arrow[d, "g"] \\
LRB \arrow[d, "\epsilon_B"'] & RB \arrow[l, mapsto] \\
B \arrow[ur, mapsto] &
\end{tikzcd}$$

$$\begin{tikzcd}
LA' \arrow[dd, "\substack{L\alpha= \\ (\alpha;\eta_A)^\flat}"'] & A' \arrow[l, mapsto] \arrow[d, "\alpha"] \\
& A \arrow[dl, mapsto] \arrow[d, "\eta_A"] \\
LA \arrow[r, mapsto] & RLA
\end{tikzcd}$$

$$\begin{tikzcd}
LRB \arrow[d, "\substack{\epsilon_B= \\ (\mathrm{id}_{RB})^\beta}"'] \arrow[dr, phantom, "\longleftarrow\!\!\dashv"] & RB \arrow[l, mapsto] \arrow[d, "\mathrm{id}_{RB}"] \\
B \arrow[r, mapsto] & RB
\end{tikzcd}$$

$$\begin{tikzcd}
& A \arrow[dl, mapsto] \arrow[d, "\eta_A"] \arrow[dd, bend left=60, "\substack{f^\sharp= \\ \eta_A;Rf}"] \\
LA \arrow[r, mapsto] \arrow[d, "f"'] & RLA \arrow[d, "Rf"] \\
B \arrow[r, mapsto] & RB
\end{tikzcd}$$

$$\begin{tikzcd}
LRB \arrow[r, mapsto] \arrow[d, "\epsilon_B"'] & RB \arrow[dd, "\substack{R\beta= \\ \epsilon_B;\beta}"] \\
B \arrow[ur, mapsto] \arrow[d, "\beta"'] & \\
B' \arrow[r, mapsto] & RB'
\end{tikzcd}$$

Theorem 2. $\langle L, R, \sharp \rangle : \mathbf{A} \to \mathbf{B}$ is completely determined by:
(i) L, R, η, with each η_A universal
(ii) G, F_0 and universal arrows η_A
(iii) F, G, ϵ with each ϵ_a universal
(iv) F, G_0 and universal arrows ϵ_a
(v)

CWM2
VI. Monads and Algebras
p.137: Monads

Fix X, $T : X \to X$, $\mu : T^2 \dot{\to} T$, $\eta : I_X \dot{\to} T$.
Then we can make a diagram:

$$\begin{tikzcd}
I \arrow[r, "\eta"] & T & T^2 \arrow[l, "\mu"'] & x \arrow[r, "\eta x"] & Tx & T^2x \arrow[l, "\mu x"'] \\
T \arrow[r, "T\eta"] \arrow[d, "\eta T"'] \arrow[dr, "\mathrm{id}"] & T^2 \arrow[d, "\mu"] & T^3 \arrow[l, "T\mu"'] \arrow[d, "\mu T"] & Tx \arrow[r, "T(\eta x)"] \arrow[d, "\eta(Tx)"'] \arrow[dr, "\mathrm{id}"] & T^2x \arrow[d, "\mu x"] & T^3x \arrow[l, "T(\mu x)"'] \arrow[d, "\mu(Tx)"] \\
T^2 \arrow[r, "\mu"'] & T & T^2 \arrow[l, "\mu"] & T^2x \arrow[r, "\mu x"'] & Tx & T^2x \arrow[l, "\mu x"]
\end{tikzcd}$$

A *monad* $T = \langle T, \eta, \mu \rangle$ *in a category* X is a triple as above that obeys $\mu \circ \eta T = I_X = \mu \circ T\eta$ and $\mu \circ T\mu = \mu \circ \mu T$.

CWM2
VI. Monads and Algebras
2. Algebras for a monad
p.140: T-algebras

Fix X and a monad $T = \langle T, \eta, \mu \rangle$ in X.
A T-*algebra* is a pair $\langle x, h \rangle$ with $x \in X$ and $h : Tx \to x$
that obeys $\mathrm{id}_x = h \circ \eta x$, $h \circ \mu x = h \circ Th$:

$$\begin{tikzcd}
x \arrow[r, "\eta x"] \arrow[dr, "\mathrm{id}"'] & Tx \arrow[d, "h"] & T^2x \arrow[l, "\mu x"'] \arrow[d, "Th"] \\
 & x & Tx \arrow[l, "h"]
\end{tikzcd}$$

or

$$\begin{tikzcd}
x \arrow[r, "\eta x"] \arrow[d, "\mathrm{id}"'] & Tx \arrow[dl, "h"] & T^2x \arrow[l, "\mu x"'] \arrow[dl, "Th"] \\
x & Tx \arrow[l, "h"] &
\end{tikzcd}$$

A morphism $f : \langle x, h \rangle \to \langle x', h' \rangle$ (in the category X^T of T-algebras)
is a morphism $f : x \to x'$ obeying $f \circ h = h' \circ Tf$.

$$\begin{tikzcd}
x \arrow[d, "f"'] & Tx \arrow[l, "h"'] \arrow[d, "Tf"] \\
x' & Tx' \arrow[l, "h'"]
\end{tikzcd}$$

CWM2
VI. Monads and Algebras
First examples

Let M be a monoid.
We will call its identity e and its elements a, b, c, etc.
Multiplication in M will be written as ab.
Let Q, R, S be (arbitrary) sets.
Then $T = (\times M) : \mathbf{Set} \to \mathbf{Set}$ and $\langle \times M, \eta, \mu \rangle$ is a monad on $\mathbf{Set}$, where:

$$\begin{array}{ccccc} \eta S & : & S & \to & S \times M \\ & & s & \mapsto & \langle s, e \rangle \end{array} \quad \text{and} \quad \begin{array}{ccccc} \mu S & : & (S \times M) \times M & \to & S \times M \\ & & \langle \langle s, a \rangle, b \rangle & \mapsto & \langle s, ab \rangle. \end{array}$$

In λ-notation they are $\eta = \lambda S.\lambda s.\langle s, e \rangle$ and $\mu = \lambda S.\lambda \langle \langle s, a \rangle, b \rangle.\langle s, ab \rangle$.
Note that the conditions on η and μ, that we gave abstractly as:

$$\begin{tikzcd}
x \arrow[r, "\eta x"] & Tx & T^2 x \arrow[l, "\mu x"']
\end{tikzcd}$$

$$\begin{tikzcd}
Tx \arrow[r, "T(\eta x)"] \arrow[d, "\eta(Tx)"'] \arrow[dr, "\mathrm{id}" description] & T^2 x \arrow[d, "\mu x"] & T^3 x \arrow[l, "T(\mu x)"'] \arrow[d, "\mu(Tx)"] \\
T^2 x \arrow[r, "\mu x"'] & Tx & T^2 x \arrow[l, "\mu x"]
\end{tikzcd}$$

become:

$$\begin{tikzcd}
q \arrow[r, mapsto] & \langle q, e \rangle & \langle q, ab \rangle & \langle \langle q, a \rangle, b \rangle \arrow[l, mapsto]
\end{tikzcd}$$

$$\begin{tikzcd}
\langle q, a \rangle \arrow[r, mapsto] \arrow[ddd, mapsto] \arrow[ddr, mapsto] & \langle \langle q, e \rangle, a \rangle \arrow[d, mapsto] & \langle \langle q, ab \rangle, c \rangle \arrow[d, mapsto] & & \langle \langle \langle q, a \rangle, b \rangle, c \rangle \arrow[ll, mapsto] \arrow[ddd, mapsto] \\
& \langle q, ae \rangle & \langle q, (ab)c \rangle & & \\
& \langle q, a \rangle & & & \\
\langle \langle q, a \rangle, e \rangle \arrow[r, mapsto] & \langle q, ea \rangle & & \langle q, a(bc) \rangle & \langle \langle q, a \rangle, bc \rangle \arrow[l, mapsto]
\end{tikzcd}$$

CWM2
VI. Monads and Algebras
First examples (2)

Fix a monoid M and sets Q, R.
An *action of M on a set Q* is a map
$$\begin{array}{rcrcl} h & : & Q \times M & \to & Q \\ & & \langle q, a \rangle & \mapsto & qa \end{array}$$
obeying $q(ab) = (qa)b$ and $qe = q$.
An *action of M on a set R* is a map
$$\begin{array}{rcrcl} h' & : & R \times M & \to & R \\ & & \langle r, a \rangle & \mapsto & qa \end{array}$$
obeying $r(ab) = (ra)b$ and $re = r$.
Note that we don't *write* h or h'.

Kan extensions in my notation

Archetypal example: the functor $\Delta : \mathbf{Set} \to \mathbf{Set}^{\bullet\bullet}$
has both adjoints: $\mathrm{Colim} \dashv \Delta \dashv \mathrm{Lim}$.
I will refer to the unit η of $\mathrm{Colim} \dashv \Delta$ as injs
and to the counit ϵ of $\Delta \dashv \mathrm{Lim}$ as projs.

$$\begin{tikzcd}
(B\quad C) \arrow[d, "\mathrm{injs}_{(B\quad C)}"'] \\
(B{+}C\quad B{+}C) \\
(E{\times}F\quad E{\times}F) \arrow[d, "\mathrm{projs}_{(E\quad F)}"'] \\
(E\quad F)
\end{tikzcd}
\qquad
\begin{tikzcd}
(B\quad C) \arrow[r, mapsto] \arrow[d] & B{+}C \arrow[d] \\
(D\quad D) \arrow[d] & D \arrow[l, mapsto] \arrow[d] \\
(E\quad F) \arrow[r, mapsto] & E{\times}F
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{Set}^{\bullet\bullet} \arrow[r, shift left=2, "\mathrm{Colim}"] \arrow[r, shift right=2, "\mathrm{Lim}"'] & \mathbf{Set} \arrow[l, "\Delta" description]
\end{tikzcd}$$

$$\begin{tikzcd}
\bullet \arrow[r, mapsto] \arrow[d, equal] & (B\quad C) \arrow[r, "(i\quad i')"] \arrow[d, equal] & (B{+}C\quad B{+}C) \arrow[d] & B{+}C \arrow[l, mapsto] \arrow[d] \\
\bullet \arrow[r, mapsto] & (B\quad C) \arrow[r] & (D\quad D) & D \arrow[l, mapsto] \\
1 \arrow[r, "\mathrm{S}_{(B\quad C)}"] & \mathbf{Set}^{\bullet\bullet} \arrow[r, equal] & \mathbf{Set}^{\bullet\bullet} & \mathbf{Set} \arrow[l, "\Delta"']
\end{tikzcd}$$

$\Leftarrow$ $(\bullet, (i\quad i'), B{+}C)$
is an initial object in
$(\mathrm{S}_{(B\quad C)} \downarrow \Delta)$

$$\begin{tikzcd}
D \arrow[r, mapsto] \arrow[d, equal] & (D\quad D) \arrow[r] \arrow[d, equal] & (E\quad F) \arrow[d] & \bullet \arrow[l, mapsto] \arrow[d] \\
E{\times}F \arrow[r, mapsto] & (E{\times}F\quad E{\times}F) \arrow[r, "(\pi\quad \pi')"] & (E\quad F) & \bullet \arrow[l, mapsto] \\
\mathbf{Set} \arrow[r, "\Delta"] & \mathbf{Set}^{\bullet\bullet} \arrow[r, equal] & \mathbf{Set}^{\bullet\bullet} & 1 \arrow[l, "\mathrm{S}_{(E\quad F)}"']
\end{tikzcd}$$

$\Leftarrow$ $(E{\times}F, (\pi\quad \pi'), \bullet)$
is a terminal object in
$(\Delta \downarrow \mathrm{S}_{(E\quad F)})$

Kan extensions in my notation

Archetypal example: the functor $f^* : \mathbf{Set}^4 \to \mathbf{Set}^6$ has both adjoints: Colim $\dashv f^* \dashv$ Lim.
I will refer to the unit η of Colim $\dashv f^*$ as injs and to the counit ϵ of $f^* \dashv$ Lim as projs.

$$\begin{tikzcd}[ampersand replacement=\&]
\begin{pmatrix} & & C_2 \\ & & \downarrow \\ C_3 & \to & C_4 \\ \downarrow & & \\ C_5 & & \end{pmatrix} \arrow[r, mapsto] \arrow[d] \& \begin{pmatrix} 0 & \to & C_2 \\ \downarrow & & \downarrow \\ C_3 & \to & C_4 \\ \downarrow & & \downarrow \\ C_5 & \to & C_{\mathrm{po}} \end{pmatrix} \arrow[d] \& \begin{pmatrix} 0 & \to & D_2 \\ \downarrow & & \downarrow \\ D_3 & \to & D_4 \\ \downarrow & & \downarrow \\ D_5 & \to & D_{\mathrm{po}} \end{pmatrix} \arrow[d, "\mathrm{injs}_{(B\quad C)}"] \\
\begin{pmatrix} & & D_2 \\ & & \downarrow \\ D_3 & \to & D_4 \\ \downarrow & & \\ D_5 & & \end{pmatrix} \arrow[d] \& \begin{pmatrix} D_1 & \to & D_2 \\ \downarrow & & \downarrow \\ D_3 & \to & D_4 \\ \downarrow & & \downarrow \\ D_5 & \to & D_6 \end{pmatrix} \arrow[l, mapsto] \arrow[d] \& \begin{pmatrix} D_1 & \to & D_2 \\ \downarrow & & \downarrow \\ D_3 & \to & D_4 \\ \downarrow & & \downarrow \\ D_5 & \to & D_6 \end{pmatrix} \arrow[d, "\mathrm{projs}_{(E\quad F)}"] \\
\begin{pmatrix} & & E_2 \\ & & \downarrow \\ E_3 & \to & E_4 \\ \downarrow & & \\ E_5 & & \end{pmatrix} \arrow[r, mapsto] \& \begin{pmatrix} E_{\mathrm{pb}} & \to & E_2 \\ \downarrow & & \downarrow \\ E_3 & \to & E_4 \\ \downarrow & & \downarrow \\ E_5 & \to & 1 \end{pmatrix} \& \begin{pmatrix} D_{\mathrm{pb}} & \to & D_2 \\ \downarrow & & \downarrow \\ D_3 & \to & D_4 \\ \downarrow & & \downarrow \\ D_5 & \to & 1 \end{pmatrix}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{Set}^4 \arrow[r, shift left=2, "\mathrm{Colim}"] \arrow[r, shift right=2, "\mathrm{Lim}"'] & \mathbf{Set}^6 \arrow[l, "f^*" description]
\end{tikzcd}$$

Kan extensions in my notation

$$\begin{tikzcd}
SK \arrow[d, "\sigma K"'] \arrow[dd, bend right=50, "\alpha"'] & S \arrow[l, mapsto, "A^K"'] \arrow[d, "\sigma"] \\
RK \arrow[d, "\epsilon"'] & R \arrow[l, mapsto, "A^K"'] \arrow[r, equal] & \mathrm{Ran}_K T \\
T
\end{tikzcd}$$

$$\begin{tikzcd}
A^M & A^C \arrow[l, "A^K"']
\end{tikzcd}$$

$$\begin{tikzcd}
M \arrow[r, "K"] & C
\end{tikzcd}$$