Notes on notation: Elephant

Eduardo Ochs, 2017
Version at the bottom of the page.
eduardoochs@gmail.com

http://angg.twu.net/LATEX/2017elephant.pdf
http://angg.twu.net/math-b.html#notes-on-notation

All the extracts from page 3 onwards are from Peter Johstone's "Sketches of an Elephant", vol.1, sections A1 ("Regular and Cartesian Closed Categories") and A4 ("Geometric Morphisms - Basic Theory"). They are interspersed with my notes about what are the "missing diagrams" in the book; the idea of "missing diagrams" is explained here:
http://angg.twu.net/math-b.html#logic-for-children-unilog-2018
http://angg.twu.net/LATEX/2017vichy-workshop.pdf
http://www.uni-log.org/wk6-logic-for-children.html

The diagrams in the first pages are for the third paper in this series:
http://angg.twu.net/math-b.html#zhas-for-children-2

Surjections (defined in A4.2.6(iv)),
inclusions (defined in A4.2.9),
and some examples:

$$
\begin{tikzcd}
f^*D \arrow[d] & D \arrow[l, mapsto] \arrow[d] & D \arrow[d, "\eta", "\text{(monic)}"'] \\
E \arrow[r, mapsto] & f_*E & f_*f^*D \\
\mathcal{F} \arrow[r, "f", "\text{(surj)}"'] & \mathcal{E} &
\end{tikzcd}
\qquad
\begin{tikzcd}
h^*h_*E \arrow[d, "\epsilon"', "\text{(iso)}"] & h^*D \arrow[d] & D \arrow[l, mapsto] \arrow[d] \\
E & E \arrow[r, mapsto] & h_*E \\
& \mathcal{F} \arrow[r, "h", "\text{(incl)}"'] & \mathcal{E}
\end{tikzcd}
$$

$$
\begin{tikzcd}
(D_{12} \quad D_{12}) \arrow[d] & D_{12} \arrow[l, mapsto] \arrow[d] & D_{12} \arrow[d, "\eta", "\text{(monic)}"'] \\
(E_1 \quad E_2) \arrow[r, mapsto] & E_1 \times E_2 & D_{12} \times D_{12} \\
\mathbf{Set}^{(1 \quad 2)} \arrow[r, "f", "\text{(surj)}"'] & \mathbf{Set}^{(12)} &
\end{tikzcd}
\qquad
\begin{tikzcd}
(E_6) \arrow[d, "\epsilon"', "\text{(iso)}"] & (D_6) \arrow[d] & (D_5 \to D_6 \to D_7) \arrow[l, mapsto] \arrow[d] \\
(E_6) & (E_6) \arrow[r, mapsto] & (E_6 \to E_6 \to 1) \\
& \mathbf{Set}^{(6)} \arrow[r, "h", "\text{(incl)}"'] & \mathbf{Set}^{(5 \to 6 \to 7)}
\end{tikzcd}
$$

$$
\begin{tikzcd}
(D_3 \quad D_4) \arrow[d] & (D_3 \to D_4) \arrow[l, mapsto] \arrow[d] & (D_3 \to D_4) \arrow[d, "\eta", "\text{(monic)}"'] \\
(E_3 \quad E_4) \arrow[r, mapsto] & (E_3 \times E_4 \to E_4) & (D_3 \times D_4 \to D_4) \\
\mathbf{Set}^{(3 \quad 4)} \arrow[r, "f", "\text{(surj)}"'] & \mathbf{Set}^{(3 \to 4)} &
\end{tikzcd}
\qquad
\begin{tikzcd}
(E_8) \arrow[d, "\epsilon"', "\text{(iso)}"] & (D_8) \arrow[d] & (D_8 \quad D_9) \arrow[l, mapsto] \arrow[d] \\
(E_8) & (E_8) \arrow[r, mapsto] & (E_8 \quad 1) \\
& \mathbf{Set}^{(8)} \arrow[r, "h", "\text{(incl)}"'] & \mathbf{Set}^{(8 \quad 9)}
\end{tikzcd}
$$

The factorization (theorem A4.2.10),
and an example:

$$
\mathbf{Set}^{\left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \\ 5 & \end{smallmatrix}\right)} \xrightarrow{\qquad\qquad f \qquad\qquad} \mathbf{Set}^{\left(\begin{smallmatrix} 12 & & \\ 3 & \to & 4 \\ 5 & & 6 \end{smallmatrix}\right)}
$$

$$
\mathbf{Set}^{\left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \\ 5 & \end{smallmatrix}\right)} \xrightarrow[\text{(surj)}]{g} \mathbf{Set}^{\left(\begin{smallmatrix} 12 & & \\ 3 & \to & 4 \\ 5 & & \end{smallmatrix}\right)} \xrightarrow[\text{(incl)}]{h} \mathbf{Set}^{\left(\begin{smallmatrix} 12 & & \\ 3 & \to & 4 \\ 5 & & 6 \end{smallmatrix}\right)}
$$

$$
\mathbf{Set}^{\left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \\ & 6 \end{smallmatrix}\right)} \xrightarrow{\qquad\qquad\qquad\qquad f \qquad\qquad\qquad\qquad} \mathbf{Set}^{\left(\begin{smallmatrix} 12 & & & & \\ 3 & \to & 4 & & \\ 5 & \to & 6 & \to & 7 \end{smallmatrix}\right)}
$$

$$
\mathbf{Set}^{\left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \\ & 6 \end{smallmatrix}\right)} \xrightarrow[\text{(surj)}]{g} \mathbf{Set}^{\left(\begin{smallmatrix} 12 & & \\ 3 & \to & 4 \\ & & 6 \end{smallmatrix}\right)} \xrightarrow[\qquad\qquad \text{(incl)} \qquad\qquad]{h} \mathbf{Set}^{\left(\begin{smallmatrix} 12 & & & & \\ 3 & \to & 4 & & \\ 5 & \to & 6 & \to & 7 \end{smallmatrix}\right)}
$$

$$
\mathbf{Set}^{\left(\begin{smallmatrix} 12 & & \\ 3 & \to & 4 \\ & & 6 \end{smallmatrix}\right)} \xrightarrow[\text{(dense)}]{i} \mathbf{Set}^{\left(\begin{smallmatrix} 12 & & & & \\ 3 & \to & 4 & & \\ & & 6 & \to & 7 \end{smallmatrix}\right)} \xrightarrow[\text{(closed)}]{j} \mathbf{Set}^{\left(\begin{smallmatrix} 12 & & & & \\ 3 & \to & 4 & & \\ 5 & \to & 6 & \to & 7 \end{smallmatrix}\right)}
$$

$\mathbf{Set}^{(6)} \to \mathbf{Set}^{(5\to 6\to 7)}$:
$\eta cD : cD \to g_* g^* cD$ is $(D_6 \to D_6 \to D_6) \to (D_6 \to D_6 \to 1)$ (not monic)
$\mathrm{Ran}_g(0) = (0 \to 0 \to 0)$ (not initial)

$$\begin{tikzcd}
(E_6) \arrow[d, "\epsilon"', "(\mathrm{iso})"'] & (D_6) \arrow[d] & (D_5 \to D_6 \to D_7) \arrow[l, mapsto] \arrow[d] & (D_6 \to D_6 \to D_6) \arrow[d, "\eta", "\text{not monic} =("'] \\
(E_6) & (E_6) \arrow[r, mapsto] & (E_6 \to E_6 \to 1) & (D_6 \to D_6 \to 1)
\end{tikzcd}$$

$$(0) \xrightarrow[=(]{} (0 \to 0 \to 1)$$

$$\mathbf{Set}^{(6)} \xrightarrow[(\mathrm{incl})]{h} \mathbf{Set}^{(5\to 6\to 7)}$$

$\mathbf{Set}^{(6)} \to \mathbf{Set}^{(5\to 6)}$:
$\eta cD : cD \to g_* g^* cD$ is $(D_6 \to D_6) \to (D_6 \to D_6)$ (monic)
$\mathrm{Ran}_g(0) = (0 \to 0)$ (initial)

$$\begin{tikzcd}
(E_6) \arrow[d, "\epsilon"', "(\mathrm{iso})"'] & (D_6) \arrow[d] & (D_5 \to D_6) \arrow[l, mapsto] \arrow[d] & (D_6 \to D_6) \arrow[d, "\eta", "(\mathrm{monic})"'] \\
(E_6) & (E_6) \arrow[r, mapsto] & (E_6 \to E_6) & (D_6 \to D_6)
\end{tikzcd}$$

$$(0) \xrightarrow[=)]{} (0 \to 0)$$

$$\mathbf{Set}^{(6)} \xrightarrow[(\mathrm{incl})]{h} \mathbf{Set}^{(5\to 6)}$$

$\mathbf{Set}^{(6)} \to \mathbf{Set}^{(6\to 7)}$:
$\eta cD : cD \to g_* g^* cD$ is $(D_6 \to D_6) \to (D_6 \to D_6)$ (monic)
$\mathrm{Ran}_g(0) = (0 \to 0)$ (initial)

$$\begin{tikzcd}
(E_6) \arrow[d, "\epsilon"', "(\mathrm{iso})"'] & (D_6) \arrow[d] & (D_6 \to D_7) \arrow[l, mapsto] \arrow[d] & (D_6 \to D_7) \arrow[d, "\eta", "\text{not monic} =("'] \\
(E_6) & (E_6) \arrow[r, mapsto] & (E_6 \to 1) & (D_6 \to 1)
\end{tikzcd}$$

$$(0) \xrightarrow[=(]{} (0 \to 1)$$

$$\mathbf{Set}^{(6)} \xrightarrow[(\mathrm{incl})]{h} \mathbf{Set}^{(6\to 7)}$$

$\mathbf{Set}^{(5\to 6)} \to \mathbf{Set}^{(5\to 6\to 7)}$:
$\eta c D : cD \to g_* g^* cD$ is $(D_6 \to D_6 \to D_6) \to (D_6 \to D_6 \to 1)$ (not monic)
$\mathrm{Ran}_g(0 \to 0) = (0 \to 0 \to 1)$ (not initial)

$$\begin{tikzcd}
(E_5 \to E_6) \arrow[d, "\epsilon"', "\text{(iso)}"'] & (D_5 \to D_6) \arrow[d] & (D_5 \to D_6 \to D_7) \arrow[l, mapsto] \arrow[d] & (D_6 \to D_6 \to D_6) \arrow[d, "\eta", "\text{not monic} =("'] \\
(E_5 \to E_6) & (E_5 \to E_6) \arrow[r, mapsto] & (E_6 \to E_6 \to 1) & (D_6 \to D_6 \to 1)
\end{tikzcd}$$

$$(0 \to 0) \xrightarrow[=(]{} (0 \to 0 \to 1)$$

$$\mathbf{Set}^{(5\to 6)} \xrightarrow[\text{(incl)}]{h} \mathbf{Set}^{(5\to 6\to 7)}$$

$\mathbf{Set}^{(6\to 7)} \to \mathbf{Set}^{(5\to 6\to 7)}$:
$\eta c D : cD \to g_* g^* cD$ is $(D_6 \to D_6 \to D_6)(D_6 \to D_6 \to D_6)$ (monic)
$\mathrm{Ran}_g(0 \to 0) = (0 \to 0 \to 0)$ (initial)

$$\begin{tikzcd}
(E_6 \to E_7) \arrow[d, "\epsilon"', "\text{(iso)}"'] & (D_6 \to D_7) \arrow[d] & (D_5 \to D_6 \to D_7) \arrow[l, mapsto] \arrow[d] & (D_6 \to D_6 \to D_6) \arrow[d, "\eta", "\text{monic}"'] \\
(E_6 \to E_7) & (E_6 \to E_7) \arrow[r, mapsto] & (E_6 \to E_6 \to E_7) & (D_6 \to D_6 \to D_6)
\end{tikzcd}$$

$$(0 \to 0) \xrightarrow[=)]{} (0 \to 0 \to 0)$$

$$\mathbf{Set}^{(6\to 7)} \xrightarrow[\text{(incl)}]{h} \mathbf{Set}^{(5\to 6\to 7)}$$

Dense-closed factorization (A4.5.20)

A geometric morphism f is an inclusion when all counit maps $\epsilon E : f^* f_* E \to E$ are isos (A4.2.9); a geometric inclusion is dense exactly when all the unit maps on constant presheaves, $\eta cD : cD \to g_* g^* cD$, are monics, and closed when all the counit maps, $\epsilon E : h^* h_* E \to E$, are isos (Peter Arndt, 5.pdf, p.8)...

Problems (Peter, help, please!):

1) I can't find these characterizations of dense and closed in the Elephant,

2) "closed inclusion" *should be* stricter than "inclusion"!...

http://angg.twu.net/LATEX/5.pdf

$$
\begin{tikzcd}
f^* f_* E \arrow[d, "\epsilon"', "\text{(iso)}"'] & f^* D \arrow[d] & & D \arrow[ll, mapsto] \arrow[d] \\
E & E \arrow[rr, mapsto] & & f_* E \\
& \mathcal{E}' \arrow[rr, "f", "\text{(inclusion)}"'] & & \mathcal{E}
\end{tikzcd}
$$

$$
\begin{tikzcd}
g^* D \arrow[d] & D \arrow[l, mapsto] \arrow[d] & cD \arrow[d, "\eta", "\text{(monic)}"'] \\
E \arrow[r, mapsto] & g_* E & g_* g^* cD \\
\mathcal{E}' \arrow[r, "g", "\text{(dense)}"'] & \mathcal{E}'' &
\end{tikzcd}
\qquad
\begin{tikzcd}
h^* D \arrow[d] & D \arrow[l, mapsto] \arrow[d] \\
E \arrow[r, mapsto] & h_* E \\
0 \arrow[r, mapsto] & h_* 0 = 0 \\
\mathcal{E}'' \arrow[r, "h", "\text{(closed)}"'] & \mathcal{E}
\end{tikzcd}
$$

A1.1 Preliminary assumptions

(...)

A *full* subcategory, of course, is one whose inclusion functor is full; but when dealing with subcategories we shall generally assume (sometimes without saying so explicitly) that they are also *replete*, i.e., that any object of the ambient category isomorphic to one in the subcategory is itself in the subcategory. The full subcategories of $\mathcal{C}$ correspond to classes of objects of $\mathcal{C}$ which are closed under isomorphism. In particular, for us a *reflective* subcategory will always mean a full, replete subcategory whose inclusion functor has a left adjoint.

We use the term *reflection* for an adjunction whose right adjoint is full and faithful, and *reflector* for a monad which is idempotent (i.e., one whose multiplication is an isomorphism); it is well known that these three concepts are essentially the same. The following, related, result seems not to be widely known, however; and since we shall need it occasionally, we sketch its proof here.

Lemma 1.1.1 Let $F : \mathcal{C} \to \mathcal{D}$ be a functor having a right adjoint G. If there is any natural isomorphism (nor necessarily the counit of the adjunction) between FG and the identity functor on $\mathcal{D}$, then $(F \dashv G)$ is a reflection.

Reflective:

$$\begin{tikzcd} LD \arrow[d] & D \arrow[l, mapsto] \arrow[d] \\ S \arrow[r, mapsto] & S \\ \mathcal{S} \arrow[r, shift right, "(\mathrm{inc})"'] & \mathcal{D} \arrow[l, shift right, "L"'] \end{tikzcd}$$

Reflector:

$$\begin{tikzcd} C \arrow[r, "\eta C"] & GFC & GFGFC \arrow[l, "\mu C := G\epsilon FC"', "(\mathrm{iso})"] \end{tikzcd}$$

Reflection:

$$\begin{tikzcd} FGD \arrow[d, "\epsilon D"', "(\mathrm{iso})"] & GD \arrow[l, mapsto] \arrow[d] \\ D \arrow[r, mapsto] & GD \\ \mathcal{D} \arrow[r, shift right, "G"'] & \mathcal{C} \arrow[l, shift right, "F"'] \end{tikzcd}$$

Definition 4.1.1 (a) Let $\mathcal{E}$ and $\mathcal{F}$ be toposes. A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ consists of a pair of functors $f_* : \mathcal{F} \to \mathcal{E}$ (the direct image of f) and $f^* : \mathcal{E} \to \mathcal{F}$ (the inverse image of f) together with an adjunction $(f^* \dashv f_*)$, such that f^* is cartesian (i.e. preserves finite limits).

(b) Let f and $g : \mathcal{F} \to \mathcal{E}$ be geometric morphisms. A geometric transformation $\alpha : f \to g$ is defined to be a natural transformation $\alpha : f^* \to g^*$.

$$\begin{tikzcd}
f^*E \arrow[d] & E \arrow[l, mapsto] \arrow[d] \\
F \arrow[r, mapsto] & f_*F
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F} \arrow[r, shift right, "f_*"'] & \mathcal{E} \arrow[l, shift right, "f^*"']
\end{tikzcd}$$

$$\mathcal{F} \xrightarrow{f} \mathcal{E}$$

Example 4.1.4 Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between small categories. Then composition with f defines a functor $f^* : [\mathcal{D}, \mathbf{Set}] \to [\mathcal{C}, \mathbf{Set}]$, which has adjoints on both sides, the left and right *Kan extensions* along f: for example, the right Kan extension $\varprojlim_f$ sends a functor $F : \mathcal{C} \to \mathbf{Set}$ to the functor whose value at an object B of $\mathcal{D}$ is the limit of the diagram

$$(B \downarrow f) \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathbf{Set}$$

(here $(B \downarrow f)$ is the comma category whose objects are pairs (A, φ) with $\varphi : B \to fA$ in $\mathcal{D}$, and U is the forgetful functor from this category to $\mathcal{C}$). Thus f^* is the inverse image of a geometric morphism $[\mathcal{C}, \mathbf{Set}] \to [\mathcal{D}, \mathbf{Set}]$, whose direct image is $\varprojlim_f$.

$$\begin{tikzcd}
f^*D \arrow[d] & D \arrow[l, mapsto] \arrow[d] & \\
F \arrow[r, mapsto] & f_*F \arrow[r, equal] & \varprojlim_f F
\end{tikzcd}
\qquad (f_*F)(B) = \mathrm{Lim}((B \downarrow f) \xrightarrow{FU} \mathbf{Set})$$

$$\begin{tikzcd}
{[\mathcal{C}, \mathbf{Set}]} \arrow[r, shift right, "f_*"'] & {[\mathcal{D}, \mathbf{Set}]} \arrow[l, shift right, "f^*"']
\end{tikzcd}$$

$$\mathcal{C} \xrightarrow{f} \mathcal{D}$$

$$\begin{tikzcd}
(\bullet \mapsto B \overset{\varphi}{\to} fA \leftarrow\!\shortmid A) \arrow[d, "{(_,\alpha)}"] \arrow[r, mapsto] & A \arrow[r, mapsto] \arrow[d, "\alpha"] & FA \arrow[d, "F\alpha"] \\
(\bullet \mapsto B \overset{\varphi'}{\to} fA' \leftarrow\!\shortmid A') \arrow[r, mapsto] & A' \arrow[r, mapsto] & FA'
\end{tikzcd}$$

$$(1 \to \mathcal{D} = \mathcal{D} \leftarrow \mathcal{C})$$

$$(B \downarrow f) \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathbf{Set}$$

Moreover, any natural transformation $\alpha : f \to g$ between functors $\mathcal{C} \to \mathcal{D}$ induces a natural transformation $f^* \to g^*$ (whose value at F is the natural transformation $F\alpha : Ff \to Fg$), i.e. a geometric transformation $(\varprojlim_f, f^*) \to (\varprojlim_g, g^*)$. Thus the assignment $\mathcal{C} \mapsto [\mathcal{C}, \mathbf{Set}]$ can be made into a functor (that is, a 2-functor) from the 2-category $\mathfrak{Cat}$ of small categories, functors and natural transformations into $\mathfrak{Cat}$ (in fact into $\mathfrak{Cat}/\mathbf{Set}$).

(...)

We note that the geometric morphisms which arise as in 4.1.4, though not as special as those of 4.1.2, still have the property that their inverse image functors have left adjoints as well as right adjoints. We call a geometric morphism f *essential* if it has this property; we normally write $f_!$ for the left adjoint of f^*. With the aid of this notion, we can prove a partial converse to 4.1.4:

Lemma 4.1.5 Let $\mathcal{C}$ and $\mathcal{D}$ be small categories such that $\mathcal{D}$ is Cauchy-complete (cf. 1.1.10). Then every essential geometric morphism $f : [\mathcal{C}, \mathbf{Set}] \to [\mathcal{D}, \mathbf{Set}]$ is induced as in 4.1.4 by a functor $\mathcal{C} \to \mathcal{D}$.

Example 4.1.8 Let $(\mathcal{C}, T)$ be a small site, as defined in 2.1.9. The inclusion functor $\mathbf{Sh}(\mathcal{C}, T) \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ has a cartesian left adjoint (the *associated sheaf functor* — this is a special case of a result which we shall prove in 4.4.8 below), so it is the direct image of a geometric morphism.

$$\begin{tikzcd}
f^*P \arrow[d] & P \arrow[l, mapsto] \arrow[d] \\
S \arrow[r, mapsto] & f_*P
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{Sh}(\mathcal{C}, T) \arrow[r, shift right, "f_* \text{ (inc)}"'] & {[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]} \arrow[l, shift right, "f^* \text{ (asf)}"']
\end{tikzcd}$$

Example 4.1.10 Let $\mathcal{C}$ and $\mathcal{D}$ be small cartesian categories, and $f : \mathcal{C} \to \mathcal{D}$ a cartesian functor. We shall show that in this case the left Kan extension functor $\varinjlim_f [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$ is also cartesian, so that it is the inverse image of a geometric morphism $[\mathcal{D}^{\mathrm{op}}, \mathbf{Set}] \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, whose *direct* image is f^* (compare 4.1.4). To verify this, note that for any $B \in \mathrm{ob}\mathcal{D}$, the functor $\varinjlim_f(-)(B) : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Set}$ may be described as the composite

$$[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \xrightarrow{U^*} [(B \downarrow f)^{\mathrm{op}}, \mathbf{Set}] \xrightarrow{\varinjlim} \mathbf{Set}$$

where $U : (B \downarrow f) \to \mathcal{C}$ is the forgetful functor, as before.

Lemma 4.2.6 Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. The following conditions are equivalent:

(...)

(iii) f^* is faithful.
(iv) The unit η of the adjunction $(f^* \dashv f_*)$ is monic.

(...)

A geometric morphism satisfying the equivalent conditions of Lemma 4.2.6 is called a *surjection*. We next list some typical examples.

$$\begin{tikzcd} f^*E \arrow[d] & E \arrow[l, mapsto, "\text{(faithful)}"'] \arrow[d, "\eta", "\text{(monic)}"'] \\ f^*E \arrow[r, mapsto] & f_*f^*E \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{F} \arrow[r, shift right, "f_*"'] & \mathcal{E} \arrow[l, shift right, "f^*"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{F} \arrow[r, "f", "\text{(surjection)}"'] & \mathcal{E} \end{tikzcd}$$

Examples 4.2.7 (...)
(b) Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between small categories. If f is surjective on objects, then it is easily verified that the functor $f^* : [\mathcal{D}, \mathbf{Set}] \to [\mathcal{C}, \mathbf{Set}]$ is conservative; for a natural transformation a between functors $\mathcal{D} \to \mathbf{Set}$ is an isomorphism iff α_B is bijective for every object B of $\mathcal{D}$. So the geometric morphism $[\mathcal{C}, \mathbf{Set}] \to [\mathcal{D}, \mathbf{Set}]$ induced by f as in 4.1.4 is surjective.

(...)

(c) Let $f : X \to Y$ be a continuous map of topological spaces. If f is surjective, then the geometric morphism $\mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ induced by f as in 4.1.11 is a surjection.

$$\begin{tikzcd} \mathcal{F} \arrow[r, shift right, "g_*"'] & \mathcal{F}_{\mathbb{G}} \arrow[l, shift right, "g^*"'] \arrow[r, shift right, "h_*"'] & \mathcal{E} \arrow[l, shift right, "h^*"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{F} \arrow[rr, shift right, "f_*"'] & & \mathcal{E} \arrow[ll, shift right, "f^*"'] \end{tikzcd}$$

Proposition 4.2.8 With the notation established above, the counit $h^*h_* \to 1$ is an isomorphism.

A geometric morphism h satisfying the condition that the counit $h^*h_* \to 1$ is an isomorphism, or the equivalent condition that h_* is full and faithful, is called an *inclusion* (though some authors prefer the term *embedding*). We shall study inclusions in greater detail in the next three sections; for the present, we digress briefly to note an alternative characterization of them:

Lemma 4.2.9 A geometric morphism is an inclusion iff its direct image is a cartesian closed functor (i.e. preserves exponentials).

Theorem 4.2.10 Every geometric morphism can be factored, uniquely up to canonical equivalence, as a surjection followed by an inclusion.

Examples 4.2.12 (...)

(b) Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between small categories. If f is full and faithful, then the induced geometric morphism $[\mathcal{C}, \mathbf{Set}] \to [\mathcal{D}, \mathbf{Set}]$ is an inclusion; (...)

(c) Let $f : X \to Y$ be a continuous map of topological spaces. Then it is straightforward to verify that $f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(F)$ is faithful iff it is full and faithful, iff $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is surjective. If X is a subspace of Y and f is the inclusion, then the latter condition is satisfied; the converse holds (up to homeomorphism) provided Y satisfies the T_0 separation axiom, in which case the surjectivity of f^{-1} forces f to be injective. Combining this with 4.2.7(c), we see that if we apply the factorization of 4.2.10 to the morphism $\mathbf{Sh}(X) \to \mathbf{Sh}(F)$ induced by an arbitrary continuous $f : X \to Y$, we obtain $\mathbf{Sh}(I)$, where I is the image of f topologized as a subspace of Y (that is, we obtain the coimage factorization in $\mathbf{Sp}$, rather than the image factorization).

4.3 Cartesian Reflectors and Sheaves

Proposition 4.3.1 Let $\mathcal{E}$ be a cartesian closed category, and $\mathcal{L}$ a reflective subcategory of $\mathcal{E}$, corresponding to a reflector L on $\mathcal{E}$. Then L preserves finite products iff (the class of objects of) $\mathcal{E}$ is an exponential ideal in $\mathcal{E}$. Moreover, if these conditions hold then $B^\eta : B^{LA} \to B^A$ is an isomorphism for every object B of $\mathcal{L}$, where $\eta : 1_{\mathcal{E}} \to L$ is the unit of the reflection.

My way to visualize 4.3.1: choose a ZHA H and a J-operator J on it. Then H is a (posetal) CCC, and $J(H)$ is a reflective subcategory of H, corresponding to a reflector $J : H \to J(H) \subseteq H$. If $Q \in J(H)$, i.e., $Q = Q^*$, then we have this; note that in the obvious $(\to)$-cube we have $(P^* \to Q^*) \to (P \to Q^*)$, but in the full $(\to)$-cube we have $(P^* \to Q^*) \to (P \to Q^*)$.

$$\begin{tikzcd} LB \arrow[d, "\epsilon_B"', "=\mathrm{id}"'] & LA \arrow[d] & A \arrow[l, mapsto] \arrow[d] & A \arrow[d, "\eta_A"'] \arrow[dr, dashed] & & B^A \\ B & B \arrow[r, mapsto] & B & LA \arrow[r, dashed] & B & B^{LA} \arrow[u, "B^{\eta A}"', "(\mathrm{iso})"] \end{tikzcd}
\qquad
\begin{tikzcd} Q^* \arrow[d, "\epsilon_Q"', "=\mathrm{id}"'] & P^* \arrow[d] & P \arrow[l, mapsto] \arrow[d] & P \arrow[d, "\eta_P"] & (P{\to}Q) \\ Q & Q \arrow[r, mapsto] & Q & P^* & (P^*{\to}Q) \arrow[u, "(\mathrm{iso})"'] \end{tikzcd}$$

$$\mathcal{L} \underset{(\mathrm{inc})}{\overset{L}{\leftrightarrows}} \mathcal{E} \qquad\qquad J(H) \underset{(\mathrm{inc})}{\overset{J}{\leftrightarrows}} H$$

Here is a typical non-trivial inclusion, and a map $A \to LA$ on it:

$$\begin{tikzcd} \begin{pmatrix} E_2 \end{pmatrix} \arrow[d, "\eta"', "(\mathrm{iso})"'] & \begin{pmatrix} D_2 \end{pmatrix} \arrow[d] & \begin{pmatrix} D_1 \\ \downarrow \\ D_2 \\ \downarrow \\ D_3 \end{pmatrix} \arrow[l, mapsto] \arrow[d] & \begin{pmatrix} D_1 \\ \downarrow \\ D_2 \\ \downarrow \\ D_3 \end{pmatrix} \arrow[d, "\eta", "(\mathrm{iso})"] \\ \begin{pmatrix} E_2 \end{pmatrix} & \begin{pmatrix} E_2 \end{pmatrix} \arrow[r, mapsto] & \begin{pmatrix} E_2 \\ \downarrow \\ E_2 \\ \downarrow \\ 1 \end{pmatrix} & \begin{pmatrix} D_2 \\ \downarrow \\ D_2 \\ \downarrow \\ 1 \end{pmatrix} \end{tikzcd}
\qquad
\begin{pmatrix} \{47,48\} \\ \downarrow \\ \{14,15,16\} \\ \downarrow \\ \{1,2,3\} \end{pmatrix} \overset{\eta}{\to} \begin{pmatrix} \{14,15,16\} \\ \downarrow \\ \{14,15,16\} \\ \downarrow \\ \{0\} \end{pmatrix}$$

$$\mathbf{Set}^{\begin{pmatrix} 2 \end{pmatrix}} \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathbf{Set}^{\begin{pmatrix} 1 \\ \downarrow \\ 2 \\ \downarrow \\ 3 \end{pmatrix}} \qquad\qquad A \overset{\eta}{\to} LA$$

$$\begin{pmatrix} 2 \end{pmatrix} \xrightarrow{\quad f \quad} \begin{pmatrix} 1 \\ \downarrow \\ 2 \\ \downarrow \\ 3 \end{pmatrix}$$

Now suppose that $\mathcal{E}$ has pullbacks, and let L be a reflector on $\mathcal{E}$ which preserves pullbacks. Then, for any object A of $\mathcal{E}$, we may define a unary operation $c_{L,A}$ (or simply c_L) on subobjects of A, as follows: if $A' \rightarrowtail A$ is monic, then so is $LA' \rightarrowtail LA$, and we define $c_L(A')$ by the pullback diagram:

$$
\begin{tikzcd}
c_L(A') \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & LA' \arrow[d, tail] \\
A \arrow[r, "\eta_A"] & LA
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \arrow[r, tail] \arrow[rr, tail, bend left, "\eta_Q"] \arrow[dr] & JQ{\wedge}R \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & JQ \arrow[d, tail] \\
& R \arrow[r, "\eta_R"] & JR
\end{tikzcd}
\qquad
\begin{aligned}
c_{J,R}(Q) &= c_J(Q) \\
&= JQ \times_{JR} R \\
&= JQ \times R \\
&= Q^* \wedge R \\
&=: Q^{(R)}
\end{aligned}
$$

$$
\begin{tikzcd}
\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} \arrow[r, tail] \arrow[rr, tail, bend left, "\eta"] \arrow[dr] & \begin{pmatrix} P_2{\times_{R_2}}R_1 \\ P_2 \\ R_3 \end{pmatrix} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \begin{pmatrix} P_2 \\ P_2 \\ 1 \end{pmatrix} \arrow[d, tail] \\
& \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} \arrow[r, "\eta"] & \begin{pmatrix} R_2 \\ R_2 \\ 1 \end{pmatrix}
\end{tikzcd}
\qquad
\begin{tikzcd}
\begin{pmatrix} \{35\} \\ \{13\} \\ \{1\} \end{pmatrix} \arrow[r, tail] \arrow[rr, tail, bend left, "\eta"] \arrow[dr] & \begin{pmatrix} \{35,36\} \\ \{13\} \\ \{1,2\} \end{pmatrix} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \begin{pmatrix} \{13\} \\ \{13\} \\ \{0\} \end{pmatrix} \arrow[d, tail] \\
& \begin{pmatrix} \{35,36\} \\ \{13,14\} \\ \{1,2\} \end{pmatrix} \arrow[r, tail, "\eta"] & \begin{pmatrix} \{13,14\} \\ \{13,14\} \\ \{0\} \end{pmatrix}
\end{tikzcd}
$$

A way to understand how it works:

$$
\left(\begin{pmatrix} \{35\} \\ \{13\} \\ \{1\} \end{pmatrix} \twoheadrightarrow \begin{pmatrix} \{35,36\} \\ \{13,14\} \\ \{1,2\} \end{pmatrix} \right) \mapsto \begin{pmatrix} 1,0 \\ 1,0 \\ 1,0 \end{pmatrix} \mapsto \begin{pmatrix} 1,0 \end{pmatrix} \mapsto \begin{pmatrix} 1,1 \\ 1,0 \\ 1,1 \end{pmatrix} \mapsto \begin{pmatrix} \{35,36\} \\ \{13\} \\ \{1,2\} \end{pmatrix}
$$

Lemma 4.3.2 The operation c_L just defined is a closure operation on $\mathrm{Sub}(A)$; that is, it is order-preserving and satisfies $A' \leq c_L(A') \cong c_L c_L(A')$ for any A'. Moreover, c_L commutes (up to isomorphism) with pullback along an arbitrary morphism of $\mathcal{E}$.

Lemma 4.3.3 Let c be a universal closure operation on a cartesian closed category $\mathcal{E}$. Then

(i) Given a commutative square (...) where m is a dense object and n is closed, there is a unique morphism $g : A \to B'$ satisfying $ng = f$ and $gm = f'$.

$$
\begin{tikzcd}
A' \arrow[r, "f"] \arrow[d, tail, "{\substack{m \\ \text{(dense)}}}"'] & B' \arrow[d, tail, "{\substack{n \\ \text{(closed)}}}"] \\
A \arrow[r, "f"'] \arrow[ur, dashed, "\exists! g"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, tail, "{\substack{m \\ P^{(Q)}=Q}}"'] & R \arrow[d, tail, "{\substack{n \\ R^{(S)}=R}}"] \\
Q \arrow[r, "f"'] \arrow[ur, dashed, "\exists! g"] & S
\end{tikzcd}
$$

(ii) For any $A' \twoheadrightarrow A$, $c(A')$ may be characterized as the unique subobject A''

of A such that $A' \twoheadrightarrow A''$ is dense and $A'' \twoheadrightarrow A$ is closed.

$$\begin{tikzcd} A & & & R & \\ & A'' = c(A') \arrow[ul, tail, "\text{(closed)}"'] & & & Q = P^{(R)} \arrow[ul, tail, "\text{(closed)}"'] \\ A' \arrow[uu, tail] \arrow[ur, tail, "\text{(dense)}"'] & & & P \arrow[uu, tail] \arrow[ur, tail, "\text{(dense)}"'] & \end{tikzcd}$$

(iii) For subobjects $A' \twoheadrightarrow A$ and $A'' \twoheadrightarrow A$, we have $c(A' \cap A'') \cong c(A') \cap c(A'')$.

My way to visualize 4.3.3

Fix a ZHA H, a J-operator $J : H \to H$ and an element $Q \in H$; remember that we can write $[00, Q] \cap H$ for the set of elements of H below Q. The operation

$$\begin{array}{rcrcl} \cdot^{(Q)} & : & [00,Q] \cap H & \to & [00,Q] \cap H \\ & & P & \mapsto & P^{(Q)} \\ & & & := & P^* \wedge Q \end{array}$$

is a J-operator J' on the ZHA $H' := [00, Q] \cap H$, whose cuts are the same as the ones in J, except, of course, that we don't use the cuts above Q; note that some regions of J may be partly inside H' and partly outside it — if P belongs to one of these regions then $J'(P) = J(P) \wedge Q \neq J(P)$.

Definition 4.3.4 Let c be a universal closure operator on a cartesian category $\mathcal{E}$.

(a) We say an object A of $\mathcal{E}$ is *(c-)separated* if, whenever we have a diagram

$$\begin{tikzcd} B' \arrow[r, "f'"] \arrow[d, tail, "m \atop \text{(dense)}"'] & A \\ B \arrow[ur, dashed, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r] \arrow[d, tail, "P^{(Q)}=Q"'] & R \\ Q \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} R & \\ & Q = P^{(R)} \arrow[ul, tail, "! / \exists!"'] \\ P \arrow[uu, tail] \arrow[ur, tail, "\text{(dense)}"'] & \end{tikzcd}$$

where m is c-dense, there is at most one $f : B \to A$ with $fm = f'$.

(b) We say A is a (c-)sheaf if, whenever we have a diagram as above, there is exactly one f with $fm = f'$.

My way to visualize 4.3.4

All objects $R \in H$ are J-separated.

All objects $R \in J(H)$ are J-sheaves.

An object $R \notin J(H)$ is not a J-sheaf. The map $R \twoheadrightarrow R^*$ is not an iso, and we can't build the diagonal map when $P := R$ and $Q := R^*$:

$$\begin{tikzcd} R \arrow[r, "\text{(iso)}"] \arrow[d, tail, "\text{(not iso)}"'] & R \\ R^* \arrow[ur, dashed, "=("'] & \end{tikzcd}$$

Example 4.3.5

Lemma 4.3.6 Let L be a cartesian reflector on a cartesian category $\mathcal{E}$, corresponding to a reflective subcategory $\mathcal{L}$, and let c_L denote the universal closure derived from L as in 4.3.2. Let A be an object of $\mathcal{E}$. Then

(a) The following are equivalent:
(i) A is c_L-separated.
(ii) The unit map $\eta_A : A \to LA$ is monic.
(iii) A is a subobject of an object of C.
(iv) The diagonal map $A \rightarrowtail A \times A$ is c_L -closed.

(b) The following are equivalent:
(i) A is a c_L-sheaf.
(ii) The unit $\eta_a : A \to LA$ is an isomorphism.
(iii) A is an object of C.

Lemma 4.3.7

Lemma 4.3.8

Theorem 4.3.9 Let $\mathcal{E}$ be a topos, and L a cartesian reflector on $\mathcal{E}$, corresponding to a reflective subcategory $\mathcal{L}$. Then $\mathcal{L}$ is a topos, and the inclusion $\mathcal{L} \to \mathcal{E}$ is the direct image of a geometric morphism, whose inverse image is (the factorization through $\mathcal{L}$ of) L.

Remark 4.3.10

Proposition 4.3.11 Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. and L a cartesian reflector on $\mathcal{E}$. The following are equivalent:

(i) f factors (uniquely) through the inclusion $h : \mathcal{L} \to \mathcal{E}$ which corresponds to L under 4.3.9.

A4.4 Local Operators

Example 4.4.8 For a quasitopos $\mathcal{E}$, there is a bijection between reflective subcategories of $\mathcal{E}$ with cartesian reflector, and proper universal closure operations on $\mathcal{E}$. In particular, if $\mathcal{E}$ is a topos, there is a bijection between subtoposes of $\mathcal{E}$ and local operators on $\mathcal{E}$.

Example 4.5.2 Let $\mathcal{C}$ be a small category, and $\mathcal{D}$ a full subcategory of $\mathcal{C}$. Then the geometric morphism $[\mathcal{D}, \mathbf{Set}] \to [\mathcal{C}, \mathbf{Set}]$ induced by the inclusion $\mathcal{D} \to \mathcal{D}$ is an inclusion by 4.2.12(b); so it corresponds to a local operator on $[\mathcal{C}, Set]$.

Proposition 4.5.8 Let j be a local operator on a topos $\mathcal{E}$. The following conditions are equivalent:

(i) The associated sheaf functor $L : \mathcal{E} \to \mathbf{sh}_j(\mathcal{E})$ preserves the subobject classifier.

(ii) The canonical monomorphism $\Omega_j \to \Omega$ is j-dense.

(iii) For any $\varphi : A \to \Omega$, the equalizer of φ and $j\varphi$ is a j-dense subobject of A.

(iv) Every monomorphism in $\mathcal{E}$ may be factored (not necessarily uniquely) as a j-closed monomorphism followed by a j-dense one.

(v) j commutes with implication, i.e. the diagram (...) commutes.

Example 4.5.9 Let $\neg : \Omega \to \Omega$ be the Heyting negation map, i.e. the classifying map of $\perp : 1 \rightarrowtail \Omega$. It is straightforward to verify that the composite $\neg\neg$ is a local operator, i.e. that it satisfies the conditions of 4.4.1. Moreover, it satisfies the conditions of 4.5.8: to see this, observe that for any element x of a Heyting algebra if, we have $x \leq (\neg\neg x \Rightarrow x)$ and $\neg x \leq (\neg\neg x \Rightarrow x)$ (the latter since $(\neg x \wedge \neg\neg x) = \perp \leq x$), and so $(\neg\neg x \Rightarrow x) \geq (x \vee \neg x)$; hence $\neg\neg(\neg\neg x \Rightarrow x) \geq \neg\neg(x \vee \neg x) = \top$. But this is just the statement that the diagram in (vi) of 4.5-8 commutes. Alternatively, we could use condition (iv): given a subobject $A' \rightarrowtail A$, if we set $A'' = A' \cup \neg A'$, then $A' \rightarrowtail A''$ is $\neg\neg$-closed (since it is complemented) and $A'' \rightarrowtail A$ is $\neg\neg$-dense (cf. the proof of 1.4.14).

We note that the subtopos $\mathbf{sh}_{\neg\neg}(\mathcal{E})$ is Boolean; for if A is any $\neg\neg$-sheaf, its subobjects in $\mathbf{sh}_{\neg\neg}(\mathcal{E})$ are its $\neg\neg$-closed subobjects in $\mathcal{E}$, and these form a Boolean algebra. It is easy to see that it is not an open subtopos in general; for example, if X is a T_0-space (such as $\mathbb{R}$) in which no nonempty open subspace is discrete, then $\mathbf{sh}_{\neg\neg}(\mathbf{Sh}(X))$ cannot be open. We shall have more to say about Boolean subtoposes in 4.5.21 below.

We write $\mathbf{Lop}(\mathcal{E})$ for the class of all local operators on a topos $\mathcal{E}$ (note that it is a set if $\mathcal{E}$ is locally small). $\mathbf{Lop}(\mathcal{E})$ carries a natural partial order, defined by $j_1 \leq j_2$ iff $\wedge(j_1, j_2) = j_1$; this is equivalent to saving that $J_1 < J_2$ in $\mathrm{Sub}(\Omega)$, or that $\Omega_{j_2} \leq \Omega_{j_1}$, or that $\mathbf{sh}_{j_2}(\mathcal{E}) \subseteq \mathbf{sh}_{j_1}(\mathcal{E})$ as subcategories of $\mathcal{E}$ (the more dense monomorphisms we have, the more conditions an object has to satisfy to be a sheaf). We shall see eventually that $\mathbf{Lop}(\mathcal{E})$ is a Heyting algebra; for the moment, we note

Lemma 4.5.10 The partial ordering $\mathbf{Lop}(\mathcal{E})$ has greatest and least elements, and binary meets.

Lemma 4.5.19

Corollary 4.5.20 Any geometric inclusion $\mathcal{E}' \to \mathcal{E}$ has a unique factorization $\mathcal{E}' \to \mathcal{E}'' \to \mathcal{E}$, where $\mathcal{E}' \to \mathcal{E}''$ is dense and $\mathcal{E}'' \to \mathcal{E}$ is closed.

Examples 4.6.2 (a) Every inclusion is localic, for if f is an inclusion then every object of its domain is isomorphic to one of the form f^*A. More generally, if f_* is merely faithful, then the counit $f^*f_*B \to B$ is epic for all B, and so f is localic.

(...)

(c) Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between small categories. If f is faithful, then the induced geometric morphism $[\mathcal{C}, \mathbf{Set}] \to [\mathcal{D}, \mathbf{Set}]$ of 4.1.4 is localic. For every functor $\mathcal{C} \to Set$ is a quotient of a coproduct of representable functors; if f is faithful then the representable functor $\mathcal{C}(A,)$ is a subfunctor of $f^*(\mathcal{D}(f(A),))$; and f^* preserves coproducts. The converse is also true: if $\mathcal{C}(A,)$ appears as a subquotient of some $f^*(F)$, then (being projective) it actually occurs as a subobject of $f^*(F)$, and this can only happen if there exists $x \in F(f(A))$ such that $F(f\alpha)(x) \neq F(f\beta)(x)$ whenever $\alpha, \beta : A \rightrightarrows B$ are distinct morphisms of $\mathcal{C}-$ which in particular forces $f\alpha \neq f\beta$.

(d) In particular, if $\mathcal{C}$ is a preorder (so that the unique functor from $\mathcal{C}$ to the terminal category $\mathbf{1}$ is faithful), then the unique geometric morphism $[\mathcal{C}, \mathbf{Set}] \to \mathbf{Set}$ of 4.1.9 is localic.

(e) It is easy to verify that a composite of localic morphisms is localic, since the subquotient relation is transitive and inverse image functors preserve monomorphisms and epimorphisms. So, combining (a) and (d), we see that if $(\mathcal{C}, T)$ is a small site whose underlying category is a preorder, then the unique geometric morphism $\mathbf{Sh}(\mathcal{C}, T) \to \mathbf{Set}$ is localic. (We shall prove a converse to this result in B3.3.5.) In particular, for any topological space X, $\mathbf{Sh}(X) \to \mathbf{Set}$ is localic. Similarly, combining (a) and (b), we note that the surjection with Boolean domain constructed in the proof of 4.5.23 is localic.

(f) It is even easier to verify that, if

$$\mathcal{G} \xrightarrow{g} \mathcal{F} \xrightarrow{f} \mathcal{E}$$

is a composable pair of geometric morphisms and the composite fg is localic, then g is localic. Hence if $\mathcal{F}$ and $\mathcal{G}$ both admit localic morphisms to $\mathbf{Set}$, then any geometric morphism between them is localic. For example, the geometric morphism $\mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ induced by a continuous map of spaces $X \to Y$, as in 4.1.11, is always localic.

Theorem 4.6.5 Any geometric morphism can be factored, uniquely up to equivalence, as a hyperconnected morphism followed by a localic one.

Proposition 4.6.6 Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. The following are equivalent:

(i) f is hyperconnected.

(ii) f^* is full and faithful, and its image is closed under subobjects in $\mathcal{F}$.

(iii) f^* is full and faithful, and its image is closed under quotients in $\mathcal{F}$.

(iv) The unit and counit of $(f^* \dashv f_*)$ are both monic.

(v) f_* preserves Ω, i.e. the comparison map $\tau : f_*(\Omega_{\mathcal{F}}) \to \Omega_{\mathcal{E}}$ (the classifying map of $f_*(\top_{\mathcal{F}})$) is an isomorphism.

(vi) For each object A of $\mathcal{E}$, f^* induces an equivalence $\mathrm{Sub}_{\mathcal{E}}(A) \simeq Sub_{\mathcal{F}}(f^*A)$.

Inclusion:
A4.2.8: The counit $h^*h_* \to 1$ is an iso:

$$\begin{tikzcd} h^*D \arrow[d] & D \arrow[l, mapsto] \arrow[d, "\substack{\epsilon \\ (\mathrm{iso})}"] \\ h^*D \arrow[r, mapsto] & h_*h^*D \end{tikzcd}$$

$$\mathbf{Set}^A \underset{h_*}{\overset{h^*}{\leftrightarrows}} \mathbf{Set}^B$$

$$A \xrightarrow{h} B$$