Planar Heyting Algebras for Children 2: Local Operators

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Abstract

A local operator '*' on a Heyting Algebra H is a function $\cdot^* : H \to H$ obeying $P \leq P^* = P^{**}$ and $(P \land Q)^* = P^* \land Q^*$. They are also called *Lawvere-Tierney topologies*, *modalities*, and *J-operators* in the literature, and they are important in Topos Theory because every local operator on the logic of a topos can be extended to the whole topos in a way that defines a sheaf. We use the prefix "J-" in the paper: every J-operator on H induces a J-equivalence and a J-partition on H.

In this paper we use finite, planar HAs — "ZHAs", in the terminology of the preceding paper in this series — to understand *visually* how J-operators work. Our first result concerns "slashings" that cut a ZHA into equivalence classes by diagonal cuts not stopping midway; every J-partition on a ZHA is a slash-partition, and vice-versa. Our second result is about how J-operators interact with the connectives — for example, $P^* \land Q^* = (P \land Q)^*$ is always true, but $P^* \lor Q^* = (P \lor Q)^*$ has countermodels. We present three small ZHAs that can be used to remind us which sentences like these are theorems, and that yield countermodels for all those that are not theorems. Our third result is a way to visualize the algebra of J-operators on a ZHA H that yields a simple way to express every J-operator on H as a finite conjunction of "boolean quotients". Our fourth result uses that every ZHA H "is" a topology on a 2-column graph (P, A); we show that every J-operator on H corresponds to a set of "points to forget" in P and we show that this can be structured as an adjunction and as a geometric morphism, yielding an example "for children" for some theorems that the topos theory books present in a way that is very abstract.

Contents

Introduction

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In this paper we use finite, planar HAs — "ZHAs", in the terminology of the preceding paper in this series — to understand visually how local operators, or "J-operators", work.

Our first result relates J-operators to "slashings" that cut a ZHA into equivalence classes by diagonal cuts not stopping midway: the boundaries between the equivalence classes of a J-operator are slashings, and every slashing induces a J-equivalence and a J-operator.

Our second result is about how J-operators interact with the connectives — for example, $P^* \land Q^* = (P \land Q)^*$ is always true, but $P^* \lor Q^* = (P \lor Q)^*$ has countermodels. We present three small ZHAs that can be used to remind us which sentences like these are theorems, and that yield countermodels for those that are not theorems.

Our third result is simply a way to visualize the algebra of J-operators that lets us understand visually how some J-operators — especially the "closed quotients" and the "open quotients" — interact with one another. We use it to show that all J-operators on a ZHA are "polynomial" and can be expressed as finite conjunction of "boolean quotients"; the double negation is a particular case of boolean quotients.

Our fourth result uses that every ZHA H "is" the order topology on a 2-column graph (P, A); we show that every J-operator corresponds to forgetting the information on a subset Q of points of P and then reconstructing it in a maximal way. We use that to connect the previous ideas to standard ways of presenting toposes and sheaves: a J-operator can be seen as coming from an adjunction, and that adjunction can be generalized to a geometric morphism that yields a sheaf — we get a "miniature case" of geometric morphisms and sheaves, in which everything is easy to draw explicitly and to calculate with.

A note on "children". "Children" here means "people without mathematical maturity", in the sense that, for example, they prefer to start from concrete examples and only then understand the general theorems.

Many years ago I tried to learn Topos Theory starting by Peter Johnstone's first book on the subject. It was too abstract for me, and I said to my friends "I need a version for children of this!!!"... with time this half-joke became serious — I made up a definition for "children" that was good enough to characterize what would be a version "for children" of a text on Category Theory, and a handful of techniques for building the diagrams and examples that were "missing" in the original text so that we would have a presentation "for adults" and one for "for children" of the material, and they could be followed in parallel. In this paper we will only use explicitly three of these techniques, and quite briefly, to present a geometric morphism in sec.5 — a thorough discussion of the techniques will be left to the next paper on this series.

1 Slashings

A *slashing* of a ZHA H is a way to divide H into regions by diagonal cuts that "do not stop midway". In this section we will define formally cuts, slashings, slash-equivalence, slash-partitions, and slash-operators.

1.1 Piccs and slashings

A picc ("partition into contiguous classes") of an interval $I = \{0, \ldots, n\}$ is a partition P of I that obeys this condition ("picc-ness"):

$$\forall a, b, c \in \{0, \ldots, n\}.\ (a < b < c \ \ \&\ \ a \sim_P c) \to (a \sim_P b \sim_P c).$$

So $P = \{\{0\}, \{1, 2, 3\}, \{4, 5\}\}$ is a picc of $\{0, \ldots, 5\}$, and $P' = \{\{0\}, \{1, 2, 4, 5\}, \{3\}\}$ is a partition of $\{0, \ldots, 5\}$ that is not a picc.

A short notation for piccs is this:

$$0|123|45 \equiv \{\{0\}, \{1, 2, 3\}, \{4, 5\}\}$$

we list all digits in the "interval" in order, and we put bars to indicate where we change from one equivalence class to another.

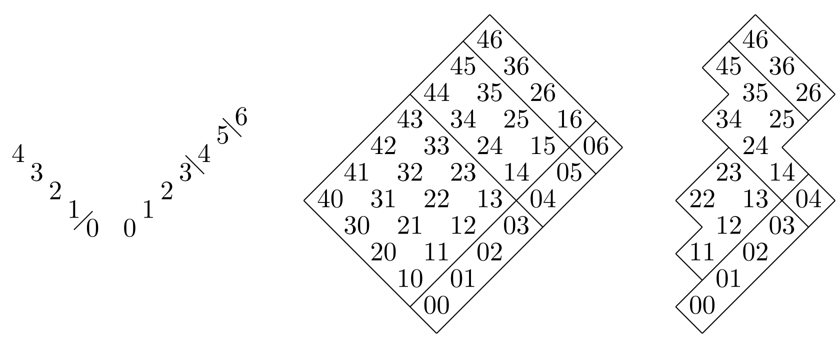
Let's define a notation for "intervals" in $\mathbb{LR}$,

$$[ab, ef] := [\langle a, b\rangle, \langle e, f\rangle] := \{\, \langle c, d\rangle \in \mathbb{LR} \mid a \le c \le e \ \ \& \ \ b \le d \le f \,\},$$

Note that it can be adapted to define "intervals" in a ZHAs H:

$$\begin{array}{rcl} [ab, ef] \cap H & := & \{\, \langle c, d\rangle \in \mathbb{LR} \mid a \le c \le e \ \ \& \ \ b \le d \le f \,\} \cap H \\ & = & \{\, \langle c, d\rangle \in H \mid a \le c \le e \ \ \& \ \ b \le d \le f \,\}. \end{array}$$

A *slashing* S on a ZHA H with top element ab is a pair of piccs, $S = (L, R)$, where L is a picc on $\{0, \ldots, a\}$ and R is a picc on $\{0, \ldots, b\}$; for example, $S = (4321/0,\ 0123\backslash 45\backslash 6)$ is a slashing on $[00, 46]$. We write the bars in L as '/'s and the bars in R as '\' as a reminder that they are to be interpreted as northeast and northwest "cuts" respectively; $S = (4321/0,\ 0123\backslash 45\backslash 6)$ is interpreted as the diagram at the left below, and it "slashes" $[00, 46]$ and the ZHA at the right below as:



A slashing $S = (L, R)$ on a ZHA H with top element ab induces an equivalence relation '$\sim_S$' on H that works like this: $\langle c, d\rangle \sim_S \langle e, f\rangle$ iff $c \sim_L e$ and $d \sim_R f$. We write

$$\begin{array}{rcl} [c]_L & := & \{\, e \in \{0, \ldots, a\} \mid c \sim_L a \,\} \\ [d]_R & := & \{\, f \in \{0, \ldots, b\} \mid d \sim_L f \,\} \\ [cd]_S & := & \{\, ef \in H \mid cd \sim_S ef \,\} \end{array}$$

for the equivalence classes, and note that

$$\begin{array}{lrcl} \text{if} & [c]_L & = & \{c', \ldots, c''\} \\ \text{and} & [d]_L & = & \{d', \ldots, d''\} \\ \text{then} & [cd]_S & = & [c'd', c''d''] \cap H; \end{array}$$

for example, in the ZHA at the right at the example above we have:

$$\begin{array}{rcl} [1]_L &=& \{1,2,3,4\}, \\ [2]_R &=& \{0,1,2,3\}, \\ [12]_S &=& [10,43] \cap H \;=\; \{11,12,13,22,23\}. \end{array}$$

We say that a slashing S on a ZHA H partitions H into *slash-regions*; later (sec.2.1) we will see that a J-operator J also partitions H, and we will refer to its equivalence classes as *J-regions*.

Slash-regions are intervals, but note that neither 10 or 43 belong to the slash-region $[12]_S = [10,43] \cap H$ above.

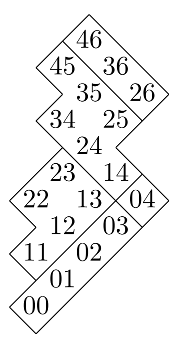
A *slash-partition* is a partition on a ZHA induced by a slashing, and a *slash-equivalence* is an equivalence relation on a ZHA induced by a slashing. Formally, a slash-partition on H is a set of subsets of H, and a slash-equivalence is subset of $H \times H$, but it is so easy to convert between partitions and equivalence relations that we will often use both terms interchangeably. Our visual representation for slash-partitions and slash-equivalences on a ZHA H will be the same: H slashed by diagonal cuts.

1.2 From slash-partitions back to slashings

We saw how to go from a slashing $S = (L, R)$ on H to an equivalence relation $\sim_S$ on H; let's see now how to recover L and R from $\sim_S$.

Let LW_H be the left wall of H, and RW_H the right wall of H. For example,

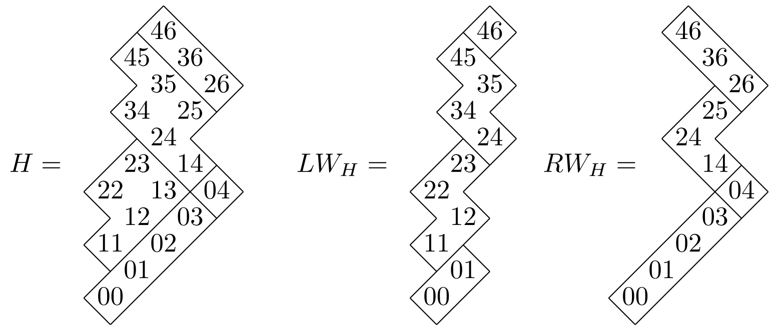
$H =$



$$LW_H = \{00,01,11,12,22,23,24,34,35,45,46\}$$
$$RW_H = \{00,01,02,03,04,14,24,25,26,36,46\}$$

To recover the picc L — which is a picc on $\{0,1,2,3,4\}$ — we need to find where we change from an L-equivalence class to another when we go from one digit to the next; and to recover the picc R — which is a picc on $\{0,1,2,3,4,5,6\}$ — we need to find where we change from an R-equivalence class to another when we go from one digit to the next.

We can recover L and R by walking LW_H (or RW_H) from bottom to top in a series of white pawns moves, and checking when we change from one S-equivalence class to another. Northwest moves give information about L, and northeast moves give information about R. Look at the example below, in which we walk on RW_H:



$$\begin{array}{lr}
 & {}_{25}\nearrow^{26} : 25 \not\sim_S 26 \ \Rightarrow\ 5 \not\sim_R 6 \ \Rightarrow\ 5\backslash 6 \\
 & {}_{24}\nearrow^{25} : 24 \sim_S 25 \ \Rightarrow\ 4 \sim_R 5 \ \Rightarrow\ 45 \\
{}^{46}\nwarrow_{36} : 36 \sim_S 46 \ \Rightarrow\ 3 \sim_L 4 \ \Rightarrow\ 43 & {}_{03}\nearrow^{04} : 03 \not\sim_S 04 \ \Rightarrow\ 3 \not\sim_R 4 \ \Rightarrow\ 3\backslash 4 \\
{}^{36}\nwarrow_{26} : 26 \sim_S 36 \ \Rightarrow\ 2 \sim_L 3 \ \Rightarrow\ 32 & {}_{02}\nearrow^{03} : 02 \sim_S 03 \ \Rightarrow\ 2 \sim_R 3 \ \Rightarrow\ 23 \\
{}^{24}\nwarrow_{14} : 14 \sim_S 24 \ \Rightarrow\ 1 \sim_L 2 \ \Rightarrow\ 21 & {}_{01}\nearrow^{02} : 01 \sim_S 02 \ \Rightarrow\ 1 \sim_R 2 \ \Rightarrow\ 12 \\
{}^{14}\nwarrow_{04} : 04 \not\sim_S 14 \ \Rightarrow\ 0 \not\sim_L 1 \ \Rightarrow\ 1/0 & {}_{00}\nearrow^{01} : 00 \sim_S 01 \ \Rightarrow\ 0 \sim_R 1 \ \Rightarrow\ 01
\end{array}$$

$$(L, R) \ = \ (4321/0, 0123\backslash 45\backslash 6)$$

1.3 Slash-regions have maximal elements

...here is how our argument will work, in a particular case:

$$\begin{array}{rcl}
[1]_L & = & \{1, 2, 3, 4\}, \\
[2]_R & = & \{0, 1, 2, 3\}, \\
I & = & [10, 43], \\
[12]_S & = & I \cap H \ = \ \{11, 12, 13, 22, 23\}.
\end{array}$$

$$\underbrace{(\underbrace{(\underbrace{(\underbrace{11 \vee 12}_{=12\in I}) \vee 13}_{=13\in I}) \vee 22}_{=23\in I}) \vee 23}_{=23\in I} \qquad \underbrace{(\underbrace{(\underbrace{(\underbrace{11 \vee 12}_{=12\in H}) \vee 13}_{=13\in H}) \vee 22}_{=23\in H}) \vee 23}_{=23\in H}$$

$$\bigvee [12]_S = \bigvee \{11, 12, 13, 22, 23\} = 11 \vee 12 \vee 13 \vee 22 \vee 23 \in I \cap H$$

$$11 \le \bigvee [12]_S,\ 12 \le \bigvee [12]_S,\ \ldots,\ 23 \le \bigvee [12]_S$$

We have $[12]_S = I \cap H$, and $\bigvee [12]_S$ belongs to $I \cap H$ and is greter-or-equal than all elements of $I \cap H$, so $\bigvee [12]_S$ is the maximal element of $[12]_S$.

Here is how we can do that in the general case. Let $S = (L, R)$ be a slashing on a ZHA H. Let P be a point of H. The equivalence class $[P]_S$ is a finite set $\{P_1, \ldots, P_n\}$, and we know that $[P]_S = H \cap I$ for some interval I. Look at the elements P_1, $P_1 \vee P_2$, $(P_1 \vee P_2) \vee P_3$, …, $((P_1 \vee P_2) \vee \ldots) \vee P_n$ We can see that all of them belong to both H and I, so we conclude that $\bigvee [P]_S = ((P_1 \vee P_2) \vee \ldots) \vee P_n$ belongs to $H \cap I$, and it is easy to see that it is greater-or-equal that all elements in $H \cap I$, so it is the maximal element of $H \cap I$.

A similar argument shows that $\bigwedge [P]_S = ((P_1 \wedge P_2) \wedge \ldots) \wedge P_n$ is the smallest element of $[P]_S$.

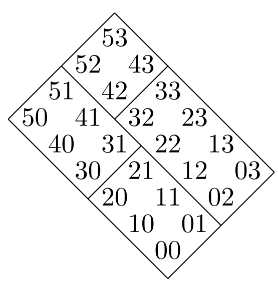
The same argument shows that if C is any non-empty set of the form $I \cap H$, where I is an interval, then $\bigvee C \in C$, $\bigwedge C \in C$, $[\bigwedge C, \bigvee C] \cap H = C$.

Remember that an *interval* in a ZHA H is any set of the form $[P, Q] \cap H$. Let's introduce a new definition: a *closed interval* in a ZHA H is a non-empty set $C \subset H$, with $\bigvee C \in C$, $\bigwedge C \in C$, $[\bigwedge C, \bigvee C] \cap H = C$; informally, a closed interval in a ZHA has a lowest and highest element, and it "is" everything between them.

1.4 Cuts stopping midway

We saw in the last section that every slash-region is a closed interval. A *partition into closed intervals* of a ZHA H is, as its name says, a partition of H whose equivalence classes are all closed intervals in H.

Some partitions into closed intervals of a ZHA are not slashings — for example, take the partition P with these equivalence classes:



Here is an easy way to prove formally that the partition above does not come from a slashing $S = (L, R)$. We will adapt the idea from sec.1.2, where we recovered L and R from northwest and northeast steps.

$$\begin{array}{ccccc}
\underbrace{21 \sim_P 31}_{\text{false}} & \leftrightarrow & \underbrace{2 \sim_L 3}_{=(} & \leftrightarrow & \underbrace{22 \sim_P 32}_{\text{true}} \\
\underbrace{31 \sim_P 41}_{\text{true}} & \leftrightarrow & \underbrace{3 \sim_L 4}_{=(} & \leftrightarrow & \underbrace{32 \sim_P 42}_{\text{false}}
\end{array}$$

The problem is that the figure above has "cuts stopping midway"... if its cuts all crossed the ZHA all the way through, we would have this for L and northeast cuts,

$$\begin{array}{ccccccccc}
0 \sim_L 1 & \leftrightarrow & 00 \sim_P 10 & \leftrightarrow & 01 \sim_P 11 & \leftrightarrow & 02 \sim_P 12 & \leftrightarrow & 03 \sim_P 13 \\
1 \sim_L 2 & \leftrightarrow & 10 \sim_P 20 & \leftrightarrow & 11 \sim_P 21 & \leftrightarrow & 12 \sim_P 22 & \leftrightarrow & 13 \sim_P 23 \\
2 \sim_L 3 & \leftrightarrow & 20 \sim_P 30 & \leftrightarrow & 21 \sim_P 31 & \leftrightarrow & 22 \sim_P 32 & \leftrightarrow & 23 \sim_P 33 \\
3 \sim_L 4 & \leftrightarrow & 30 \sim_P 40 & \leftrightarrow & 31 \sim_P 41 & \leftrightarrow & 32 \sim_P 42 & \leftrightarrow & 33 \sim_P 43 \\
4 \sim_L 5 & \leftrightarrow & 40 \sim_P 50 & \leftrightarrow & 41 \sim_P 51 & \leftrightarrow & 42 \sim_P 52 & \leftrightarrow & 43 \sim_P 53 \\
5 \sim_L 6 & \leftrightarrow & 50 \sim_P 60 & \leftrightarrow & 51 \sim_P 61 & \leftrightarrow & 52 \sim_P 62 & \leftrightarrow & 53 \sim_P 63
\end{array}$$

and something similar for R and northwest cuts.

Formally, a partition P on H has an "L-cut between c and c^+ stopping midway" if $cd \sim_P c^+d \not\leftrightarrow cd \sim_P c^+d$ for some d, and it has an "R-cut between d and d^+ stopping midway" if $cd \sim_P cd^+ \not\leftrightarrow c^+d \sim_P c^+d^+$ for some c; here we are writing x^+ for $x+1$.

Theorem: a partition of H into closed intervals is a slash-partition if and only if it doesn't have any cuts stopping midway. Proof: use the ideas above to recover L and R from $\sim_P$, and then check that $S = (L, R)$ induces an equivalence relation $\sim_S$ that coincides with $\sim_P$.

1.5 Slash-operators

We can define operations that take each each $P \in H$ to the maximal and to the minimal element of its S-equivalent class, now that we know that these maximal and minimal elements exist:

$$\begin{array}{rcll}
P^S & := & \bigvee [P]_S & \text{(maximal element)}, \\
P^{\mathrm{co}S} & := & \bigwedge [P]_S & \text{(minimal element)}.
\end{array}$$

Note that $[P]_S = [P^{\mathrm{co}S}, P^S] \cap H$.

We will use the operation $\cdot^S$ a lot and $\cdot^{\mathrm{co}S}$ very little. The 'co' in 'co$S$' means that $\cdot^{\mathrm{co}S}$ is dual to $\cdot^S$, in a sense that will be made precise later.

A *slash-operator* on a ZHA H is a function $\cdot^S : H \to H$ induced by a slashing $S = (L, R)$ on H. It is easy to see that $P \leq P^S$ ("$\cdot^S$ is non-decreasing") and that $P^S = (P^S)^S$ ("$\cdot^S$ is idempotent").

Any idempotent function $\cdot^F : H \to H$ induces an equivalence relation on H: $P \sim_F Q$ iff $P^F = Q^F$. We can use that to test if a given $\cdot^F : H \to H$ is a slash-operator: $\cdot^F$ is a slash-operator iff it obeys all this:
1) $\cdot^F$ is idempotent,
2) $\cdot^F$ is non-decreasing,
3) $\sim_F$ partitions H into closed intervals,
4) $\sim_F$ doesn't have cuts stopping midway.

1.6 Slash-operators: a property

Slash-operators obey a certain property that will be very important later. Let's state that property in five equivalent ways:

1) If $cd \sim_S c'd'$ and $ef \sim_S e'f'$ then $cd \land ef \sim_S c'd' \land e'f'$.
2) If $P \sim_S P'$ and $Q \sim_S Q'$ then $P \land Q \sim_S P' \land Q'$.
3) If $P \sim_S P'$ and $Q \sim_S Q'$ then $(P \land Q)^S = (P' \land Q')^S$.
4) If $P \sim_S P'$ and $Q \sim_S Q'$ then

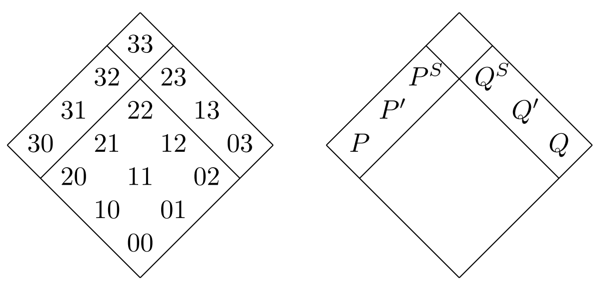
$$\begin{array}{rcll} (P \land Q)^S & = & (P^S \land Q^S)^S & \text{(a)} \\ & = & ((P')^S \land (Q')^S)^S & \text{(b)} \\ & = & (P' \land Q')^S & \text{(c)} \end{array}$$

5) $(P \land Q)^S = (P^S \land Q^S)^S$.

Here's a proof of $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4 \leftrightarrow 5$.

$1 \leftrightarrow 2$: we just changed notation,
$2 \leftrightarrow 3$: because $A \sim_S B$ iff $A^S = B^S$,
$3 \to 5$: make the substitution $\begin{bmatrix} P' := P^S \\ Q' := Q^S \end{bmatrix}$ in 3,
$5 \to 4$: 4a is just a copy of 5, and 4c is a copy of 5 with $\begin{bmatrix} P := P' \\ Q := Q' \end{bmatrix}$. For 4b, note that $P \sim_P P'$ implies $P^S = (P')^S$ and $Q \sim_P Q'$ implies $Q^S = (Q')^S$,
$4 \to 3$: 4 is an equality between more expressions than 3,

...and here is a way to visualize what is going on:



$$\underbrace{(\underbrace{\underbrace{P}_{30} \land \underbrace{Q}_{03}}_{00})^S}_{22} = \underbrace{(\underbrace{\underbrace{\underbrace{P}_{30}{}^S}_{32} \land \underbrace{\underbrace{Q}_{03}{}^S}_{23}}_{22})^S}_{22} = \underbrace{(\underbrace{\underbrace{\underbrace{P'}_{31}{}^S}_{32} \land \underbrace{\underbrace{Q'}_{13}{}^S}_{23}}_{22})^S}_{22} = \underbrace{(\underbrace{\underbrace{P'}_{31} \land \underbrace{Q'}_{13}}_{11})^S}_{22}$$

Note that all subexpressions belong to three S-regions: a region with P, P', $P^S = P'^S$, another with Q, Q', $Q^S = Q'^S$, and one with all the '$\land$'s. If we had cuts stopping midway then some of the '$\land$'s could be in different regions.

I think that the clearest way to show (1) is by putting its proof in tree form:

$$\dfrac{\dfrac{\dfrac{cd \sim_S c'd'}{c \sim_L c'} \quad \dfrac{ef \sim_S e'f'}{e \sim_L e'}}{\min(c,e) \sim_L \min(c',e')} \quad \dfrac{\dfrac{cd \sim_S c'd'}{d \sim_R d'} \quad \dfrac{ef \sim_S e'f'}{f \sim_R f'}}{\min(d,f) \sim_L \min(d',f')}}{\dfrac{\min(c,e)\min(d,f) \sim_S \min(c',e')\min(d',f')}{cd \wedge ef \sim_S c'd' \wedge e'f'}}$$

2 J-operators

A *J-operator* on a Heyting Algebra $H = (\Omega, \leq, \top, \bot, \wedge, \vee, \to, \leftrightarrow, \neg)$ is a function $J : \Omega \to \Omega$ that obeys the axioms J1, J2, J3 below; we usually write J as $\cdot^* : \Omega \to \Omega$, and write the axioms as rules.

$$\dfrac{}{P \leq P^*}\,\mathsf{J1} \qquad \dfrac{}{P^* = P^{**}}\,\mathsf{J2} \qquad \dfrac{}{(P\&Q)^* = P^*\&Q^*}\,\mathsf{J3}$$

J1 says that the operation $\cdot^*$ is non-decreasing.
J2 says that the operation $\cdot^*$ is idempotent.
J3 is a bit mysterious but will have interesting consequences.

In secs.1.5 and 1.6 we saw that slash-operators are J-operators, and in sec.2.2 we will see that all J-operators on ZHAs are slash-operators — but the idea of a J-operator makes sense on all Heyting Algebras, not only ZHAs.

J-operators are called *local operators* in [Joh02] (section A4.4), *modalities* in [Bel88] (chapter 5), *Lawvere-Tierney topologies* or just *topologies* in [MM92] (V.1) and [Joh77] (3.1). We will refer to them as J-operators following [FS79] (p.324) because "J-" works well as a prefix.

2.1 J-operators and J-regions

A J-operator induces an equivalence relation and equivalence classes on Ω, like slashings do:

$$\begin{array}{rcl} P \sim_J Q & \text{iff} & P^* = Q^* \\ [P]^J & := & \{\, Q \in \Omega \mid P^* = Q^* \,\} \end{array}$$

The axioms J1, J2, J3 have many consequences. The first ones are listed in Figure 1 as derived rules, whose names mean:

Mop (monotonicity for products): a lemma used to prove Mo,
Mo (monotonicity): $P \leq Q$ implies $P^* \leq Q^*$,
Sand (sandwiching): all truth values between P and P^* are equivalent,
EC&: equivalence classes are closed by '&',
EC∨: equivalence classes are closed by '∨',
ECS: equivalence classes are closed by sandwiching,

Take a J-equivalence class, $[P]^J$, and list its elements: $[P]^J = \{P_1, \ldots, P_n\}$. Let $P_\wedge := ((P_1 \wedge P_2) \wedge \ldots) \wedge P_n$ and Let $P_\vee := ((P_1 \vee P_2) \vee \ldots) \vee P_n$. It turns out that $[P]^J = [P_\wedge, P_\vee] \cap \Omega$; let's prove that by doing '⊆' first, then '⊇'.

$$\dfrac{}{(P\&Q)^* \le Q^*}\,\mathsf{Mop} \quad := \quad \dfrac{\dfrac{}{(P\&Q)^* = P^*\&Q^*}\,\mathsf{J3} \qquad \dfrac{}{P^*\&Q^* \le Q^*}}{(P\&Q)^* \le Q^*}$$

$$\dfrac{P \le Q}{P^* \le Q^*}\,\mathsf{Mo} \quad := \quad \dfrac{\dfrac{\dfrac{P \le Q}{P = P\&Q}}{P^* = (P\&Q)^*} \qquad \dfrac{}{(P\&Q)^* \le Q^*}\,\mathsf{Mop}}{P^* \le Q^*}$$

$$\dfrac{P \le Q \le P^*}{P^* = Q^*}\,\mathsf{Sand} \quad := \quad \dfrac{\dfrac{P \le Q}{P^* \le Q^*}\,Mo \qquad \dfrac{\dfrac{Q \le P^*}{Q^* \le P^{**}}\,\mathsf{Mo} \qquad \dfrac{}{P^{**} = P^*}\,\mathsf{J2}}{Q^* \le P^*}}{P^* = Q^*}$$

$$\dfrac{P^* = Q^*}{P^* = Q^* = (P\&Q)^*}\,\mathsf{EC\&} \quad := \quad \dfrac{\dfrac{P^* = Q^*}{P^* = Q^* = P^*\&Q^*} \qquad \dfrac{}{P^*\&Q^* = (P\&Q)^*}\,\mathsf{J3}}{P^* = Q^* = (P\&Q)^*}$$

$$\dfrac{P^* = Q^*}{P^* = Q^* = (P \vee Q)^*}\,\mathsf{EC\vee} \quad := \quad \dfrac{P^* = Q^* \qquad \dfrac{\dfrac{\dfrac{}{P \le P \vee Q} \qquad \dfrac{\dfrac{}{P \le P^*}\,\mathsf{J1} \qquad \dfrac{\dfrac{}{Q \le Q^*}\,\mathsf{J1} \qquad \dfrac{P^* = Q^*}{Q^* = P^*}}{Q \le P^*}}{P \vee Q \le P^*}}{P \le P \vee Q \le P^*}}{P^* = (P \vee Q)^*}\,\mathsf{Sand}}{P^* = Q^* = (P \vee Q)^*}$$

$$\dfrac{P \le Q \le R \qquad P^* = R^*}{P^* = Q^* = R^*}\,\mathsf{ECS} \quad := \quad \dfrac{\dfrac{\dfrac{P \le Q \le R \qquad \dfrac{}{R \le R^*}\,\mathsf{J1} \qquad \dfrac{P^* = R^*}{R^* = P^*}}{P \le Q \le P^*}}{P^* = Q^*}\,\mathsf{Sand} \qquad P^* = R^*}{P^* = Q^* = R^*}$$

Figure 1: J-operators: basic derived rules

Using $\mathsf{EC\&}$ and $\mathsf{EC}\vee$ several times we see that

$$
\begin{array}{rr}
P_1 \land P_2 \sim_J P & P_1 \lor P_2 \sim_J P \\
(P_1 \land P_2) \land P_3 \sim_J P & (P_1 \lor P_2) \lor P_3 \sim_J P \\
\vdots & \vdots \\
((P_1 \land P_2) \land \ldots) \land P_n \sim_J P & ((P_1 \lor P_2) \lor \ldots) \lor P_n \sim_J P
\end{array}
$$

so $P_\land \sim_J P_\lor \sim_J P$, and by the sandwich lemma $([P_\land, P_\lor] \cap \Omega) \subseteq [P]^J$.

For any $P_i \in [P]^J$ we have $P_\land \le P_i \le P_\lor$, which means that:

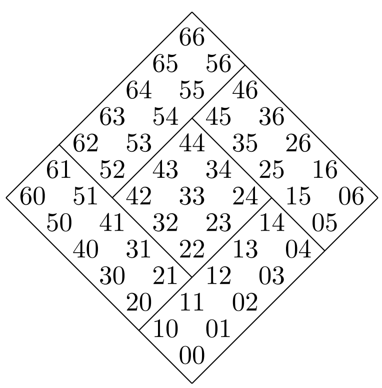
$$
\begin{array}{rcl}
[P]^J & = & \{P_1, \ldots, P_n\} \\
 & \subseteq & \{\, Q \in \Omega \mid P_\land \le Q \le P_\lor \,\} \\
 & = & [P_\land, P_\lor] \cap \Omega,
\end{array}
$$

so $[P]^J \subseteq [P_\land, P_\lor] \cap \Omega$.

As the operation '$\cdot^*$' is increasing and idempotent, each equivalence class $[P]^J$ has exactly one maximal element, which is P^*; but $P_\lor$ is also the maximal element of $[P]^J$, so $P_\lor = P^*$, and we can interpret the operation '$\cdot *$' as "take each P to the top element in its equivalence class", which is similar to how we defined an(other) operation '$\cdot^*$' on slashings in the previous section.

The operation "take each P to the bottom element in its equivalence class" will be useful in a few occasions; we will call it '$\cdot^{\text{co}*}$' to indicate that it is dual to '$\cdot^*$' in some sense. Note that $P^{\text{co}*} = P_\land$.

Look at the figure below, that shows a partition of a ZHA $A = [00, 66]$ into five regions, each region being an interval; this partition does not come from a slashing, as it has cuts that stop midway. Define an operation '$\cdot^*$' on A, that works by taking each truth-value P in it to the top element of its region; for example, $30^* = 61$.



It is easy to see that '$\cdot^*$' obeys $\mathsf{J1}$ and $\mathsf{J2}$; however, it does *not* obey $\mathsf{J3}$ — we will prove that in sec.2.2. As we will see, *the partitons of a ZHA into intervals that obey* $\mathsf{J1}$*,* $\mathsf{J2}$*,* $\mathsf{J3}$ *ae exactly the slashings;* or, in other words, *every J-operator comes from a slashing.*

2.2 The are no Y-cuts and no λ-cuts

We want to see that if a partition of a ZHA H into intervals has “Y-cuts” or “λ-cuts” like these parts of the last diagram in the last section,

$$\begin{array}{ccc} & 22 & \\ 21 & \diagup & 12 \\ & 11 & \end{array} \quad \Leftarrow \quad \text{this is a Y-cut}$$

$$\begin{array}{ccc} & 25 & \\ 24 & \diagdown & 15 \\ & 14 & \end{array} \quad \Leftarrow \quad \text{this is a } \lambda\text{-cut}$$

then it operation J that takes each element to the top of its equivalence class cannot obey J1, J2 and J3 at the same time. We will prove that by deriving rules that say that if $11 \sim_J 12$ then $21 \sim_J 22$, and that if $15 \sim_J 25$ then $14 \sim_J 24$; actually, our rules will say that if $11^* = 12^*$ then $(11 \vee 21)^* = (12 \vee 21)^*$, and that if $15^* = 25^*$ then $(15 \wedge 24)^* = (25 \wedge 24)^*$. The rules are:

$$\dfrac{P^* = Q^*}{(P \vee R)^* = (Q \vee R)^*}\ \mathsf{NoYcuts} \quad := \quad \dfrac{\dfrac{\dfrac{P^* = Q^*}{P \vee R^* = Q \vee R^*}}{(P \vee R^*)^* = (Q \vee R^*)^*}}{(P \vee R)^* = (Q \vee R)^*}\ \vee^*\mathsf{Cube}$$

$$\dfrac{P^* = Q^*}{(P \& R)^* = (Q \& R)^*}\ \mathsf{No}\lambda\mathsf{cuts} \quad := \quad \dfrac{\dfrac{P^* = Q^*}{P^* \& R^* = Q^* \& R^*}}{(P \& R)^* = (Q \& R)^*}\ \mathsf{J3}$$

The top derivation mentions a rule called ‘$\vee^*$Cube’, which will be defined and proved in sec.2.4.

2.3 How J-operators interact with connectives: the obvious cubes

It is easy to prove each one of the arrows below ($A \longrightarrow B$ means $A \leq B$):

$$\begin{tikzcd}[column sep=tiny]
 & (P^* \& Q^*)^* & \\
(P^* \& Q)^* \arrow[ur] & P^* \& Q^* \arrow[u] & (P \& Q^*)^* \arrow[ul] \\
P^* \& Q \arrow[u] \arrow[ur] & (P \& Q)^* \arrow[ul] \arrow[ur] & P \& Q^* \arrow[ul] \arrow[u] \\
 & P \& Q \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
 & (P^* \vee Q^*)^* & \\
(P^* \vee Q)^* \arrow[ur] & P^* \vee Q^* \arrow[u] & (P \vee Q^*)^* \arrow[ul] \\
P^* \vee Q \arrow[u] \arrow[ur] & (P \vee Q)^* \arrow[ul] \arrow[ur] & P \vee Q^* \arrow[ul] \arrow[u] \\
 & P \vee Q \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
 & (P^* \to Q^*)^* \arrow[dr] & \\
(P^* \to Q)^* \arrow[ur] \arrow[dr] & P^* \to Q^* \arrow[u] \arrow[dr] & (P \to Q^*)^* \\
P^* \to Q \arrow[u] \arrow[ur] \arrow[dr] & (P \to Q)^* \arrow[ur] & P \to Q^* \arrow[u] \\
 & P \to Q \arrow[u] \arrow[ur] &
\end{tikzcd}$$

The cubes above will be called the “obvious and-cube”, the “obvious or-cube”, and the “obvious implication-cube”, and they show partial orders between expressions of the form $(P^? \odot Q^?)^?$, where the ‘$\odot$’ stands for one of the connectives '$\wedge$', '$\vee$' or '$\to$', and each '?' marks a place where we can put either a '*' or nothing; let's be more precise.

The "cube of $\wedge$-expressions", $\mathsf{ECube}_\wedge$, is the set of eight expressions of the form $(P^? \wedge Q^?)^?$; $\mathsf{ECube}_\vee$ is the set of eight expressions of the form $(P^? \vee Q^?)^?$, and $\mathsf{ECube}_\to$ the set of eight expressions of the form $(P^? \to Q^?)^?$.

The "obvious $\wedge$-cube", $\mathsf{OCube}_\wedge$, is the directed graph shown above, with 12 arrows between elements of $\mathsf{ECube}_\wedge$. Its transitive closure, $\mathsf{OCube}^*_\wedge$, is a partial order on $\mathsf{ECube}_\wedge$. We define $\mathsf{OCube}_\vee$, $\mathsf{OCube}^*_\vee$, $\mathsf{OCube}_\to$, and $\mathsf{OCube}^*_\to$ similarly.

If we establish that the three '?'s in $(P^? \odot Q^?)^?$ are "worth" 1, 2 and 4 respectively, we get a way to number the elements in $\mathsf{ECube}_\wedge$ from 0 to 7. We define $(\wedge)_0, \ldots, (\wedge)_7$ as:

$$
\begin{array}{rcl@{\qquad}rcl}
(\wedge)_0 & = & P \wedge Q, & (\wedge)_4 & = & (P \wedge Q)^*, \\
(\wedge)_1 & = & P^* \wedge Q, & (\wedge)_{1+4} & = & (P^* \wedge Q)^*, \\
(\wedge)_2 & = & P \wedge Q^*, & (\wedge)_{2+4} & = & (P \wedge Q^*)^*, \\
(\wedge)_{1+2} & = & P^* \wedge Q^*, & (\wedge)_{1+2+4} & = & (P^* \wedge Q^*)^*,
\end{array}
$$

and we do the same for $(\vee)_0, \ldots, (\vee)_7, (\to)_0, \ldots, (\to)_7$. We always draw the '$(\odot)_i$'s in this position:

$$
\begin{array}{ccc}
 & (\odot)_7 & \\
(\odot)_5 & (\odot)_3 & (\odot)_6 \\
(\odot)_1 & (\odot)_4 & (\odot)_2 \\
 & (\odot)_0 &
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & 7 & \\
5 & 3 & 6 \\
1 & 4 & 2 \\
 & 0 &
\end{array}
$$

With this numbering we can reinterpret the cubes as subsets of $\{0, \ldots, 7\}^2$; $\{0, \ldots, 7\}^2$ is a ZSet, and so we can use the positional notation and interpret each cube as a grid of '0's and '1's. For example,

$$
\begin{tikzcd}[sep=small]
 & 7 & \\
5 \arrow[ur] & 3 \arrow[u] & 6 \arrow[ul] \\
1 \arrow[u] \arrow[ur] & 4 \arrow[ul] \arrow[ur] & 2 \arrow[u] \arrow[ul] \\
 & 0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
\quad = \quad
\left\{
\begin{array}{c}
(0,1), (2,3), (4,5), (6,7), \\
(0,2), (1,3), (4,6), (5,7), \\
(0,4), (1,5), (2,6), (3,7)
\end{array}
\right\}
\quad = \quad
\begin{array}{cccccccc}
0&0&0&1&0&1&1&0 \\
0&0&1&0&1&0&0&0 \\
0&1&0&0&1&0&0&0 \\
1&0&0&0&0&0&0&0 \\
0&1&1&0&0&0&0&0 \\
1&0&0&0&0&0&0&0 \\
1&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0
\end{array}
$$

The transitive-reflexive closure of a cube yields a different grid:

$$
\left(
\begin{tikzcd}[sep=small]
 & 7 & \\
5 \arrow[ur] & 3 \arrow[u] & 6 \arrow[ul] \\
1 \arrow[u] \arrow[ur] & 4 \arrow[ul] \arrow[ur] & 2 \arrow[u] \arrow[ul] \\
 & 0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
\right)^*
\quad = \quad
\begin{array}{cccccccc}
1&1&1&1&1&1&1&1 \\
1&0&1&0&1&0&1&0 \\
1&0&0&0&1&1&0&0 \\
1&1&0&0&1&0&0&0 \\
1&1&1&1&0&0&0&0 \\
1&0&1&0&0&0&0&0 \\
1&1&0&0&0&0&0&0 \\
1&0&0&0&0&0&0&0
\end{array}
$$

Note that the grids for $\mathsf{OCube}_\land$ and $\mathsf{OCube}_\lor$ are equal, but the grid for $\mathsf{OCube}_\to$ is different. Also, note that $\mathsf{OCube}_\land$, $\mathsf{OCube}^*_\land$, etc, are directed graphs; sometimes we will need to regard them as pairs, and we will use a lowercase notation for their sets of arrows: $\mathsf{OCube}_\land = (\mathsf{ECube}_\land, \mathsf{ocube}_\land)$, $\mathsf{OCube}^*_\to = (\mathsf{ECube}_\to, \mathsf{ocube}^*_\to)$, etc.

2.4 How J-operators interact with connectives: the full cubes

We can prove these new derived rules,

$$\frac{}{(P^*\&Q^*)^* = P^*\&Q^* = (P\&Q)^*}\,\&^*\mathsf{C}_0 \quad := \quad \cfrac{\cfrac{\cfrac{}{P^{**}=P^*}\,\mathsf{J2} \qquad \cfrac{}{Q^{**}=Q^*}\,\mathsf{J2}}{(P^*\&Q^*)^* = P^{**}\&Q^{**} = P^*\&Q^* = (P\&Q)^*}\,\mathsf{J3}}{(P^*\&Q^*)^* = P^*\&Q^* = (P\&Q)^*}$$

$$\frac{}{(P^*\lor Q^*)^* \le (P\lor Q)^*}\,\lor^*\mathsf{C}_0 \quad := \quad \cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{P \le P\lor Q}}{P^* \le (P\lor Q)^*}\,\mathsf{Mo} \qquad \cfrac{\cfrac{}{Q \le P\lor Q}}{Q^* \le (P\lor Q)^*}\,\mathsf{Mo}}{P^*\lor Q^* \le (P\lor Q)^*}}{(P^*\lor Q^*)^* \le (P\lor Q)^{**}}\,\mathsf{Mo}}{(P^*\lor Q^*)^* \le (P\lor Q)^*}\,\mathsf{J2}$$

$$\frac{}{(P\to Q^*)^* \le P^*\to Q^*}\,\to^*\mathsf{C}_0 \quad := \quad \cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{P\to Q^* \le P\to Q^*}}{(P\to Q^*)\&P \le Q^*}}{((P\to Q^*)\&P)^* \le Q^{**}}\,\mathsf{Mo}}{((P\to Q^*)\&P)^* \le Q^*}\,\mathsf{J2}}{(P\to Q^*)^*\&P^* \le Q^*}\,\mathsf{J3}}{(P\to Q^*)^* \le P^*\to Q^*}$$

and interpret them as extra arrows on the cubes. The "full $\land$-cube", $\mathsf{FCube}_\land$, is $\mathsf{OCube}_\land$ plus these arrows:

$$(P^*\land Q^*)^* \leftrightarrows P^*\land Q^* \leftrightarrows (P\land Q)^*$$

The "full $\lor$-cube", $\mathsf{FCube}_\lor$, is $\mathsf{OCube}_\lor$ plus this,

$$(P^*\lor Q^*)^* \longrightarrow (P\lor Q)^*$$

and the "full $\to$-cube", $\mathsf{FCube}_\to$, is $\mathsf{OCube}_\to$ plus this,

$$(P\to Q^*)^* \longrightarrow (P^*\to Q^*)$$

We are interested in the transitive-reflexive closures of these full cubes. $\mathsf{FCube}^*_\land$ yields a *non-strict* partial order on $\mathsf{ECube}_\land$ that identifies five of its elements, and $\mathsf{FCube}^*_\lor$ and $\mathsf{FCube}^*_\to$ yield non-strict partial orders that identify four elements each. My favorite way to represent these non-strict partial orders is by the diagrams at the right below, that show very clearly which elements are identified:

$$
\left(\begin{tikzcd}
 & & (\wedge)_7 \arrow[d, shift left] & & \\
(\wedge)_5 \arrow[urr] & & (\wedge)_3 \arrow[u, shift left] \arrow[d, shift left] & & (\wedge)_6 \arrow[ull] \\
(\wedge)_1 \arrow[u] \arrow[urr] & & (\wedge)_4 \arrow[ull] \arrow[urr] \arrow[u, shift left] & & (\wedge)_2 \arrow[u] \arrow[ull] \\
 & & (\wedge)_0 \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}\right)^* = \left(\begin{tikzcd}
 & & (\wedge)_7 & & \\
(\wedge)_5 \arrow[urr, equal] & & (\wedge)_3 \arrow[u, equal] & & (\wedge)_6 \arrow[ull, equal] \\
(\wedge)_1 \arrow[u] \arrow[urr] & & (\wedge)_4 \arrow[ull, equal] \arrow[urr, equal] & & (\wedge)_2 \arrow[u] \arrow[ull] \\
 & & (\wedge)_0 \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}\right)^*
$$

$$
\left(\begin{tikzcd}
 & & (\vee)_7 \arrow[dd, shift left] & & \\
(\vee)_5 \arrow[urr] & & (\vee)_3 \arrow[u] & & (\vee)_6 \arrow[ull] \\
(\vee)_1 \arrow[u] \arrow[urr] & & (\vee)_4 \arrow[ull] \arrow[urr] & & (\vee)_2 \arrow[u] \arrow[ull] \\
 & & (\vee)_0 \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}\right)^* = \left(\begin{tikzcd}
 & & (\vee)_7 & & \\
(\vee)_5 \arrow[urr, equal] & & (\vee)_3 \arrow[u] & & (\vee)_6 \arrow[ull, equal] \\
(\vee)_1 \arrow[u] \arrow[urr] & & (\vee)_4 \arrow[ull, equal] \arrow[urr, equal] & & (\vee)_2 \arrow[u] \arrow[ull] \\
 & & (\vee)_0 \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}\right)^*
$$

$$
\left(\begin{tikzcd}
 & & (\to)_7 \arrow[drr] & & \\
(\to)_5 \arrow[urr] \arrow[drr] & & (\to)_3 \arrow[u] \arrow[drr] & & (\to)_6 \arrow[ll] \\
(\to)_1 \arrow[u] \arrow[urr] \arrow[drr] & & (\to)_4 \arrow[urr] & & (\to)_2 \arrow[u] \\
 & & (\to)_0 \arrow[u] \arrow[urr] & &
\end{tikzcd}\right)^* = \left(\begin{tikzcd}
 & & (\to)_7 & & \\
(\to)_5 \arrow[urr] \arrow[drr] & & (\to)_3 \arrow[u, equal] \arrow[drr, equal] & & (\to)_6 \arrow[ull, equal] \\
(\to)_1 \arrow[u] \arrow[urr] \arrow[drr] & & (\to)_4 \arrow[urr] & & (\to)_2 \arrow[u, equal] \\
 & & (\to)_0 \arrow[u] \arrow[urr] & &
\end{tikzcd}\right)^*
$$

When the arrow $(\wedge)_i \longrightarrow (\wedge)_j$ belongs to $\mathsf{FCube}^*_\wedge$ we say that $(\wedge)_i \le (\wedge)_j$ is true "by the full and-cube". We write this as a derived rule as

$$\frac{}{(\wedge)_i \le (\wedge)_j}\ \&^*\mathsf{Cube}_{ij} \qquad \text{or just as:} \qquad \frac{}{(\wedge)_i \le (\wedge)_j}\ \&^*\mathsf{Cube}\ ,$$

and when the arrows $(\wedge)_i \rightleftarrows (\wedge)_j$ belongs to $\mathsf{FCube}^*_\wedge$ we say that $(\wedge)_i = (\wedge)_j$ is true "by the full and-cube", and we write that as:

$$\frac{}{(\wedge)_i = (\wedge)_j}\ \&^*\mathsf{Cube}_{ij} \qquad \text{or just as:} \qquad \frac{}{(\wedge)_i = (\wedge)_j}\ \&^*\mathsf{Cube}\ ,$$

and we do the same for '$\vee$' and '$\to$'.

The double-bar rule in sec.2.2 is a contraction of:

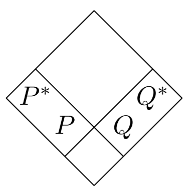
$$\frac{(P \vee Q^*)^* = (P \vee R^*)^* \qquad \dfrac{}{(P \vee R^*)^* = (P \vee R)^*}\ \vee^*\mathsf{Cube}_{64}}{(P \vee Q)^* = (P \vee R)^*}$$

2.5 How J-operators interact with connectives: ZHA*-valuations

Let's write $\mathsf{Exprs}(\mathsf{V})$ for the set of well-formed expressions built from a set of variables V, constants $\top$ and $\bot$, and operations $\land$, $\lor$, $\to$, $\leftrightarrow$, $\neg$, $\cdot^*$; each one of the sets $\mathsf{ECube}_\land$, $\mathsf{ECube}_\lor$ and $\mathsf{ECube}_\to$ of the last sections is an 8-element subset of $\mathsf{Exprs}(\{P,Q\})$.

If $\mathsf{E} \subseteq \mathsf{Exprs}(\mathsf{V})$, a *ZHA*-valuation for* E, or an E-*valuation*, is a triple (H,J,v), where H is a ZHA, J is a J-operator on H, and $v : \mathsf{V} \to H$ is a function that assigns a truth-value in H to each variable in V. There is a natural way to extend v to a function $v' : \mathsf{Exprs}(\mathsf{V}) \to H$, and we can restrict v' to a function $v'' : \mathsf{E} \to H$.

We can draw all components of an $\mathsf{ECube}_\lor$-valuation (H,J,v) together by writing 'P' and 'Q' on the positions $v(P)$ and $v(Q)$ on (H,J), as we did in sec.1.6. We will often also write 'P^*' and 'Q^*' on the positions $v'(P^*)$ and $v'(Q^*)$ for clarity. For example:



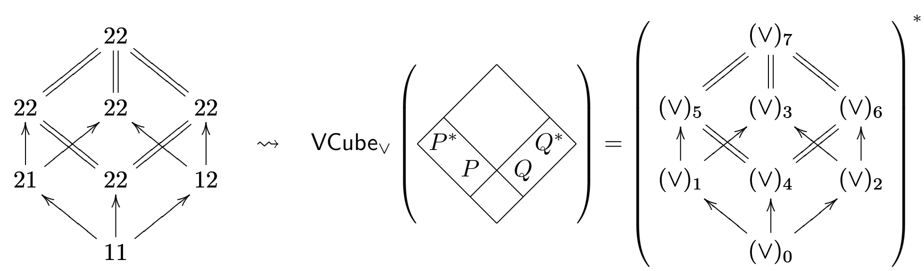
$$\begin{array}{rcl} v(P) &=& 10 \\ v(Q) &=& 01 \\ v'(P^*) &=& 20 \\ v'(Q^*) &=& 02 \end{array} \qquad \begin{array}{rcccl} v'(P \lor Q) &=& 11 &=& v''((\lor)_0) \\ v'(P^* \lor Q) &=& 21 &=& v''((\lor)_1) \\ v'(P \lor Q^*) &=& 12 &=& v''((\lor)_2) \\ v'(P^* \lor Q^*) &=& 22 &=& v''((\lor)_3) \\ v'((P \lor Q)^*) &=& 22 &=& v''((\lor)_4) \\ v'((P^* \lor Q)^*) &=& 22 &=& v''((\lor)_5) \\ v'((P \lor Q^*)^*) &=& 22 &=& v''((\lor)_6) \\ v'((P^* \lor Q^*)^*) &=& 22 &=& v''((\lor)_7) \end{array}$$

Each $\mathsf{ECube}_\lor$-valuation (H,J,v) induces a non-strict partial order on $\mathsf{ECube}_\lor$, in which $(\lor)_i \leq (\lor)_j$ iff $v''((\lor)_i) \leq v''((\lor)_j)$. We will write that partial order as

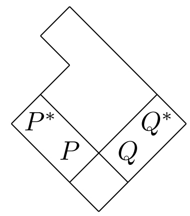
$$\begin{array}{rcll} \mathsf{VCube}_\lor(H,J,v) &=& (\mathsf{ECube}_\lor, \mathsf{vcube}_\lor(H,J,v)) & \quad\text{or:} \\ \mathsf{VCube}_\lor(v) &=& (\mathsf{ECube}_\lor, \mathsf{vcube}_\lor(v)) & \end{array}$$

We will often omit the 'H' and the 'J' and write just $\mathsf{VCube}_\lor(v)$.

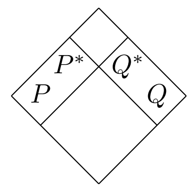
It is easy to calculate by hand the partial orders $\mathsf{VCube}_\lor(v)$, $\mathsf{VCube}_\land(v)$ or $\mathsf{VCube}_\to(v)$ associated to a given valuation (H,J,v): we write in the position corresponding to each '$(\odot)_i$' of the cube the value of the corresponding $v''((\odot)_i)$, then we draw the arrows — some of them will be '='s —, then transfer the arrows to the cube with '$(\odot)_i$'s. For example:



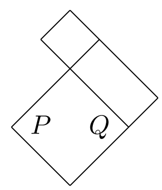
A very important fact. For any i and j,

$(\vee)_i \leq (\vee)_j$ is a theorem iff it is true in 

,

$(\wedge)_i \leq (\wedge)_j$ is a theorem iff it is true in 

,

$(\to)_i \leq (\to)_j$ is a theorem iff it is true in 

.

We will call the valuations at the right above $(H_\vee, J_\vee, v_\vee)$, $(H_\wedge, J_\wedge, v_\wedge)$, $(H_\to, J_\to, v_\to)$. In the language of partial orders, the very important fact can be stated as:

$$\begin{array}{rcl} \mathsf{FCube}^*_\vee &=& \mathsf{VCube}_\vee(v_\vee), \\ \mathsf{FCube}^*_\wedge &=& \mathsf{VCube}_\wedge(v_\wedge), \\ \mathsf{FCube}^*_\to &=& \mathsf{VCube}_\to(v_\to). \end{array}$$

Suppose that (H_1, J_1, v_1), (H_2, J_2, v_2), $\ldots$ are valuations on — say — $\mathsf{ECube}_\to$. This always holds

$$\mathsf{FCube}^*_\to \subseteq \mathsf{VCube}_\to(v_i),$$

because all ZHA*-theorems are true in all valuations. We say that:

$$\begin{array}{rll} v_i \text{ is } \textit{good} & \text{when} & \mathsf{FCube}^*_\to = \mathsf{VCube}_\to(v_i), \\ v_i \text{ and } v_j \text{ are } \textit{equivalent} & \text{when} & \mathsf{VCube}_\to(v_i) = \mathsf{VCube}_\to(v_j), \\ v_i \text{ is } \textit{better than } v_j & \text{when} & \mathsf{VCube}_\to(v_i) \subseteq \mathsf{VCube}_\to(v_j). \end{array}$$

Also, a *non-theorem* is an arrow $(\to)_i \leq (\to)_j$ that is not in $\mathsf{FCube}^*_\to$; a *countermodel* for a non-theorem $(\to)_i \leq (\to)_j$ is a valuation that "falsifies" $(\to)_i \leq (\to)_j$, i.e., a valuation in which $(\to)_i \leq (\to)_j$ is not true. Note that a valuation is "good" when it is a countermodel for all non-theorems at once, and a valuation v_1 is strictly better than v_2 when v_1 falsifies all non-theorems that v_2 falsifies, plus some.

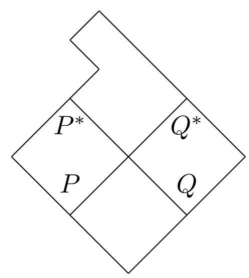
In sec.18 of [PH1] we saw that ZHAs do not distinguish as many sentences as arbitrary Heyting Algebras; we saw a sentence $S_P \vee S_Q \vee S_R$ that had a countermodel in a HA, but that ZHAs "think" that its value is always $\top$. To formalize and extend this idea we need a slight abuse of language. We will say that an E-valuation (H, J, v) "distinguishes all elements of E", or "distinguishes E", instead of the more precise "is a countermodel for all non-theorems of the form $E_i \leq E_j$ at once"; and we will say that v_1 "distinguishes more elements of E" than v_2 when v_1 is better than v_2. A set of expressions E is *ZHA*-good* when there is a valuation that distinguishes all elements of E. So:

$$\begin{array}{rcl}
\{S_P \vee S_Q \vee S_R, \top\} & \text{is not} & \text{ZHA}^*\text{-good}, \\
\mathsf{ECube}_\vee & \text{is} & \text{ZHA}^*\text{-good}, \\
\mathsf{ECube}_\wedge & \text{is} & \text{ZHA}^*\text{-good}, \\
\mathsf{ECube}_\to & \text{is} & \text{ZHA}^*\text{-good}.
\end{array}$$

ZHAs with J-operators do not distinguish all sets of expressions, but they distinguish some sets, like $\mathsf{ECube}_\vee$, $\mathsf{ECube}_\wedge$, $\mathsf{ECube}_\to$, that are *very* useful.

Note that this valuation

$$(H_{\wedge\vee}, J_{\wedge\vee}, v_{\wedge\vee}) \quad =$$



distinguishes $\mathsf{ECube}_\vee \cup \mathsf{ECube}_\wedge$, but it does not distinguish $\mathsf{ECube}_\to$ — it thinks that $P \to Q$ and $P^* \to Q$ are equal.

An observation. I arrived at the cubes $\mathsf{FCube}^*_\wedge$, $\mathsf{FCube}^*_\vee$, $\mathsf{FCube}^*_\to$ by taking the material in the corollary 5.3 of chapter 5 in [Bel88] and trying to make it fit into less mental space (as discussed in [Och13]); after that I wanted to be sure that each arrow that is not in a full cube has a countermodel, and I found the countermodels one by one; then I wondered if I could find a single countermodel for all non-theorems in $\mathsf{FCube}^*_\wedge$ (and the same for $\mathsf{FCube}^*_\vee$ and $\mathsf{FCube}^*_\to$), and I tried to start with a valuation that distinguished *some* elements in $\mathsf{ECube}_\wedge$, and change it bit by bit, getting valuations that distinguished more elements at every step. Eventually I arrived at $v_\wedge$, $v_\vee$ and at $v_\to$, and at the — surprisingly nice — "very important fact".

2.6 Good valuations

If $(\vee)_i \leq (\vee)_j$ is true in $\mathsf{FCube}^*_\vee$ then it is a theorem, and it holds in every $\mathsf{ECube}_\vee$-valuation (H, J, v) — so $\mathsf{FCube}^*_\vee \subseteq \mathsf{VCube}_\vee(H, J, v)$. The important information that a ZHA*-valuation carries is in its '$\not\leq$'s, as they say that something *cannot* be a theorem and that (H, J, v) is a countermodel showing that. For example, in $(H_\vee, J_\vee, v_\vee)$ we had $(\vee)_7 \not\leq (\vee)_3$; if we could prove, using new derived rules like the ones in sec.2.4, that $(\vee)_7 \leq (\vee)_3$ is a theorem, then we would have $(\vee)_7 \leq (\vee)_3$ in all valuations, which is incompatible with the $(\vee)_7 \not\leq (\vee)_3$ in $\mathsf{VCube}_\vee(H_\vee, J_\vee, v_\vee)$.

Note that this means that: 1) that if a statement of the form $(\vee)_i \leq (\vee)_j$ is not in $\mathsf{FCube}^*_\vee$ then it cannot be proved, i.e., all attempts to find a tree-proof for that $(\vee)_i \leq (\vee)_j$ using the HA rules and J1, J2, J3 are bound to fail; 2) the theorems of the form $(\vee)_i \leq (\vee)_j$ are exactly the ones that are true in $\mathsf{VCube}_\vee(H_\vee, J_\vee, v_\vee)$, so we can use $(H_\vee, J_\vee, v_\vee)$ as a *reminder* for which sentences of the form $(\vee)_i \leq (\vee)_j$ are theorems — and the same for '$\wedge$' and '$\to$'.

3 Visualizing the algebra of J-operators

The J-operators on a Heyting Algebra H, $\mathsf{J\text{-}ops}(H)$, have a natural lattice structure, in which the bottom element is the identity function and whose top element is the operator that takes all elements to $\top$. The bottom element of $\mathsf{J\text{-}ops}(H)$ is the "quotient" (in the terminology of sec.3.1) with the maximum number of equivalence classes, the top element is the "quotient" with a single equivalence class. We can refer to them as $\bot, \top \in \mathsf{J\text{-}ops}(H)$, and define operations $\wedge, \vee : \mathsf{J\text{-}ops}(H)^2 \to \mathsf{J\text{-}ops}(H)$; this is the *algebra of J-operators* on H.

Some important J-operators are called "closed quotients", "open quotients" and "forcing quotients". In this section we will see how to visualize the algebra $\mathsf{J\text{-}ops}(H)$ when H is a ZHA, and how to visualize these special J-operators and understand how they interact — including a way to factor arbitrary J-operators on a ZHA as a conjunction of finitely many basic ("polynomial") J-operators.

3.1 Polynomial J-operators

It is not hard to check that for any Heyting Algebra H and any $Q, R \in H$ the operations $(\neg\neg)$, ..., $(\vee Q \wedge {\to}R)$ below are J-operators:

$$\begin{array}{rcl}
(\neg\neg)(P) &=& \neg\neg P \\
(\to\to R)(P) &=& (P \to R) \to R \\
(\vee Q)(P) &=& P \vee Q \\
(\to R)(P) &=& P \to R \\
(\vee Q \wedge \to R)(P) &=& (P \vee Q) \wedge (P \to R)
\end{array}$$

Checking that they are J-operators means checking that each of them obeys J1, J2, J3. Let's define formally what are J1, J2 and J3 "for a given $F : H \to H$":

$$\begin{array}{rcc}
\mathsf{J1}_F &:=& (P \leq F(P)) \\
\mathsf{J2}_F &:=& (F(P) = F(F(P)) \\
\mathsf{J3}_F &:=& (F(P \wedge P') = F(P) \wedge F(P'))
\end{array}$$

and:

$$\mathsf{J123}_F \quad := \quad \mathsf{J1}_F \wedge \mathsf{J2}_F \wedge \mathsf{J3}_F .$$

Checking that $(\neg\neg)$ obeys J1, J2, J3 means proving $\mathsf{J123}_{(\neg\neg)}$ using only the rules from intuitionist logic from sec.**??**; we will leave the proof of this, of and $\mathsf{J123}_{(\to\to R)}$, $\mathsf{J123}_{(\vee Q)}$, and so on, to the reader.

The J-operator $(\vee Q \wedge \to R)$ is a particular case of building more complex J-operators from simpler ones. If $J, K : H \to H$, we define:

$$(J \wedge K) := \lambda P{:}H.(J(P) \wedge K(P))$$

it not hard to prove $\mathsf{J123}_{(J\wedge K)}$ from $\mathsf{J123}_J$ and $\mathsf{J123}_K$ using only the rules from intuitionistic logic.

The J-operators above are the first examples of J-operators in Fourman and Scott's "Sheaves and Logic" ([FS79]); they appear in pages 329–331, but with these names (our notation for them is at the right):

(i) *The closed quotient,*

$$J_a p = a \vee p \qquad J_Q = (\vee Q).$$

(ii) *The open quotient,*

$$J^a p = a \to p \qquad J^R = (\to R).$$

(iii) *The Boolean quotient.*

$$B_a p = (p \to a) \to a \qquad B_R = (\to\to R).$$

(iv) *The forcing quotient.*

$$(J_a \wedge J^b)p = (a \vee p) \wedge (b \to p) \qquad (J_Q \wedge J^R) = (\vee Q \wedge \to R).$$

(vi) *A mixed quotient.*

$$(B_a \wedge J^a)p = (p \to a) \to p \qquad (B_Q \wedge J^Q) = (\to\to Q \wedge \to Q).$$

The last one is tricky. From the definition of B_a and J^a what we have is

$$(B_a \wedge J^a)p = ((p \to a) \to a) \wedge (a \to p),$$

but it is possible to prove

$$((p \to a) \to a) \wedge (a \to p) \quad \leftrightarrow \quad ((p \to a) \to p)$$

intuitionistically.

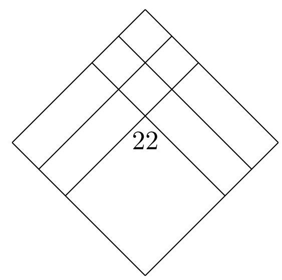
The operators above are "polynomials on $P, Q, R, \to, \wedge, \vee, \bot$" in the terminology of Fourman/Scott: "If we take a polynomial in $\to, \wedge, \vee, \bot$, say, $f(p, a, b, \ldots)$, it is a decidable question whether for all $a, b, \ldots$ it defines a J-operator" (p.331).

When I started studying sheaves I spent several years without any visual intuition about the J-operators above. I was saved by ZHAs and brute force —

and the brute force method also helps in testing if a polynomial (in the sense above) is a J-operator in a particular case. For example, take the operators $\lambda P{:}H.(P \wedge 22)$ and $(\vee 22)$ on $H = [00, 44]$:

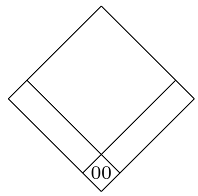
$$
\lambda P{:}H.(P \wedge 22) \quad = \quad
\begin{array}{ccccccccc}
 & & & & 22 & & & & \\
 & & & 22 & & 22 & & & \\
 & & 22 & & 22 & & 22 & & \\
 & 21 & & 22 & & 22 & & 12 & \\
20 & & 21 & & 22 & & 12 & & 02 \\
 & 20 & & 21 & & 12 & & 02 & \\
 & & 20 & & 11 & & 02 & & \\
 & & & 10 & & 01 & & & \\
 & & & & 00 & & & &
\end{array}
$$

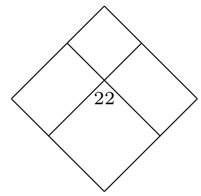
$$
(\vee 22) \quad = \quad
\begin{array}{ccccccccc}
 & & & & 44 & & & & \\
 & & & 43 & & 34 & & & \\
 & & 42 & & 33 & & 24 & & \\
 & 42 & & 32 & & 23 & & 24 & \\
42 & & 32 & & 22 & & 23 & & 24 \\
 & 32 & & 22 & & 22 & & 23 & \\
 & & 22 & & 22 & & 22 & & \\
 & & & 22 & & 22 & & & \\
 & & & & 22 & & & &
\end{array}
\quad =
$$

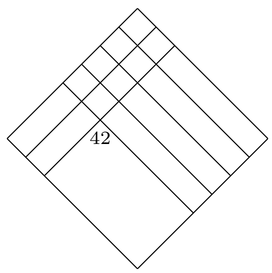


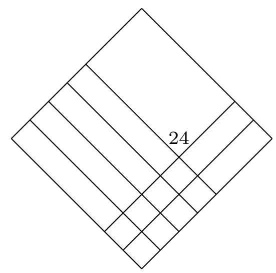
The first one, $\lambda P{:}H.(P \wedge 22)$, is not a J-operator; one easy way to see that is to look at the region in which the result is 22 — its top element is 44, and this violates the conditions on slash-operators in sec.1.5. The second operator, $(\vee 22)$, is a slash operator and a J-operator; at the right we introduce a convenient notation for visualizing the action of a polynomial slash-operator, in which we draw only the contours of the equivalence classes and the constants that appear in the polynomial.

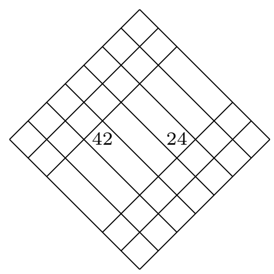
Using this new notation, we have:

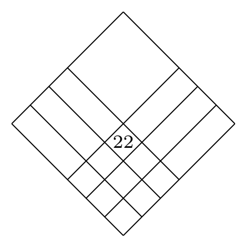
$(\neg\neg) \;=\; (\to\to 00) \;=$ 

$(\to\to 22) \;=$ 

$(\vee 42) \;=$ 

$(\to 24) \;=$ 

$(\vee 42 \wedge \to 24) \;=$ 

$(\to\to 22 \wedge \to 22) \;=$ 

Note that the slashing for $(\vee 42 \wedge \to 24)$ has all the cuts for $(\vee 42)$ plus all the cuts for $(\to 24)$, and $(\vee 42 \wedge \to 24)$ "forces $42 \le 24$" in the following sense: if $P^* = (\vee 42 \wedge \to 24)(P)$ then $42^* \le 24^*$.

3.2 An algebra of piccs

We saw in the last section a case in which $(J \wedge K)$ has all the cuts from J plus all the cuts from K; this suggests that we *may* have an operation dual to that, that behaves as this: $(J \vee K)$ has exactly the cuts that are both in J and in K:

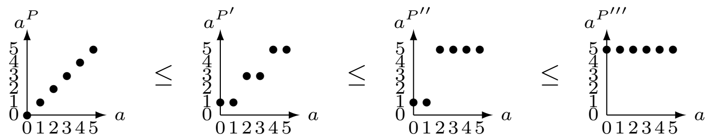
$$\begin{array}{rcl} \mathsf{Cuts}(J \wedge K) & = & \mathsf{Cuts}(J) \cup \mathsf{Cuts}(K) \\ \mathsf{Cuts}(J \vee K) & = & \mathsf{Cuts}(J) \cap \mathsf{Cuts}(K) \end{array}$$

And it $J_1, \ldots, J_n$ are all the slash-operators on a given ZHA, then

$$\begin{array}{rclcl}
\mathsf{Cuts}(J_1 \wedge \ldots \wedge J_n) & = & \mathsf{Cuts}(J_1) \cup \ldots \cup \mathsf{Cuts}(J_k) & = & \text{(all cuts)} \\
\mathsf{Cuts}(J_1 \vee \ldots \vee J_n) & = & \mathsf{Cuts}(J_1) \cap \ldots \cap \mathsf{Cuts}(J_k) & = & \text{(no cuts)}
\end{array}$$

yield the minimal element and the maximal element, respectively, of an algebra of slash-operators; note that the slash-operator with "all cuts" is the identity map $\lambda P\colon H.P$, and the slash-operator with "no cuts" is the one that takes all elements to $\top$: $\lambda P\colon H.\top$. This yields a lattice of slash-operators, in which the partial order is $J \leq K$ iff $\mathsf{Cuts}(J) \supseteq \mathsf{Cuts}(K)$. This is somewhat counterintuitive if we think in terms of cuts — the order seems to be reversed — but it makes a lot of sense if we think in terms of piccs (sec.1.1) instead.

Each picc P on $\{0, \ldots, n\}$ has an associated function $\cdot^P$ that takes each element to the top element of its equivalence class. If we define $P \leq P'$ to mean $\forall a \in \{0, \ldots, n\}.\, a^P \leq a^{P'}$, then we have this:



$$\begin{array}{ccccccc}
0|1|2|3|4|5 & \leq & 01|23|45 & \leq & 01|2345 & \leq & 012345 \\
P & \leq & P' & \leq & P'' & \leq & P'''
\end{array}$$

This yields a partial order on piccs, whose bottom element is the identity function $0|1|2|\ldots|n$, and the top element is $012\ldots n$, that takes all elements to n.

The piccs on $\{0, \ldots, n\}$ form a Heyting Algebra, where $\top = 01\ldots n$, $\bot = 0|1|\ldots|n$, and '$\wedge$' and '$\vee$' are the operations that we have discussed above; it is possible to define a '$\to$' there, but this '$\to$' is not going to be useful for us and we are mentioning it just as a curiosity. We have, for example:

$$\begin{tikzcd}[column sep=tiny]
& 01234 & \\
& 01|234 \arrow[u] & \\
0|1|234 \arrow[ur] & & 01|2|34 \arrow[ul] \\
& 0|1|2|34 \arrow[ul] \arrow[ur] & \\
& 0|1|2|3|4 \arrow[u] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
& \top & \\
& P \vee Q \arrow[u] & \\
P \arrow[ur] & & Q \arrow[ul] \\
& P \wedge Q \arrow[ul] \arrow[ur] & \\
& \bot \arrow[u] &
\end{tikzcd}$$

3.3 An algebra of J-operators

Fourman and Scott define the operations $\wedge$ and $\vee$ on J-operators in pages 325 and 329 ([FS79]), and in page 331 they list ten properties of the algebra of

J-operators:

(i)	$J_a \vee J_b$	$=$	$J_{a\vee b}$	$(\vee 21) \vee (\vee 12) = (\vee 22)$
(ii)	$J^a \vee J^b$	$=$	$J^{a\wedge b}$	$(\to 32) \vee (\to 23) = (\to 22)$
(iii)	$J_a \wedge J_b$	$=$	$J_{a\wedge b}$	$(\vee 21) \wedge (\vee 12) = (\vee 11)$
(iv)	$J^a \wedge J^b$	$=$	$J^{a\vee b}$	$(\to 32) \wedge (\to 23) = (\to 33)$
(v)	$J_a \wedge J^a$	$=$	$\bot$	$(\vee 22) \wedge (\to 22) = (\bot)$
(vi)	$J_a \vee J^a$	$=$	$\top$	$(\vee 22) \vee (\to 22) = (\top)$
(vii)	$J_a \vee K$	$=$	$K \circ J_a$	
(viii)	$J^a \vee K$	$=$	$J^a \circ K$	
(ix)	$J_a \vee B_a$	$=$	B_a	
(x)	$J^a \vee B_b$	$=$	$B_{a\to b}$	

The first six are easy to visualize; we won't treat the four last ones. In the right column of the table above we've put a particular case of (i), …, (vi) in our notation, and the figures below put all together.

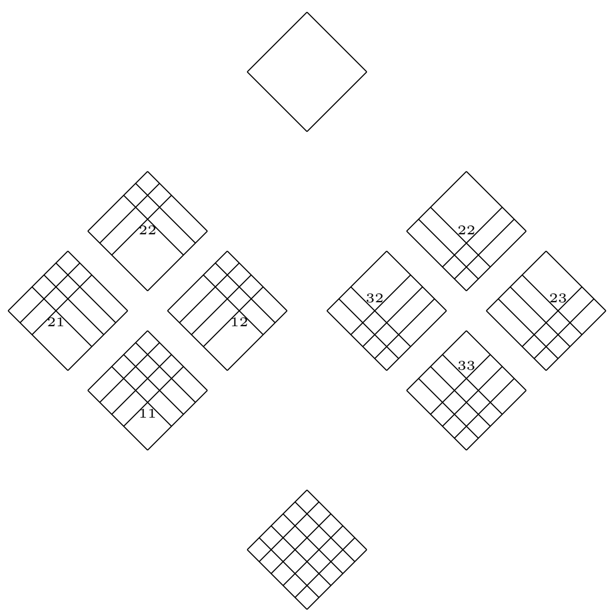
In Fourman and Scott's notation,

$$
\begin{tikzcd}
 & & & J_\top = \top = J^\bot & & & \\
 & J_{22} \arrow[urr] & & & & J^{22} \arrow[ull] & \\
J_{21} \arrow[ur] & & J_{12} \arrow[ul] & & J^{32} \arrow[ur] & & J^{23} \arrow[ul] \\
 & J_{11} \arrow[ul] \arrow[ur] & & & & J^{11} \arrow[ul] \arrow[ur] & \\
 & & & J_\bot = \bot = J^\top \arrow[ull] \arrow[urr] & & &
\end{tikzcd}
$$

in our notation,

$$\begin{tikzcd}[column sep=tiny]
 & & & (\top\vee) = (\lambda P.\top) = (\bot{\to}) & & & \\
 & (22\vee) \arrow[urr] & & & & (22{\to}) \arrow[ull] & \\
(21\vee) \arrow[ur] & & (12\vee) \arrow[ul] & & (32{\to}) \arrow[ur] & & (23{\to}) \arrow[ul] \\
 & (11\vee) \arrow[ul] \arrow[ur] & & & & (33{\to}) \arrow[ul] \arrow[ur] & \\
 & & & (\bot\vee) = (\lambda P.P) = (\top{\to}) \arrow[ull] \arrow[urr] & & &
\end{tikzcd}$$

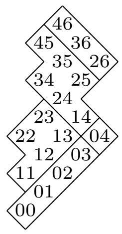
and drawing the polynomial J-operators as in sec.3.1:

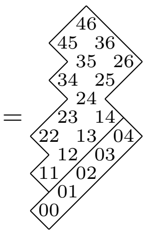


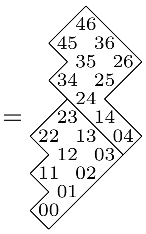
3.4 All slash-operators are polynomial

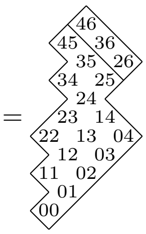
Here is an easy way to see that all slashings — i.e., J-operators on ZHAs — are polynomial. Every slashing J has only a finite number of cuts; call them

$J_1, \ldots, J_n$. For example:

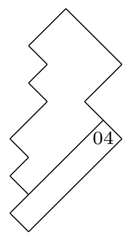
$J =$ 

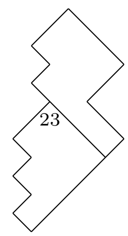
$J_1 =$ 

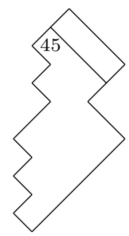
$J_2 =$ 

$J_3 =$ 

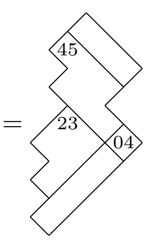
Each cut J_i divides the ZHA into an upper region and a lower region, and $J_i(00)$ yields the top element of the lower region. Also, $(\to\!\!\to J_i(00))$ is a polynomial way of expressing that cut:

$J_1 =$
$(\to\!\!\to 04) =$ 

$J_2 =$
$(\to\!\!\to 23) =$ 

$J_3 =$
$(\to\!\!\to 45) =$ 

The conjunction of these '$(\to\!\!\to J_i(00))$'s yields the original slashing:

$(\to\!\!\to 04) \land (\to\!\!\to 23) \land (\to\!\!\to 45) =$  $= J$

4 Question marks

Every ZHA H is equivalent — by the constructions explained in sections 14–17 of [Och17] — to a 2-column graph (P, A). To be more precise, each ZHA H has an associated 2CG (P, A), such that this holds: the partial order $(H, \leq)$ is equivalent to $(\mathcal{O}_A(P), \subseteq)$, where $\mathcal{O}_A(P)$ is the "order topology" on P (see sections 12–13 of [Och17]). We will use squiggly arrows to mean "is associated to”:

$$
\begin{array}{ccccc}
 & 46 & & & \\
45 & & 36 & & \\
 & 35 & & 26 & \\
34 & & 25 & & \\
 & 24 & & & \\
23 & & 14 & & \\
22 & 13 & & 04 & \\
 & 12 & & 03 & \\
11 & & 02 & & \\
 & 01 & & & \\
00 & & & &
\end{array}
\quad \leftrightsquigarrow \quad
\left(
\begin{tikzcd}[row sep=small]
 & \_6 \arrow[d] \\
 & \_5 \arrow[d] \arrow[ddl] \\
4\_ \arrow[ur] \arrow[d] & \_4 \arrow[d] \\
3\_ \arrow[ur] \arrow[d] & \_3 \arrow[d] \\
2\_ \arrow[r] \arrow[d] & \_2 \arrow[d] \\
1\_ \arrow[r] & \_1
\end{tikzcd}
\right).
$$

Choose a subset $Q \subseteq P$ — the “set of question marks”. We will represent Q graphically by writing a ‘?’ next to each point of P that is in Q. For example, if $Q = \{\_2, \_3\}$ in the 2CG (P, A) above, then

$$
((P,A),Q) = \left(
\begin{tikzcd}[row sep=small]
 & \_6 \arrow[d] & \\
 & \_5 \arrow[d] \arrow[ddl] & \\
4\_ \arrow[ur] \arrow[d] & \_4 \arrow[d] & \\
3\_ \arrow[ur] \arrow[d] & \_3 \arrow[d] & ? \\
2\_ \arrow[r] \arrow[d] & \_2 \arrow[d] & ? \\
1\_ \arrow[r] & \_1 &
\end{tikzcd}
\right)
$$

Each choice of a subset $Q \subseteq P$ induces an operation that “erases the information at question marks”, that works like this. Each element $B \in H$ corresponds to an open subset $B' \subseteq P$, and to a characteristic function $B'' : P \to \{0,1\}$:

$$
23 \quad \leftrightsquigarrow \quad \mathsf{pile}(23) = \{2\_, 1\_, \ \_3, \_2, \_1\} = \left(
\begin{tikzcd}[row sep=tiny, column sep=small]
 & 0 \\
 & 0 \arrow[ddl] \\
0 \arrow[ur] & 0 \\
0 \arrow[ur] & 1 \\
1 \arrow[r] & 1 \\
1 \arrow[r] & 1
\end{tikzcd}
\right)
$$

If we replace the information on the points of $Q \subseteq P$ by question marks we get another function, $B''' : P \to \{0, ?, 1\}$:

$$
\left(
\begin{tikzcd}[row sep=tiny, column sep=small]
 & 0 \\
 & 0 \arrow[ddl] \\
0 \arrow[ur] & 0 \\
0 \arrow[ur] & 1 \\
1 \arrow[r] & 1 \\
1 \arrow[r] & 1
\end{tikzcd}
\right)
\longmapsto
\left(
\begin{tikzcd}[row sep=tiny, column sep=small]
 & 0 \\
 & 0 \arrow[ddl] \\
0 \arrow[ur] & 0 \\
0 \arrow[ur] & ? \\
1 \arrow[r] & ? \\
1 \arrow[r] & 1
\end{tikzcd}
\right)
$$

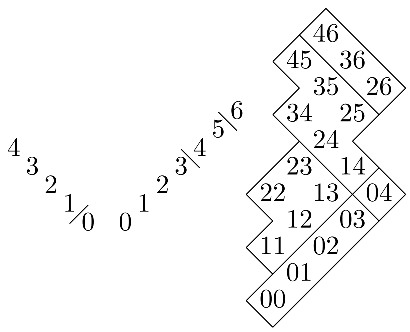
Take another element $C \in H$ and “erase its information on the question marks” according to the same procedure; this yields a function $C''' : P \to \{0, ?, 1\}$. We can now understand the equivalence relation induced by Q. We will say that B and C are Q-equivalent (for this choice of Q; notation: $B \sim_Q C$) if and only if $B''' = C'''$. With the Q of the example we have $23 \sim_Q 22$.

4.1 Q-equivalences and slashings

Take two neighboring elements of H; their 'pile's differ by a single element. For example, take 22 and 23: we have $\mathsf{pile}(23)\backslash\mathsf{pile}(22) = \{\_3\}$. They are Q-equivalent if and only if $\_3 \in Q$. Actually, what we have, in the H of the example, is:

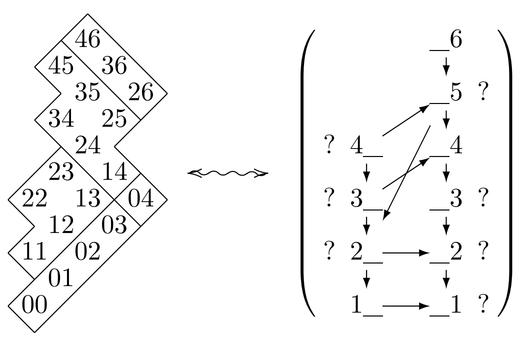
$$\begin{array}{lcl@{\qquad\qquad}lcl}
22 \sim_P 23 & \leftrightarrow & _3 \in Q & 22 \not\sim_P 23 & \leftrightarrow & _3 \notin Q \\
12 \sim_P 13 & \leftrightarrow & _3 \in Q & 12 \not\sim_P 13 & \leftrightarrow & _3 \notin Q \\
02 \sim_P 03 & \leftrightarrow & _3 \in Q & 02 \not\sim_P 03 & \leftrightarrow & _3 \notin Q
\end{array}$$

so — by the ideas of sections 1.2 and 1.4 — $\_3 \notin Q$ is equivalent to a northwest cut 2/3! This gives us a way to convert between 'Q's and slashings. In our favorite example,



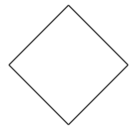
$$\begin{array}{ll@{\qquad}ll}
 & 5 \not\sim_R 6 & & _6 \notin Q \\
 & 4 \sim_R 5 & & _5 \in Q \\
3 \sim_L 4 & 3 \not\sim_R 4 & 4_ \in Q & _4 \notin Q \\
2 \sim_L 3 & 2 \sim_R 3 & 3_ \in Q & _3 \in Q \\
1 \sim_L 2 & 1 \sim_R 2 & 2_ \in Q & _2 \in Q \\
0 \not\sim_L 1 & 0 \sim_R 1 & 1_ \notin Q & _1 \in Q
\end{array}$$

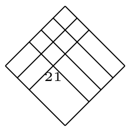
So the slashing $S = (4321/0,\ 0123\backslash 45\backslash 6)$ corresponds to $Q = \{4\_, 3\_, 2\_, \_1, \_2, \_3, \_5\}$ — we have $(\sim_S) = (\sim_Q)$. We can represent that with a figure:



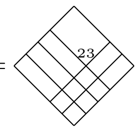
4.2 An algebra of question marks

We can translate the diagrams from sec.3.3 to the language of question marks. Let's draw four points of the lattice:

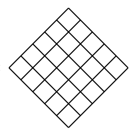
$(\top\vee) = (\lambda P.\top) = (\bot\to) =$  $= \begin{pmatrix} ? & 4\_ & \_4 & ? \\ ? & 3\_ & \_3 & ? \\ ? & 2\_ & \_2 & ? \\ ? & 1\_ & \_1 & ? \end{pmatrix}$

$(21\vee) =$ 

$= \begin{pmatrix} & 4\_ & \_4 & \\ & 3\_ & \_3 & \\ ? & 2\_ & \_2 & \\ ? & 1\_ & \_1 & ? \end{pmatrix}$

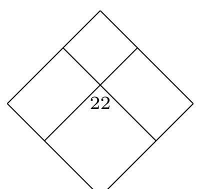
$(23\to) =$ 

$= \begin{pmatrix} ? & 4\_ & \_4 & ? \\ ? & 3\_ & \_3 & \\ & 2\_ & \_2 & \\ & 1\_ & \_1 & \end{pmatrix}$

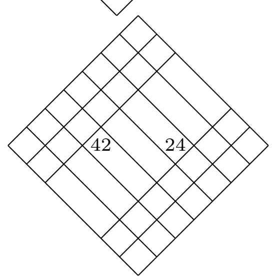
$(\bot\vee) = (\lambda P.P) = (\top\to) =$  $= \begin{pmatrix} & 4\_ & \_4 & \\ & 3\_ & \_3 & \\ & 2\_ & \_2 & \\ & 1\_ & \_1 & \end{pmatrix}$

Let's write $\mathsf{qmarks}(J)$ for the set of question marks of a J-operator J. It's easy to see that $\mathsf{qmarks}(J \wedge K) = \mathsf{qmarks}(J) \cap \mathsf{qmarks}(K)$ and $\mathsf{qmarks}(J \vee K) = \mathsf{qmarks}(J) \cup \mathsf{qmarks}(K)$.

Here are how the boolean quotients and the forcing quotients of sec.3.1 look when translated to question marks:

$(\to\to 22) \quad =$ 

$= \begin{pmatrix} ? & 4\_ & \_4 & ? \\ & 3\_ & \_3 & \\ ? & 2\_ & \_2 & ? \\ ? & 1\_ & \_1 & ? \end{pmatrix}$

$(\vee 42 \wedge \to 24) \quad =$ 

$= \begin{pmatrix} & 6\_ & \_6 \\ & 5\_ & \_5 \\ ? & 4\_ & \_4 \\ ? & 3\_ & \_3 \\ & 2\_ & \_2 \\ & 1\_ & \_1 \end{pmatrix}$

4.3 Open sets of certain form

Fix a ZHA H and a slashing S on H. Let (P, A) be the 2-column graph associated to H and let Q be the set of question marks associated to S.

In section 4 we started with an element $B \in H$, converted it to an open set B', took the characteristic function B'' of that set, and replaced some of the '0's and '1's by '?'s in B'' to make B'''. We can think of the B''' as a specification, and ask: which open sets C' are of the form B''', i.e., what characteristic functions of *open* sets can be obtained by taking B''' and replacing some of its '?'s by '0's and all the other '?'s in it by '1'? In other words, what are the possible ways to start with B''' and replace all its '?'s by '0's and '1's without getting an arrow '$1 \to 0'$'?

Let's write $\mathsf{Opens}(B''')$ for "the set of all open sets of the form B'''". From this point on, and until the end of this section, we will be a bit sloppy; elements of $\mathsf{Opens}(B''')$ will be thought as being either open sets, or their characteristic functions, or the elements of H associated to those open sets.

It is easy to see that $\mathsf{Opens}(B''')$ is the S-equivalence class of B, and the Q-equivalence class of B. We know that S-equivalence classes have maximal and minimal elements.

There is an easy way to calculate the maximal and the minimum elements of $\mathsf{Opens}(B''')$ by hand by working only on the '0's, '1's and '?'s drawn on the 2-column graph. First we get rid of the '?'s that point to '1's by replacing them with '1's, and we get rid of the '?'s that have '0's pointing to them by replacing them with '0's, and we call the result B''''. Here is an example, starting with $B = 12$ in our favorite ZHA with a slashing:

$$B \;=\; 12 \rightsquigarrow \left(\begin{tikzcd} & 0 \arrow[d] \\ & 0 \arrow[d] \arrow[ddl] \\ 0 \arrow[d] \arrow[ur] & 0 \arrow[d] \\ 0 \arrow[d] \arrow[ur] & 0 \arrow[d] \\ 0 \arrow[d] \arrow[r] & 1 \arrow[d] \\ 1 \arrow[r] & 1 \end{tikzcd}\right) \rightsquigarrow \left(\begin{tikzcd} & 0 \arrow[d] \\ & ? \arrow[d] \arrow[ddl] \\ ? \arrow[d] \arrow[ur] & 0 \arrow[d] \\ ? \arrow[d] \arrow[ur] & ? \arrow[d] \\ ? \arrow[d] \arrow[r] & ? \arrow[d] \\ 1 \arrow[r] & ? \end{tikzcd}\right) \rightsquigarrow \left(\begin{tikzcd} & 0 \arrow[d] \\ & 0 \arrow[d] \arrow[ddl] \\ 0 \arrow[d] \arrow[ur] & 0 \arrow[d] \\ 0 \arrow[d] \arrow[ur] & ? \arrow[d] \\ ? \arrow[d] \arrow[r] & ? \arrow[d] \\ 1 \arrow[r] & 1 \end{tikzcd}\right) \;=\; B''''$$

It turns out that we can replace all '?'s in B'''' by '1's and obtain an open set, and this yields the maximum element of $\mathsf{Opens}(B''')$ — 23 in the example —, and we can also replace all the '?'s in B'''' by '0's and this also yields an open set, that this time is the minimum element of $\mathsf{Opens}(B''')$ — 11 in the example.

These are our two first examples of methods for for reconstructing information after erasing it. One method reconstructs it in a maximal way, and returns the maximal possible result; the other methods reconstructs it in a minimal way and returns the minimal possible result.

4.4 Reconstructions are adjoint to erasings

Let's give names to the operations of the last section.

The operation that erases the information on Q will be called "$/Q$"; we have $B''' = B/Q$. The "manual" methods for getting rid of all '?'s will be called $\mathsf{manualmax}$ and $\mathsf{manualmin}$. We always have $\mathsf{manualmin}(B/Q) \le B \le \mathsf{manualmax}(B/Q)$.

Let $\mathcal{O}(P)/Q = \{\, B/Q \mid B \in \mathcal{O}(P)\,\}$. All elements in $\mathcal{O}(P)/Q$ have question marks in the same positions, so we can define a partial order in $\mathcal{O}(P)/Q$ like this: if $B''', C''' \in \mathcal{O}(P)/Q$ then $B''' \leq C'''$ if and only the set of points with '1's in B''' is contained or equal to the set of point with '1's in C'''.

The operations $\mathsf{manualmax}$, $/Q$, and $\mathsf{manualmin}$ are order-preserving maps between $\mathcal{O}(P)$ and $\mathcal{O}(P)/Q$, going in these directions:

$$\begin{tikzcd}
\mathcal{O}(P)/Q \arrow[r, shift left=2, "\mathsf{manualmin}"] \arrow[r, shift right=2, "\mathsf{manualmax}"'] & \mathcal{O}(P) \arrow[l, "/Q" description]
\end{tikzcd}$$

We will now show that we have adjunctions $\mathsf{manualmax} \dashv /Q \dashv \mathsf{manualmin}$. We will suppose that the reader knows enough about adjunctions and Galois connections — for example at least section 9.4 of [Awo06] ("Order Adjoints"). The conventions for drawing the diagrams would be practically the same as in section 13 of [Och13]; the dashed vertical arrows are morphisms in preorder categories, and each horizontal bijection arrow between dashed vertical arrows means that the arrow at the left exists if and only if the arrow of the right exists.

The diagram at the left below is the particular case that we saw in last section; the diagram at the right is a generalization of it. The arrows marked 'id' at the left side of each diagram always exist, and the horizontal bijection arrow assure us that the dashed arrows at the right side exist too; they are the units and counits of the adjunctions. The operations '${}^{\mathrm{co}*}$' and '*' of sec.1.5 can be recovered from these adjunctions: $B^{\mathrm{co}*} = \mathsf{src}(\epsilon_B)$, $B^* = \mathsf{tgt}(\eta_B)$.

$$\begin{tikzcd}
12/Q \arrow[r, mapsto] \arrow[d, dashed, "\mathrm{id}"'] & 11 \arrow[r, equal] \arrow[d, dashed, "\epsilon_{12}"] & 12^{\mathrm{co}*} \\
12/Q \arrow[d, dashed, "\mathrm{id}"'] & 12 \arrow[l, mapsto] \arrow[d, dashed, "\eta_{12}"] & \\
12/Q \arrow[r, mapsto] & 23 \arrow[r, equal] & 12^*
\end{tikzcd}
\qquad
\begin{tikzcd}
B/Q \arrow[r, mapsto] \arrow[d, dashed, "\mathrm{id}"'] & \mathsf{manualmin}(B/Q) = B^{\mathrm{co}*} \arrow[d, dashed, "\epsilon_B"] \\
B/Q \arrow[d, dashed, "\mathrm{id}"'] & B \arrow[l, mapsto] \arrow[d, dashed, "\eta_B"] \\
B/Q \arrow[r, mapsto] & \mathsf{manualmax}(B/Q) = B^*
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{O}(P)/Q \arrow[r, shift left=2, "\mathsf{manualmin}"] \arrow[r, shift right=2, "\mathsf{manualmin}"'] & \mathcal{O}(P) \arrow[l, "/Q" description]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}(P)/Q \arrow[r, shift left=2, "\mathsf{manualmin}"] \arrow[r, shift right=2, "\mathsf{manualmin}"'] & \mathcal{O}(P) \arrow[l, "/Q" description]
\end{tikzcd}$$

If we lift the restriction that the dashed maps at the left have to be identities we get this diagram:

$$\begin{tikzcd}
B''' \arrow[r, mapsto] \arrow[d, dashed] \arrow[dr, phantom, "\leftrightarrow"] & \mathsf{manualmin}(B''') \arrow[d, dashed] \\
C/Q \arrow[dr, phantom, "\leftrightarrow"] \arrow[d, dashed] & C \arrow[l, mapsto] \arrow[d, dashed] \\
D''' \arrow[r, mapsto] & \mathsf{manualmax}(D''')
\end{tikzcd}$$

$$\mathcal{O}(P)/Q \;\underset{\mathsf{manualmin}}{\overset{\mathsf{manualmin}}{\underset{\xleftarrow{/Q}}{\rightrightarrows}}}\; \mathcal{O}(P)$$

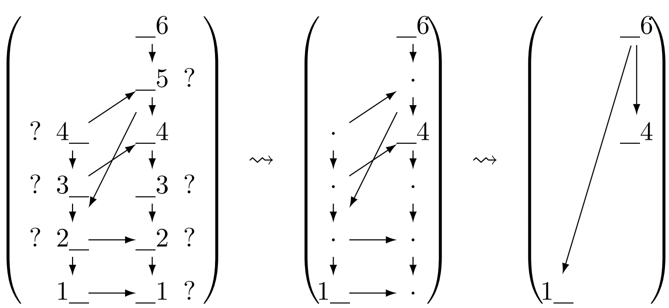
in which all dashed arrows now stand for morphisms that may or may not exist, and the horizontal biections says that the one at the left exists if and only if the one at the right exists. It is easy to verify that its horizontal bijections are true, i.e., that for any $B, C, D \in \mathcal{O}(P)$ we have:

$$\begin{array}{rcl} B/Q \le C/Q & \leftrightarrow & \mathsf{manualmin}(B/Q) \le C \\ C/Q \le D/Q & \leftrightarrow & C \le \mathsf{manualmax}(D/Q) \end{array}$$

so we have $\mathsf{manualmin} \dashv /Q \dashv \mathsf{manualmax}$.

4.5 A partial order on the non-erased points

The is a way to replace the partial order on $\mathcal{O}(P)/Q$ of the last section with something more familiar: a partial order on the set $P \backslash Q$ of points of P without question marks. That order has to be the one inherited from $\mathcal{O}(P)$, or, to use the full notation from sections 12 and 15 of [Och17], from $\mathcal{O}_A(P)$; an example:



Here is a way to obtain the "best" set of arrows on $P \backslash Q$. Let R be the transitive closure of the set of arrows A; note that it will not have any arrow of the form aRa. Let R' be R minus its "superfluous arrows" (see sec.17 of [Och17]), which are the ones of the form aRc for which we have $aRbRc$ for some b; then R' is the smallest set of arrows on $P \backslash Q$ that generates the order inherited from $\mathcal{O}_A(P)$. By abuse of language, let's denote this set of arrows on $P\backslash Q$ by $A\backslash Q$; its (order) topology is $\mathcal{O}_{A\backslash Q}(P\backslash Q)$. The functor

$$\mathcal{O}_{A\backslash Q}(P\backslash Q) \xleftarrow{\mathsf{restr}} \mathcal{O}_A(P)$$

is naturally isomorhic to the functor $\mathcal{O}(P)\backslash Q \longleftarrow \mathcal{O}_A(P)$ of the last section, and we can rewrite the adjunctions as:

$$\begin{tikzcd}
W \arrow[r, mapsto] \arrow[d, dashed] \arrow[dr, phantom, "\leftrightarrow"] & f_!W \arrow[d, dashed] \\
f^*V \arrow[d, dashed] \arrow[dr, phantom, "\leftrightarrow"] & V \arrow[l, mapsto] \arrow[d, dashed] \\
U \arrow[r, mapsto] & f_*U
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{O}_{A\backslash Q}(P\backslash Q) \arrow[r, shift left=2, "f_!"] \arrow[r, shift right=2, "f_*"'] & \mathcal{O}_A(P) \arrow[l, "f^*" description]
\end{tikzcd}$$

5 Sheaves for children

We can use the adjunctions of the last section to understand sheaves — if we are like the children ("people without mathematical maturity") of the introduction, who need concrete examples to understand an abstract definition.

We will draw an adjunction $L \dashv R$ between categories $\mathbf{C}$ and $\mathbf{D}$ like this:

$$\begin{tikzcd}
LC \arrow[d] \arrow[dr, phantom, "\leftrightarrow"] & C \arrow[l, mapsto] \arrow[d] \\
D \arrow[r, mapsto] & RD
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{D} \arrow[r, shift right, "R"'] & \mathbf{C} \arrow[l, shift right, "L"']
\end{tikzcd}$$

The left adjoint L goes left, the right adjoint R goes right, and the horizontal bijection arrow '$\leftrightarrow$' represents the natural isomorphism $\mathrm{Hom}_{\mathbf{D}}(L-,-) \cong \mathrm{Hom}_{\mathbf{C}}(-,R-)$.

We will follow the Elephant ([Joh02]). In A4.1.1 it defines a *geometric morphism* $f : \mathcal{F} \to \mathcal{E}$ between toposes $\mathcal{E}$ and $\mathcal{F}$ as an adjunction $f^* \dashv f_*$ like this,

$$\begin{tikzcd}
f^*E \arrow[d] \arrow[dr, phantom, "\leftrightarrow"] & E \arrow[l, mapsto] \arrow[d] \\
F \arrow[r, mapsto] & f_*F
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F} \arrow[r, shift right, "f_*"'] & \mathcal{E} \arrow[l, shift right, "f^*"']
\end{tikzcd}$$

$$\mathcal{F} \xrightarrow{f} \mathcal{E}$$

in which the functor f^* preserves finite limits, which is a condition weaker than requiring that f^* has a left adjoint. When f^* has a left adjoint the convention (see its Example A4.1.4) is to call it $f_!$, and to say that the geometric morphism f is *essential*.

The example A4.1.4 of the Elephant starts with a functor $f : \mathcal{C} \to \mathcal{D}$ between small categories and shows that it induces an essential geometric morphism $f = (f_! \dashv f^* \dashv f_*)$ between the toposes $[\mathcal{C}, \mathbf{Set}]$ and $[\mathcal{D}, \mathbf{Set}]$, where f^* is "composition with f" and f_* can be built by calculating the right Kan extension $\varprojlim_f$. Here is a diagram comparing the Elephant's notation, at the left, with the one that we will use:

$$\begin{tikzcd}
f^*D \arrow[d] & D \arrow[l, mapsto] \arrow[d] \\
F \arrow[r, mapsto] \arrow[ur, phantom, "\leftrightarrow"] & f_*F \arrow[r, equal] & \varprojlim_f F \\
{[\mathcal{C}, \mathbf{Set}]} \arrow[r, shift right, "f_*"'] & {[\mathcal{D}, \mathbf{Set}]} \arrow[l, shift right, "f^*"'] \\
\mathcal{C} \arrow[r, "f"] & \mathcal{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
f^*F \arrow[d] & F \arrow[l, mapsto] \arrow[d] \\
G \arrow[r, mapsto] \arrow[ur, phantom, "\leftrightarrow"] & f_*G \\
\mathbf{Set}^{\mathbf{A}} \arrow[r, shift right, "f_*"'] & \mathbf{Set}^{\mathbf{B}} \arrow[l, shift right, "f^*"'] \\
\mathbf{A} \arrow[r, "f"] & \mathbf{B}
\end{tikzcd}$$

Let $\mathbf{A}$ and $\mathbf{B}$ be these categories (preorders), and let $f : \mathbf{A} \to \mathbf{B}$ be the inclusion:

$$\mathbf{A} = \left(\begin{tikzcd}
2 \arrow[dr] & & 3 \arrow[dl] \arrow[dr] & \\
& 4 & & 5
\end{tikzcd}\right)
\qquad
\mathbf{B} = \left(\begin{tikzcd}
& 1 \arrow[dl] \arrow[dr] & & \\
2 \arrow[dr] & & 3 \arrow[dl] \arrow[dr] & \\
& 4 \arrow[dr] & & 5 \arrow[dl] \\
& & 6 &
\end{tikzcd}\right)$$

Then an object G of $\mathbf{Set}^{\mathbf{A}}$ and an object F of $\mathbf{Set}^{\mathbf{B}}$ and can be drawn as this,

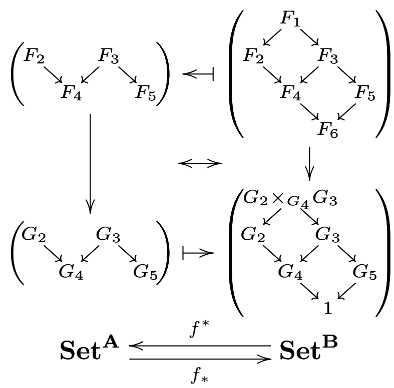
$$G = \left(\begin{tikzcd}
G_2 \arrow[dr] & & G_3 \arrow[dl] \arrow[dr] & \\
& G_4 & & G_5
\end{tikzcd}\right)
\qquad
F = \left(\begin{tikzcd}
& F_1 \arrow[dl] \arrow[dr] & & \\
F_2 \arrow[dr] & & F_3 \arrow[dl] \arrow[dr] & \\
& F_4 \arrow[dr] & & F_5 \arrow[dl] \\
& & F_6 &
\end{tikzcd}\right)$$

where $G_2, \dots, G_5, F_1, \dots, F_6$ are sets and the arrows are functions between sets. The image of F by f^* is very easy to obtain, it is just a restriction of the diagram of F. The image of F, f_*G, is harder; we can calculate f_* by Kan extensions, but we know that all the right adjoints to f_* are naturally isomorphic, so we can also obtain the right adjoint by guess-and-test... it turns out that we can define f_*G for an arbitrary G in $\mathbf{Set}^{\mathbf{A}}$ as this,

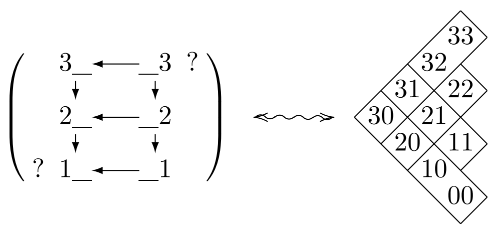
$$f_* \left(\begin{tikzcd}
G_2 \arrow[dr] & & G_3 \arrow[dl] \arrow[dr] & \\
& G_4 & & G_5
\end{tikzcd}\right)
=
\left(\begin{tikzcd}
& G_2 \times_{G_4} G_3 \arrow[dl] \arrow[dr] & & \\
G_2 \arrow[dr] & & G_3 \arrow[dl] \arrow[dr] & \\
& G_4 \arrow[dr] & & G_5 \arrow[dl] \\
& & 1 &
\end{tikzcd}\right)$$

which yields a result equivalent to using Kan extensions, but with simpler formulas; $G_2 \times_{G_4} G_3$ is a pullback.

The geometric morphism induced by our $f : \mathbf{A} \to \mathbf{B}$ can be depicted as:



Note that $\mathbf{B}$ is a two-column graph drawn in a tilted way, and $\mathbf{A}$ is the restriction of its partial order to the subset $\{2,3,4,5\} \subseteq \{1,2,3,4,5,6\}$; this is exactly like we did in sec.4.5, but with a different choice of a 2CG (P, A), and using $\{1, 6\}$ as the set of question marks. The associated J-operator is this:



If the sets $G_2, \ldots, G_5, F_1, \ldots, F_6$ are only allowed to be either singleton sets or empty sets, denoted '1' and '0' respectively, then the formula that we obtained for f_* yields exactly the "biggest way to reconstruct the missing information" that we discussed using question marks; if we allow $G_2, \ldots, G_5, F_1, \ldots, F_6$ to be arbitrary sets then this formula for f_* yields something new — an extension of the idea of J-operator, that was something that acted only on truth-values, to something that takes a functor F in $\mathbf{Set}^{\mathbf{B}}$ and produces another one.

The functor $f : \mathbf{A} \to \mathbf{B}$ that we chose has extra properties. Its induced geometric morphism is an *inclusion* in the sense of Elephant's A4.2.8 and A4.2.9, and every inclusion induces a notion of *sheaf* (A4.3) — an object $F \in \mathbf{Set}^{\mathbf{B}}$ is a sheaf iff F is isomorphic to $f_* f^* F$ — and a category of sheaves; but we will leave the discussion of this to the next paper in this series, in which we will see in details how to do categories "for children" and "for adults" in parallel.

5.1 Another example

Let's switch to a simpler example, the inclusion of "vee" into "kite":

$$\mathbf{A} = \begin{pmatrix} 2 & & 3 \\ & \searrow \ \swarrow & \\ & 4 & \end{pmatrix} \xrightarrow{f} \begin{pmatrix} & 1 & \\ \swarrow & & \searrow \\ 2 & & 3 \\ \searrow & & \swarrow \\ & 4 & \\ & \downarrow & \\ & 5 & \end{pmatrix} = \mathbf{B}$$

The unit of the adjunction $f^* \dashv f_*$ reconstructs F_1 as a pullback and F_5 as a singleton set:

$$\begin{tikzcd}
\begin{pmatrix} F_2 & & F_3 \\ & \searrow \ \swarrow & \\ & F_4 & \end{pmatrix} \arrow[dd] & \begin{pmatrix} & F_1 & \\ \swarrow & & \searrow \\ F_2 & & F_3 \\ \searrow & & \swarrow \\ & F_4 & \\ & \downarrow & \\ & F_5 & \end{pmatrix} \arrow[l, mapsto] \arrow[dd] & \begin{pmatrix} & F_1 & \\ \swarrow & & \searrow \\ F_2 & & F_3 \\ \searrow & & \swarrow \\ & F_4 & \\ & \downarrow & \\ & F_5 & \end{pmatrix} \arrow[dd, "\eta"] \\
\arrow[r, phantom, "\longleftrightarrow"] & & \\
\begin{pmatrix} G_2 & & G_3 \\ & \searrow \ \swarrow & \\ & G_4 & \end{pmatrix} \arrow[r, mapsto] & \begin{pmatrix} & G_2 \times_{G_4} G_3 & \\ \swarrow & & \searrow \\ G_2 & & G_3 \\ \searrow & & \swarrow \\ & G_4 & \\ & \downarrow & \\ & 1 & \end{pmatrix} & \begin{pmatrix} & F_2 \times_{F_4} F_3 & \\ \swarrow & & \searrow \\ F_2 & & F_3 \\ \searrow & & \swarrow \\ & F_4 & \\ & \downarrow & \\ & 1 & \end{pmatrix} \\
\mathbf{Set}^{\mathbf{A}} \arrow[r, shift right, "f_*"'] & \mathbf{Set}^{\mathbf{B}} \arrow[l, shift right, "f^*"'] &
\end{tikzcd}$$

For some 'F's in $\mathbf{Set}^{\mathbf{B}}$ the map η is not monic. For example, here, where the maps from F_1 to F_2 and F_3 drop a digit and the map from F_4 to F_5 takes 3 to 1:

$$F = \begin{pmatrix} & \{46, 64\} & \\ \swarrow & & \searrow \\ \{4, 5\} & & \{6\} \\ \searrow & & \swarrow \\ & \{3\} & \\ & \downarrow & \\ & \{1, 2\} & \end{pmatrix} \xrightarrow{\eta} \begin{pmatrix} & \{(4, 6), (5, 6)\} & \\ \swarrow & & \searrow \\ \{4, 5\} & & \{6\} \\ \searrow & & \swarrow \\ & \{3\} & \\ & \downarrow & \\ & \{*\} & \end{pmatrix} = f_* f^* F$$

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