Planar Heyting Algebras for Children 3: Visualizing Geometric Morphisms

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Warning: *this paper does not exist yet!*
Its first part — on internal views — has been moved to the beginning of:
http://angg.twu.net/LATEX/2017yoneda.pdf
Links:
Parts 1, 2, and 3 (this one) in this series of papers:
http://angg.twu.net/math-b.html#zhas-for-children-2
A workshop very much related to this:
http://angg.twu.net/logic-for-children-2018.html
A big part of the core of this paper will be on putting some of the diagrams on "Notes on notation: Elephant" in readable form.
http://angg.twu.net/LATEX/2017elephant.pdf
http://angg.twu.net/math-b.html#notes-on-notation

1 "Children" and "adults"

Different people have different ways of remembering theorems. A person with a very visual mind may remember a theorem in Category Theory mainly by the shape of a diagram and the order in which its objects are constructed. For such a person most books on Category Theory feel as if they have lots of missing diagrams, that she has to reconstruct if she wants to understand the subject.

The shape of a categorical diagram remains the same if we specialize it to a particular case — and this means that we can sometimes remember a general diagram, and the theorems associated to it, from the diagram of a particular case. A diagram for a particular case becomes a skeleton, or a scaffold, from which we reconstruct mentally a general statement when we need it.

In this paper we will use "missing diagrams", "particular cases", and a third technique called "internal views", to show how to obtain a version "for children" of some theorems about topos (actually about factorizations of geometrical morphisms between toposes). Our aim on a higher level, though, is to sketch some techniques for doing "maths for children" and "maths for adults" in parallel, with tools to creating a — somewhat faulty, but still useful — bridge between the two languages.

Let's *define* what we mean by "children" and "adults". "Children" means (for us!!!) "people without mathematical maturity", which in its turn means people who:

- have trouble with very abstract definitions,
- prefer starting from particular cases (and then generalize),
- handle diagrams better than algebraic notations,
- like to use diagrams and analogies,
- like to work with finite cases where all calculations can be done explicitly "by brute force".

People who prefer to always work on a very high-level abstract language, and who frown on working out examples in details, are "adults", or "mathematicians".

With these terms, categorical definitions are "for adults", because they may be very abstract, and particular cases, diagrams, and analogies are "for children". "Children" are willing to use "tools for children" to do mathematics, *even if they will have to translate everything to a language "for adults" to make their results dependable and publishable*, and even if the bridge between their tools "for children" and "for adults" is somewhat defective, i.e., if the translation only works on simple cases...

1.1 Positional notations

1.2 Internal views

1.3 Parallel diagrams

1.4 ZGraphs

1.5 ZCategories

1.6 ZPresheaves

1.7 ZToposes

1.8 Particular cases

1.9 Monads

A monad $1 \xrightarrow{\eta} T \xleftarrow{\mu} T^2$:

$$\begin{tikzcd}
I \arrow[r, "\eta"] & T & T^2 \arrow[l, "\mu"'] & X \arrow[r, "\eta X"] & TX & T^2X \arrow[l, "\mu X"']
\end{tikzcd}$$

$$\begin{tikzcd}
T \arrow[r, "T\eta"] \arrow[d, "\eta T"'] \arrow[dr, "\mathrm{id}" description] & T^2 \arrow[d, "\mu"] & T^3 \arrow[l, "T\mu"'] \arrow[d, "\mu T"] & TX \arrow[r, "T(\eta X)"] \arrow[d, "\eta(TX)"'] \arrow[dr, "\mathrm{id}" description] & T^2X \arrow[d, "\mu X"] & T^3X \arrow[l, "T(\mu X)"'] \arrow[d, "\mu(TX)"] \\
T^2 \arrow[r, "\mu"'] & T & T^2 \arrow[l, "\mu"] & T^2X \arrow[r, "\mu X"'] & TX & T^2X \arrow[l, "\mu X"]
\end{tikzcd}$$

An algebra $A \xleftarrow{\alpha} TA$:

$$\begin{tikzcd}
A \arrow[r, "\eta A"] \arrow[d, "\mathrm{id}"'] & TA \arrow[dl, "\alpha"] & T^2A \arrow[l, "\mu A"'] \arrow[dl, "T\alpha"] \\
A & TA \arrow[l, "h"]
\end{tikzcd}$$

1.10 Comonads

A comonad $1 \xleftarrow{\epsilon} \mathbb{G} \xrightarrow{\nu} \mathbb{G}^2$:

$$\begin{tikzcd}
I & \mathbb{G} \arrow[l, "\epsilon"'] \arrow[r, "\nu"] & \mathbb{G}^2 & X & \mathbb{G}X \arrow[l, "\epsilon X"'] \arrow[r, "\nu X"] & \mathbb{G}^2X
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{G} & \mathbb{G}^2 \arrow[l, "\mathbb{G}\epsilon"'] \arrow[r, "\mathbb{G}\nu"] & \mathbb{G}^3 & \mathbb{G}X & \mathbb{G}^2X \arrow[l, "\mathbb{G}(\epsilon X)"'] \arrow[r, "\mathbb{G}(\nu X)"] & \mathbb{G}^3X \\
\mathbb{G}^2 \arrow[u, "\epsilon\mathbb{G}"] & \mathbb{G} \arrow[l, "\nu"] \arrow[r, "\nu"'] \arrow[u, "\nu"'] \arrow[ul, "\mathrm{id}" description] & \mathbb{G}^2 \arrow[u, "\nu\mathbb{G}"'] & \mathbb{G}^2X \arrow[u, "\epsilon(\mathbb{G}X)"] & \mathbb{G}X \arrow[l, "\nu X"] \arrow[r, "\nu X"'] \arrow[u, "\nu X"'] \arrow[ul, "\mathrm{id}" description] & \mathbb{G}^2X \arrow[u, "\nu(\mathbb{G}X)"']
\end{tikzcd}$$

A coalgebra $A \xrightarrow{\alpha} \mathbb{G}A$:

$$\begin{tikzcd}
A & \mathbb{G}A \arrow[l, "\epsilon A"'] \arrow[r, "\nu A"] & \mathbb{G}^2A \\
A \arrow[u, "\mathrm{id}"] \arrow[ur, "\alpha"'] \arrow[r, "\alpha"'] & \mathbb{G}A \arrow[ur, "\mathbb{G}\alpha"']
\end{tikzcd}$$

The surjection-inclusion factorization for children

$$\begin{tikzcd}
g^*I \arrow[d] & & & I \arrow[lll, mapsto] \arrow[d] \\
F \arrow[rrr, mapsto] & & \arrow[phantom, u, "\leftrightarrow"] & g_*F \\
\mathbf{Set}^{\mathbf{A}} \arrow[rrr, "g \text{ (any g.m.)}"] & & & \mathbf{Set}^{\mathbf{D}}
\end{tikzcd}$$

$$\begin{tikzcd}
s^*G \arrow[d] \arrow[r, phantom, "\leftrightarrow" below] & G \arrow[l, mapsto] \arrow[d] & G \arrow[d, "\substack{\eta G \\ \text{(monic)}}"] & i^*i_*G \arrow[d, "\substack{\epsilon G \\ \text{(iso)}}"'] & i^*I \arrow[d] \arrow[r, phantom, "\leftrightarrow" below] & I \arrow[l, mapsto] \arrow[d] \\
F \arrow[r, mapsto] & s_*F & s_*s^*G & G & G \arrow[r, mapsto] & i_*G \\
\mathbf{Set}^{\mathbf{A}} \arrow[r, "s \text{ (surjection)}"] & \mathbf{Set}^{\mathbf{B}} & & & \mathbf{Set}^{\mathbf{B}} \arrow[r, "i \text{ (inclusion)}"] & \mathbf{Set}^{\mathbf{D}}
\end{tikzcd}$$

We will use this particular case.
Its s takes both 1 and 2 to 2.

$$\begin{tikzcd}
\begin{pmatrix} I_2 \\ I_2 \end{pmatrix} \arrow[d] & & & \begin{pmatrix} I_2 \\ I_3 \end{pmatrix} \arrow[lll, mapsto] \arrow[d] \\
\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \arrow[rrr, mapsto] & & \arrow[phantom, u, "\leftrightarrow"] & \begin{pmatrix} F_1 \times F_2 \\ 1 \end{pmatrix} \\
\mathbf{Set}^{\mathbf{A}} \arrow[rrr, "g"] & & & \mathbf{Set}^{\mathbf{D}}
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{pmatrix} G_2 \\ G_2 \end{pmatrix} \arrow[d] \arrow[r, phantom, "\leftrightarrow" below] & \begin{pmatrix} G_2 \end{pmatrix} \arrow[l, mapsto] \arrow[d] & \begin{pmatrix} G_2 \end{pmatrix} \arrow[d, "\substack{\eta G \\ \text{(monic)}}"] & \begin{pmatrix} G_2 \end{pmatrix} \arrow[d, "\substack{\epsilon G \\ \text{(iso)}}"'] & \begin{pmatrix} I_2 \end{pmatrix} \arrow[d] \arrow[r, phantom, "\leftrightarrow" below] & \begin{pmatrix} I_2 \\ I_3 \end{pmatrix} \arrow[l, mapsto] \arrow[d] \\
\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \arrow[r, mapsto] & \begin{pmatrix} F_1 \times F_2 \end{pmatrix} & \begin{pmatrix} G_1 \times G_2 \end{pmatrix} & \begin{pmatrix} G_2 \end{pmatrix} & \begin{pmatrix} G_2 \end{pmatrix} \arrow[r, mapsto] & \begin{pmatrix} G_2 \\ 1 \end{pmatrix} \\
\mathbf{Set}^{\mathbf{A}} \arrow[r, "s"] & \mathbf{Set}^{\mathbf{B}} & & & \mathbf{Set}^{\mathbf{B}} \arrow[r, "i"] & \mathbf{Set}^{\mathbf{D}} \\
\begin{pmatrix} 1 \\ 2 \end{pmatrix} \arrow[r, "s"] & \begin{pmatrix} 2 \end{pmatrix} & & & \begin{pmatrix} 2 \end{pmatrix} \arrow[r, "i"] & \begin{pmatrix} 2 \\ 3 \end{pmatrix}
\end{tikzcd}$$

The adjunction $g^* \dashv g_*$ induces this comonad on $\mathbf{Set^A}$:

$1 \xleftarrow{\epsilon} \mathbb{G} \xrightarrow{\nu} \mathbb{G}^2$

$F \xleftarrow{\epsilon F} \mathbb{G}F \xrightarrow{\nu F} \mathbb{G}\mathbb{G}F$

$\mathbb{G}F = g^* g_* F = g^* g_* \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = g^* \begin{pmatrix} F_1 \times F_2 \\ 1 \end{pmatrix} = \begin{pmatrix} F_1 \times F_2 \\ F_1 \times F_2 \end{pmatrix}$

$\mathbb{G}\mathbb{G}F = \mathbb{G} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \mathbb{G} \begin{pmatrix} F_1 \times F_2 \\ F_1 \times F_2 \end{pmatrix} = \begin{pmatrix} (F_1 \times F_2) \times (F_1 \times F_2) \\ (F_1 \times F_2) \times (F_1 \times F_2) \end{pmatrix}$

$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \xleftarrow{\epsilon F} \begin{pmatrix} F_1 \times F_2 \\ F_1 \times F_2 \end{pmatrix} \xrightarrow{\nu F} \begin{pmatrix} (F_1 \times F_2) \times (F_1 \times F_2) \\ (F_1 \times F_2) \times (F_1 \times F_2) \end{pmatrix}$

$\begin{pmatrix} a \\ d \end{pmatrix} \xleftarrow{\epsilon F = \begin{pmatrix} \pi \\ \pi' \end{pmatrix}} \begin{pmatrix} (a,b) \\ (c,d) \end{pmatrix} \qquad \begin{pmatrix} (a,b) \\ (c,d) \end{pmatrix} \xmapsto{\nu F = \begin{pmatrix} \delta \\ \delta \end{pmatrix}} \begin{pmatrix} ((a,b),(a,b)) \\ ((c,d),(c,d)) \end{pmatrix}$

Let $F \xrightarrow{\varphi} F'$ be a morphism in $\mathbf{Set^A}$. Suppose it is $\begin{pmatrix} A \\ B \end{pmatrix} \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} \begin{pmatrix} C \\ D \end{pmatrix}$.

Its image by $\mathbb{G}$ is a morphism $\mathbb{G}F \xrightarrow{\mathbb{G}\varphi} \mathbb{G}F'$,

this one: $\begin{pmatrix} A \times B \\ A \times B \end{pmatrix} \xrightarrow{\begin{pmatrix} f \times g \\ f \times g \end{pmatrix}} \begin{pmatrix} C \times D \\ C \times D \end{pmatrix}$, $\begin{pmatrix} (a,b) \\ (a',b') \end{pmatrix} \xmapsto{\begin{pmatrix} f \times g \\ f \times g \end{pmatrix}} \begin{pmatrix} (f(a),g(b)) \\ (f(a'),g(b')) \end{pmatrix}$.

A coalgebra $F \xrightarrow{\alpha} \mathbb{G}F$ is a map α that makes this commute:

$$\begin{tikzcd}
F & \mathbb{G}F \arrow[l, "\epsilon F"'] \arrow[r, "\nu F"] & \mathbb{G}^2 F \\
F \arrow[u, "\mathrm{id}"] \arrow[ur, "\alpha"'] \arrow[r, "\alpha"'] & \mathbb{G}F \arrow[ur, "\mathbb{G}\alpha"'] &
\end{tikzcd}$$

Suppose that α is $\begin{pmatrix} A \\ B \end{pmatrix} \xrightarrow{\alpha} \begin{pmatrix} A \times B \\ A \times B \end{pmatrix}$, $\begin{pmatrix} a \\ b \end{pmatrix} \xmapsto{\begin{pmatrix} \langle f,g \rangle \\ \langle h,k \rangle \end{pmatrix}} \begin{pmatrix} (f(a),g(a)) \\ (h(b),k(b)) \end{pmatrix}$.

Then $\mathbb{G}\alpha$ is $\begin{pmatrix} A \times B \\ A \times B \end{pmatrix} \xrightarrow{\mathbb{G}\alpha = \begin{pmatrix} \langle f,g \rangle \times \langle h,k \rangle \\ \langle f,g \rangle \times \langle h,k \rangle \end{pmatrix}} \begin{pmatrix} (A \times B) \times (A \times B) \\ (A \times B) \times (A \times B) \end{pmatrix}$...

$$\begin{tikzcd}
\begin{pmatrix} fa \\ kb \end{pmatrix} & \begin{pmatrix} (fa,ga) \\ (hb,kb) \end{pmatrix} \arrow[l, "\epsilon F = \begin{pmatrix} \pi \\ \pi' \end{pmatrix}"'] \arrow[r, "\nu F = \begin{pmatrix} \delta \\ \delta \end{pmatrix}"] & \begin{pmatrix} ((fa,ga),(fa,ga)) \\ ((hb,kb),(hb,kb)) \end{pmatrix} \\
\begin{pmatrix} a \\ b \end{pmatrix} & & \begin{pmatrix} ((ffa,gfa),(hga,kga)) \\ ((fhb,ghb),(hkb,kkb)) \end{pmatrix} \\
\begin{pmatrix} a \\ b \end{pmatrix} \arrow[u, "\mathrm{id}"] \arrow[uur, "\alpha"'] \arrow[r, "\alpha = \begin{pmatrix} \langle f,g \rangle \\ \langle h,k \rangle \end{pmatrix}"'] & \begin{pmatrix} (fa,ga) \\ (hb,kb) \end{pmatrix} \arrow[ur, "\mathbb{G}\alpha = \begin{pmatrix} \langle f,g \rangle \times \langle h,k \rangle \\ \langle f,g \rangle \times \langle h,k \rangle \end{pmatrix}"'] &
\end{tikzcd}$$