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Planar Heyting Algebras for Children

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Abstract

This paper shows a way to interpret (propositional) intuitionistic logic *visually* using finite Planar Heyting Algebras (that we call "ZHAs"), that are certain subsets of $\mathbb{Z}^2$. We also show the connection between ZHAs and the familiar semantics for IPL where the truth-values are open sets in a finite topological space $(P, \mathcal{O}(P))$: every ZHA is an "order topology on a 2-column graph".

In the second part of the paper we show how each closure operator $J : H \to H$ on a ZHA $H \subseteq \mathbb{Z}^2$ corresponds to a) a way to "slash" H using diagonal cuts, and b) a choice of a subset $S \subseteq P$; J can then be recovered from the inclusion $f : S \to P$ as a restriction map $f^* : \mathcal{O}(P) \to \mathcal{O}(S)$ followed by a map $f_* : \mathcal{O}(S) \to \mathcal{O}(P)$ that reconstructs the missing information "in the biggest way possible".

When a mathematical paper is written "for children" this means either that it is written for people without lots of mathematical *knowledge* or that it doesn't require mathematical *maturity*; we follow the second, stronger, meaning of the term. "Children" for us means people who are not able to understand structures that are too abstract straight away, they need particular cases first — and they also have trouble with infinite objects, and with theorems about things that they can't calculate: *calculating* is much more basic for them than *proving*. Writing "for children" makes us enforce a style where everything is constructive and finite and all the main examples are objects that are easy to draw explicitly.

Closure operators on Heyting Algebras are very important on Topos Theory: they generate geometric morphisms and sheaves. This paper introduces several tools that can be used to explain some parts of Topos Theory to "children", but here we stop just before categories and toposes — when we move to categories and (pre)sheaves we have to replace most of the '0's and '1's in our diagrams by sets.

Keywords: Heyting Algebras, Intuitionistic Logic, diagrammatic reasoning, geometric morphisms.

Introduction

This paper shows a way to interpret (propositional) intuitionistic logic *visually* (sec.8) using finite Planar Heyting Algebras ("ZHAs", sec.5), that are certain subsets of $\mathbb{Z}^2$. The "for children" of the title means "for people without mathematical maturity" (sec.1).

In sections 12–19 we show the connection between ZHAs and the familiar semantics for IPL where the truth-values are open sets in a topological space $(P, \mathcal{O}(P))$, and in sections 20–36 we discuss how each closure operator on a ZHA $H \subseteq \mathbb{Z}^2$ corresponds to a way to "slash" H using diagonal cuts; in sections 37–42 we show how each closure operator correspond to a subset $S \subseteq P$, or rather to a restriction map $\mathcal{O}(P) \to \mathcal{O}(S)$ followed by a map $\mathcal{O}(S) \to \mathcal{O}(P)$ that reconstructs the missing information "in the biggest way possible".

1 Children

The "children" in the title of this paper means: "people without mathematical maturity". "Children" in this sense are not able to understand structures that are too abstract straight away, they need particular cases first; and they also don't deal well with infinite objects or with expressions like "for every proposition $P(x)$", or even with *theorems*...

In my experience what works best with "children" is to teach them first that "basic mathematical objects" are things built from numbers, sets, and lists — like this (our first logic!):

$$\begin{array}{l}
\mathsf{CL} = (\Omega, \top, \bot, \wedge, \vee, \to, \leftrightarrow, \neg) = \\
\left(\left\{ \begin{matrix} 0, \\ 1 \end{matrix} \right\}, 1, 0, \left\{ \begin{matrix} ((0,0),0), \\ ((0,1),0), \\ ((1,0),0), \\ ((1,1),1) \end{matrix} \right\}, \left\{ \begin{matrix} ((0,0),0), \\ ((0,1),1), \\ ((1,0),1), \\ ((1,1),1) \end{matrix} \right\}, \left\{ \begin{matrix} ((0,0),1), \\ ((0,1),1), \\ ((1,0),0), \\ ((1,1),1) \end{matrix} \right\}, \left\{ \begin{matrix} ((0,0),1), \\ ((0,1),0), \\ ((1,0),0), \\ ((1,1),1) \end{matrix} \right\}, \left\{ \begin{matrix} (0,1), \\ (1,0) \end{matrix} \right\} \right) \quad ,
\end{array}$$

and then teach them how to calculate with functions, set comprehension, quantification and λ-notation when the domains are all finite; only after they acquire some practice, speed and intuition about *calculations* we can state some theorems as *propositions* whose results can be calculated by brute force, and then discuss why some of these propositions-theorems always yield "true".

Except for two last sections all the rest of this paper has been written to be readable by "children" in the sense above, and huge parts of it have been tested on "real children" of mainly two kinds: a group of "older children", who are Computer Science students who had already completed a course on Discrete Mathematics, and some "little children", who are friends of mine who are students of Psychology or Social Sciences. The text has benefited enormously from they feedback — especially their puzzled looks at some points, that made me modify my presentation and the exercises I was giving to them. Those exercises are not included here, though, and neither the rationale behind most style decisions.

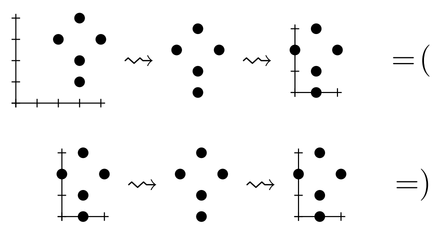
2 Positional notations

Definition: a *ZSet* is a finite, non-empty subset of $\mathbb{N}^2$ that touches both axes, i.e., that has a point of the form $(0, \_\_\_)$ and a point of the form $(\_\_\_, 0)$. We will often represent ZSets using a bullet notation, with or without the axes and ticks. For example:

$$K = \left\{ \begin{array}{ccc} & (1,3), & \\ (0,2), & & (2,2), \\ & (1,1), & \\ & (1,0) & \end{array} \right\} = \begin{array}{ccc} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{array} = \begin{array}{ccc} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{array}$$

We will use the ZSet above a lot in examples, so let's give it a short name: K ("kite").

The condition of touching both axes is what lets us represent ZSets unambiguously using just the bullets:



We can use a positional notation to represent functions *from* a ZSet. For example, if

$$\begin{array}{rrcl} f & : & K & \to & \mathbb{N} \\ & & (x,y) & \mapsto & x \end{array}$$

then

$$f = \left\{ \begin{array}{ccc} & ((1,3),1), & \\ ((0,2),0), & & ((2,2),2), \\ & ((1,1),1), & \\ & ((1,0),1) & \end{array} \right\} = \begin{array}{ccc} & 1 & \\ 0 & & 2 \\ & 1 & \\ & 1 & \end{array}$$

We will sometimes use λ-notation to represent functions compactly. For example:

$$\lambda(x,y){:}K.x = \left\{ \begin{array}{ccc} & ((1,3),1), & \\ ((0,2),0), & & ((2,2),2), \\ & ((1,1),1), & \\ & ((1,0),1) & \end{array} \right\} = \begin{array}{ccc} & 1 & \\ 0 & & 2 \\ & 1 & \\ & 1 & \end{array}$$

$$\lambda(x,y){:}K.y = \left\{ \begin{array}{ccc} & ((1,3),3), & \\ ((0,2),2), & & ((2,2),2), \\ & ((1,1),1), & \\ & ((1,0),0) & \end{array} \right\} = \begin{array}{ccc} & 3 & \\ 2 & & 2 \\ & 1 & \\ & 0 & \end{array}$$

The "reading order" on the points of a ZSet S "lists" the points of S starting from the top and going from left to right in each line. More precisely, if S has n points then $r_S : S \to \{1, \ldots, n\}$ is a bijection, and for example:

$$r_K = \begin{array}{ccc} & 1 & \\ 2 & & 3 \\ & 4 & \\ & 5 & \end{array}$$

Subsets of a ZSet are represented with a notation with '$\bullet$'s and '$\cdot$', and partial functions from a ZSet are represented with '$\cdot$'s where they are not defined. For example:

$$\begin{array}{ccc} & \bullet & \\ \cdot & & \bullet \\ & \bullet & \\ & \cdot & \end{array} \qquad \begin{array}{ccc} & 1 & \\ \cdot & & 3 \\ & 4 & \\ & \cdot & \end{array}$$

The *characteristic function* of a subset S' of a ZSet S is the function $\chi_{S'} : S \to \{0, 1\}$ that returns 1 exactly on the points of S'; for example, $\begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 1 & \\ & 0 & \end{smallmatrix}$ is the characteristic function of $\begin{smallmatrix} & \bullet & \\ \cdot & & \bullet \\ & \bullet & \\ & \cdot & \end{smallmatrix} \subset \begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{smallmatrix}$. We will sometimes denote subsets by their characteristic functions because this makes them easier to "pronounce" by reading aloud their digits in the reading order — for example, $\begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 1 & \\ & 0 & \end{smallmatrix}$ is "one-zero-one-one-zero" (see sec.13).

3 ZDAGs

We will sometimes use the bullet notation for a ZSet S as a *shorthand* for one of the two DAGs induced by S: one with its arrows going up, the other one with them going down. For example: sometimes

$$\begin{array}{ccc} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{array} \quad \text{will stand for:} \quad \begin{tikzcd}[column sep=small] & \bullet \arrow[dl] \arrow[dr] & \\ \bullet \arrow[dr] & & \bullet \arrow[dl] \\ & \bullet \arrow[d] & \\ & \bullet & \end{tikzcd} = \begin{tikzcd}[column sep=small] & (1,3) \arrow[dl] \arrow[dr] & \\ (0,2) \arrow[dr] & & (2,2) \arrow[dl] \\ & (1,1) \arrow[d] & \\ & (1,0) & \end{tikzcd} =$$

$$\left(\left\{ \begin{array}{ccc} & (1,3), & \\ (0,2), & & (2,2), \\ & (1,1), & \\ & (0,0) & \end{array} \right\}, \left\{ \begin{array}{c} ((1,3),(0,2)),((1,3),(2,2)), \\ ((0,2),(1,1)),((2,2),(1,1)), \\ ((1,1),(0,0)) \end{array} \right\} \right)$$

Let's formalize this.

Consider a game in which black and white pawns are placed on points of $\mathbb{Z}^2$, and they can move like this:

$$\begin{tikzcd}[column sep=small] & \bullet \arrow[dl] \arrow[d] \arrow[dr] & \\ \bullet & \bullet & \bullet \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] \circ & \circ & \circ \\ & \circ \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

Black pawns can move from (x, y) to $(x + k, y - 1)$ and white pawns from (x, y) to $(x + k, y + 1)$, where $k \in \{-1, 0, 1\}$. The mnemonic is that black pawns are "solid", and thus "heavy", and they "sink", so they move down; white pawns are "hollow", and thus "light", and they "float", so they move up.

Let's now restrict the board positions to a ZSet S. Black pawns can move from (x, y) to $(x+k, y-1)$ and white pawns from (x, y) to $(x+k, y+1)$, where $k \in \{-1, 0, 1\}$, but only when the starting and ending positions both belong to S. The sets of possible black pawn moves and white pawn moves on S can be defined formally as:

$$\begin{aligned}\mathsf{BPM}(S) &= \{\, ((x,y),(x',y')) \in S^2 \mid x - x' \in \{-1,0,1\}, y' = y-1 \,\} \\ \mathsf{WPM}(S) &= \{\, ((x,y),(x',y')) \in S^2 \mid x - x' \in \{-1,0,1\}, y' = y+1 \,\}\end{aligned}$$

...and now please forget everything else you expect from a game — like starting position, capturing, objective, winning... the idea of a "game" was just a tool to let us explain $\mathsf{BPM}(S)$ and $\mathsf{WPM}(S)$ quickly.

A *ZDAG* is a DAG of the form $(S, \mathsf{BPM}(S))$ or $(S, \mathsf{WPM}(S))$, where S is a ZSet.

A *ZPO* is partial order of the form $(S, \mathsf{BPM}(S)^*)$ or $(S, \mathsf{WPM}(S)^*)$, where S is a ZSet and the '*' denotes the transitive-reflexive closure of the relation.

Sometimes, when this is clear from the context, a bullet diagram like $\begin{smallmatrix}&\bullet&\\\bullet&&\bullet\\&\bullet&\\&\bullet&\end{smallmatrix}$ will stand for either the ZDAGs $(\begin{smallmatrix}&\bullet&\\\bullet&&\bullet\\&\bullet&\\&\bullet&\end{smallmatrix}, \mathsf{BPM}(\begin{smallmatrix}&\bullet&\\\bullet&&\bullet\\&\bullet&\\&\bullet&\end{smallmatrix}))$ or $(\begin{smallmatrix}&\bullet&\\\bullet&&\bullet\\&\bullet&\\&\bullet&\end{smallmatrix}, \mathsf{WPM}(\begin{smallmatrix}&\bullet&\\\bullet&&\bullet\\&\bullet&\\&\bullet&\end{smallmatrix}))$, or for the ZPOs $(\begin{smallmatrix}&\bullet&\\\bullet&&\bullet\\&\bullet&\\&\bullet&\end{smallmatrix}, \mathsf{BPM}(\begin{smallmatrix}&\bullet&\\\bullet&&\bullet\\&\bullet&\\&\bullet&\end{smallmatrix})^*)$ or $(\begin{smallmatrix}&\bullet&\\\bullet&&\bullet\\&\bullet&\\&\bullet&\end{smallmatrix}, \mathsf{WPM}(\begin{smallmatrix}&\bullet&\\\bullet&&\bullet\\&\bullet&\\&\bullet&\end{smallmatrix})^*)$ (sec.5).

4 LR-coordinates

The *lr-coordinates* are useful for working on quarter-plane of $\mathbb{Z}^2$ that looks like $\mathbb{N}^2$ turned $45°$ to the left. Let $\langle l, r \rangle := (-l+r, l+r)$; then (the bottom part of) $\{\, \langle l,r \rangle \mid l, r \in \mathbb{N} \,\}$ is:

$$\begin{array}{ccccccccc} \langle 4,0\rangle && \langle 3,1\rangle && \langle 2,2\rangle && \langle 1,3\rangle && \langle 0,4\rangle \\ & \langle 3,0\rangle && \langle 2,1\rangle && \langle 1,2\rangle && \langle 0,3\rangle & \\ && \langle 2,0\rangle && \langle 1,1\rangle && \langle 0,2\rangle && \\ &&& \langle 1,0\rangle && \langle 0,1\rangle &&& \\ &&&& \langle 0,0\rangle &&&& \end{array} \quad = \quad \begin{array}{ccccccccc} (-4,4) && (-2,4) && (0,4) && (2,4) && (4,4) \\ & (-3,3) && (-1,3) && (1,3) && (3,3) & \\ && (-2,2) && (0,2) && (2,2) && \\ &&& (-1,1) && (1,1) &&& \\ &&&& (0,0) &&&& \end{array}$$

Sometimes we will write lr instead of $\langle l, r \rangle$. So:

$$\begin{array}{ccccccccc} 40 && 31 && 22 && 13 && 04 \\ & 30 && 21 && 12 && 03 & \\ && 20 && 11 && 02 && \\ &&& 10 && 01 &&& \\ &&&& 00 &&&& \end{array} \quad = \quad \begin{array}{ccccccccc} (-4,4) && (-2,4) && (0,4) && (2,4) && (4,4) \\ & (-3,3) && (-1,3) && (1,3) && (3,3) & \\ && (-2,2) && (0,2) && (2,2) && \\ &&& (-1,1) && (1,1) &&& \\ &&&& (0,0) &&&& \end{array}$$

Let $\mathbb{LR} = \{\, \langle l, r \rangle \mid l, r \in \mathbb{N} \,\}$.

5 ZHAs

A *ZHA* is a subset of $\mathbb{LR}$ "between a left and a right wall", as we will see.

A triple (h, L, R) is a "height-left-right-wall" when:
1) $h \in \mathbb{N}$
2) $L : \{0, \ldots, h\} \to \mathbb{Z}$ and $R : \{0, \ldots, h\} \to \mathbb{Z}$
3) $L(h) = R(h)$ (the top points of the walls are the same)
4) $L(0) = R(0) = 0$ (the bottom points of the walls are the same, 0)
5) $\forall y \in \{0, \ldots, h\}. L(y) \le R(y)$ ("left" is left of "right")
6) $\forall y \in \{1, \ldots, h\}. L(y) - L(y-1) = \pm 1$ (the left wall makes no jumps)
7) $\forall y \in \{1, \ldots, h\}. R(y) - R(y-1) = \pm 1$ (the right wall makes no jumps)

The *ZHA generated by* a height-left-right-wall (h, L, R) is the set of all points of $\mathbb{LR}$ with valid height and between the left and the right walls. Formally:

$$\mathsf{ZHAG}(h, L, R) = \{\, (x, y) \in \mathbb{LR} \mid y \le h, L(y) \le x \le R(y) \,\}.$$

A *ZHA* is a set of the form $\mathsf{ZHAG}(h, L, R)$, where the triple (h, L, R) is a height-left-right-wall.

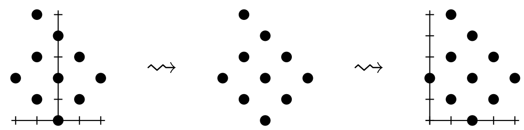
Here is an example of a ZHA (with the white pawn moves on it):

$$\begin{tikzcd}[column sep=tiny, row sep=small]
 & (-3,9) & & & & \\
(-4,8) \arrow[ur] & & (-2,8) \arrow[ul] & & & \\
 & (-3,7) \arrow[ul] \arrow[ur] & & & & \\
 & & (-2,6) \arrow[ul] & & & \\
 & & & (-1,5) \arrow[ul] & & \\
 & & (-2,4) \arrow[ur] & & (0,4) \arrow[ul] & \\
 & (-3,3) \arrow[ur] & & (-1,3) \arrow[ul] \arrow[ur] & & (1,3) \arrow[ul] \\
 & & (-2,2) \arrow[ul] \arrow[ur] & & (0,2) \arrow[ul] \arrow[ur] & \\
 & & & (-1,1) \arrow[ul] \arrow[ur] & & (1,1) \arrow[ul] \\
 & & & & (0,0) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$$\begin{array}{llll}
L(9) = -3 & R(9) = -3 & L(9) = R(9) & h = 9 \\
L(8) = -4 & R(8) = -2 & & \\
L(7) = -3 & R(7) = -3 & & \\
L(6) = -2 & R(6) = -2 & & \\
L(5) = -1 & R(5) = -1 & & \\
L(4) = -2 & R(4) = 0 & & \\
L(3) = -3 & R(3) = 1 & & \\
L(2) = -2 & R(2) = 0 & & \\
L(1) = -1 & R(1) = 1 & & \\
L(0) = 0 & R(0) = 0 & L(0) = R(0) = 0 &
\end{array}$$

We will see later (section 8) that ZHAs (with white pawn moves) are Heyting Algebras.

6 Conventions on diagrams without axes

We can use a bullet notation to denote ZHAs, but look at what happens when we start with a ZHA, erase the axes, and then add the axes back using the convention from sec.2: 

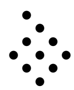
The new, restored axes are in a different position — the bottom point of the original ZHA at the left was $(0,0)$, but in the ZSet at the right the bottom point is $(2,0)$.

The convention from sec.2 is not adequate for ZHAs.

Let's modify it!

From this point on, the convention on where to draw the axes will be this one: *when it is clear from the context that a bullet diagram represents a ZHA,* then its (unique) bottom point has coordinate $(0,0)$, and we use that to draw the axes; otherwise we apply the old convention, that chooses $(0,0)$ as the point that makes the diagram fit in $\mathbb{N}^2$ and touch both axes.

The new convention with two cases also applies to functions from ZHAs, and to partial functions and subsets. For example:

$B =$  (a ZHA)

$$\lambda(x,y){:}B.x = \begin{array}{ccccc} & \text{-1} & & & \\ & & 0 & & \\ & \text{-1} & & 1 & \\ \text{-2} & & 0 & & 2 \\ & \text{-1} & & 1 & \\ & & 0 & & \end{array}$$

$$\lambda\langle l,r\rangle{:}B.l = \begin{array}{ccccc} & 3 & & & \\ & & 2 & & \\ & 2 & & 1 & \\ 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{array} \qquad \lambda\langle l,r\rangle{:}B.r = \begin{array}{ccccc} & 2 & & & \\ & & 2 & & \\ & 1 & & 2 & \\ 0 & & 1 & & 2 \\ & 0 & & 1 & \\ & & 0 & & \end{array}$$

We will often denote ZHAs by the identity function on them:

$$\lambda\langle l,r\rangle{:}B.\langle l,r\rangle \;=\; \lambda lr{:}B.lr \;=\; \begin{array}{ccccc} & 32 & & & \\ & & 22 & & \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{array} \qquad B \;=\; \begin{array}{ccccc} & 32 & & & \\ & & 22 & & \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{array}$$

Note that we are using the compact notation from the end of section 4: 'lr' instead of '$\langle l,r\rangle$'.

7 Propositional calculus

A *PC-structure* is a tuple

$$L = (\Omega, \leq, \top, \bot, \wedge, \vee, \to, \leftrightarrow, \neg),$$

where:

Ω is the "set of truth values",

$\leq$ is a relation on Ω,

$\top$ and $\bot$ are two elements of Ω,
$\land$, $\lor$, $\to$, $\leftrightarrow$ are functions from $\Omega \times \Omega$ to Ω,
$\neg$ is a function from Ω to Ω.

Classical Logic "is" a PC-structure, with $\Omega = \{0, 1\}$, $\top = 1$, $\bot = 0$, $\leq = \{(0, 0), (0, 1), (1, 0)\}$, $\land = \left\{ \begin{smallmatrix} ((0,0),0),((0,1),0), \\ ((1,0),0),((1,1),1) \end{smallmatrix} \right\}$, etc.

PC-structures let us interpret expressions from Propositional Calculus ("PC-expressions"), and let us define a notion of *tautology*. For example, in Classical Logic,

- $\neg\neg P \leftrightarrow P$ is a tautology because it is valid (i.e., it yields $\top$) for all values of P in Ω,
- $\neg(P \land Q) \to (\neg P \lor \neg Q)$ s a tautology because it is valid for all values of P and Q in Ω,
- but $P \lor Q \to P \land Q$ is *not* a tautology, because when $P = 0$ and $Q = 1$ the result is not $\top$:

$$\underbrace{\underbrace{\underbrace{P}_{0} \lor \underbrace{Q}_{1}}_{1} \to \underbrace{\underbrace{P}_{0} \land \underbrace{Q}_{1}}_{0}}_{0}$$

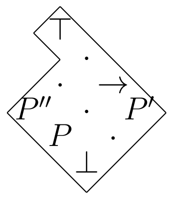
8 Propositional calculus in a ZHA

Let Ω be the set of points of a ZHA and $\leq$ the default partial order on it. The *default meanings for* $\top, \bot, \land, \lor, \to, \leftrightarrow, \neg$ are these ones:

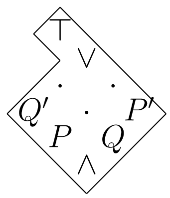
$$
\begin{array}{rcl}
\langle a,b\rangle \leq \langle c,d\rangle & := & a \leq c \land b \leq d \\
\langle a,b\rangle \geq \langle c,d\rangle & := & a \geq c \land b \geq d \\[4pt]
\langle a,b\rangle \mathsf{\ above\ } \langle c,d\rangle & := & a \geq c \land b \geq d \\
\langle a,b\rangle \mathsf{\ below\ } \langle c,d\rangle & := & a \leq c \land b \leq d \\
\langle a,b\rangle \mathsf{\ leftof\ } \langle c,d\rangle & := & a \geq c \land b \leq d \\
\langle a,b\rangle \mathsf{\ rightof\ } \langle c,d\rangle & := & a \leq c \land b \geq d \\[4pt]
\mathsf{valid}(\langle a,b\rangle) & := & \langle a,b\rangle \in \Omega \\
\mathsf{ne}(\langle a,b\rangle) & := & \mathsf{if\ valid\ }(\langle a,b+1\rangle)\mathsf{\ then\ ne}(\langle a,b+1\rangle)\mathsf{\ else\ }\langle a,b\rangle\mathsf{\ end} \\
\mathsf{nw}(\langle a,b\rangle) & := & \mathsf{if\ valid\ }(\langle a+1,b\rangle)\mathsf{\ then\ nw}(\langle a+1,b\rangle)\mathsf{\ else\ }\langle a,b\rangle\mathsf{\ end} \\[4pt]
\langle a,b\rangle \land \langle c,d\rangle & := & \langle \mathsf{min}(a,c), \mathsf{min}(b,d)\rangle \\
\langle a,b\rangle \lor \langle c,d\rangle & := & \langle \mathsf{max}(a,c), \mathsf{max}(b,d)\rangle \\[4pt]
\langle a,b\rangle \to \langle c,d\rangle & := & \begin{array}{llll}
\mathsf{if} & \langle a,b\rangle \mathsf{\ below\ } \langle c,d\rangle & \mathsf{then} & \top \\
\mathsf{elseif} & \langle a,b\rangle \mathsf{\ leftof\ } \langle c,d\rangle & \mathsf{then} & \mathsf{ne}(\langle c,d\rangle) \\
\mathsf{elseif} & \langle a,b\rangle \mathsf{\ rightof\ } \langle c,d\rangle & \mathsf{then} & \mathsf{nw}(\langle c,d\rangle) \\
\mathsf{elseif} & \langle a,b\rangle \mathsf{\ above\ } \langle c,d\rangle & \mathsf{then} & \langle c,d\rangle \\
\mathsf{end} & & &
\end{array} \\
\top & := & \mathsf{sup}(\Omega) \\
\bot & := & \langle 0,0\rangle \\
\neg\langle a,b\rangle & := & \langle a,b\rangle \to \bot \\
\langle a,b\rangle \leftrightarrow \langle c,d\rangle & := & (\langle a,b\rangle \to \langle c,d\rangle) \land (\langle c,d\rangle \to \langle a,b\rangle)
\end{array}
$$

Let Ω be the ZHA at the top left in the figure below. Then, with the default meanings for the connectives neither $\neg\neg P \to P$ nor $\neg(P \land Q) \to (\neg P \lor \neg Q)$ are tautologies, as there are valuations that make them yield results different than $\top = 32$:

$$\begin{array}{ccccc} & 32 & & & \\ & & 22 & & \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{array}$$



$$\underbrace{(\underbrace{\neg\underbrace{\neg\underbrace{P}_{10}}_{02}}_{20}) \to \underbrace{P}_{10}}_{12}$$



$$\underbrace{\underbrace{\neg(\underbrace{\underbrace{P}_{10} \land \underbrace{Q}_{01}}_{00})}_{32} \to \underbrace{(\underbrace{\neg\underbrace{P}_{10}}_{02} \lor \underbrace{\neg\underbrace{Q}_{01}}_{20})}_{22}}_{22}$$

So: *some* classical tautologies are not tautologies in this ZHA.

The somewhat arbitrary-looking definition of '$\to$' will be explained at the end of the next section.

9 Heyting Algebras

A *Heyting Algebra* is a PC-structure

$$H = (\Omega, \leq_H, \top_H, \bot_H, \land_H, \lor_H, \to_H, \leftrightarrow_H, \neg_H),$$

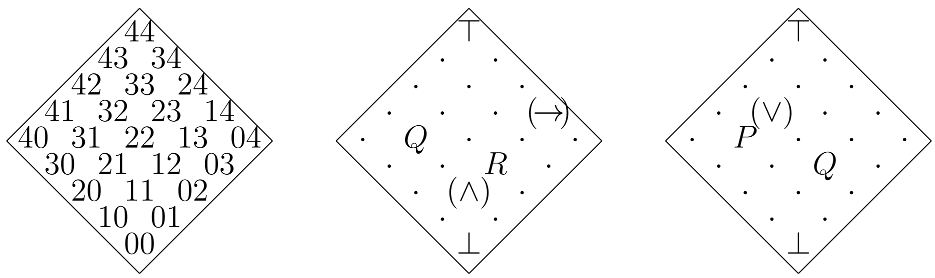
in which:

1) $(\Omega, \leq_H)$ is a partial order
2) $\top_H$ is the top element of the partial order
3) $\bot_H$ is the bottom element of the partial order
4) $P \leftrightarrow_H Q$ is the same as $(P \to_H Q) \land_H (Q \to_H P)$
5) $\neg_H P$ is the same as $P \to_H \bot_H$
6) $\forall P, Q, R \in \Omega.\ (P \leq_H (Q \land_H R)) \leftrightarrow ((P \leq_H Q) \land (P \leq_H R))$
7) $\forall P, Q, R \in \Omega.\ ((P \lor_H Q) \leq_H R) \leftrightarrow ((P \leq_H R) \land (Q \leq_H R))$
8) $\forall P, Q, R \in \Omega.\ (P \leq_H (Q \to_H R)) \leftrightarrow ((P \land_H Q) \leq_H R)$

6') $\forall Q, R \in \Omega. \exists! Y \in \Omega. \forall P \in \Omega.\ (P \leq_H Y) \leftrightarrow ((P \leq_H Q) \land (P \leq_H R))$
7') $\forall P, Q \in \Omega. \exists! X \in \Omega. \forall R \in \Omega.\ (X \leq_H R) \leftrightarrow ((P \leq_H R) \land (Q \leq_H R))$
8') $\forall Q, R \in \Omega. \exists! Y \in \Omega. \forall P \in \Omega.\ (P \leq_H Y) \leftrightarrow ((P \land_H R) \leq_H R)$

The conditions 6', 7', 8' say that there are unique elements in Ω that "behave as" $Q \land_H R$, $P \lor_H Q$ and $Q \to_H R$ for given P, Q, R; the conditions 6,7,8 say that $Q \land_H R$, $P \lor_H Q$ and $Q \to_H R$ are exactly the elements with this behavior.

The positional notation on ZHAs is very helpful for visualizing what the conditions 6',7',8',6,7,8 mean. Let Ω be the ZDAG on the left below:



we will see that

a) if $Q = 31$ and $R = 12$ then $Q \wedge_H R = 11$,
b) if $P = 31$ and $Q = 12$ then $P \vee_H Q = 32$,
c) if $Q = 31$ and $R = 12$ then $Q \to_H R = 14$.

Let's see each case separately — but, before we start, note that in 6, 7, 8, 6', 7', 8' we work part with truth values in Ω and part with standard truth values. For example, in 6, with $P = 20$, we have:

$$\underbrace{\underbrace{(\underbrace{P}_{20} \le_H \underbrace{(\underbrace{Q}_{31} \wedge_H \underbrace{R}_{12})}_{11})}_{0} \leftrightarrow \underbrace{(\underbrace{(\underbrace{P}_{20} \le_H \underbrace{Q}_{31})}_{1} \wedge \underbrace{(\underbrace{P}_{20} \le_H \underbrace{R}_{12})}_{0})}_{0}}_{1}$$

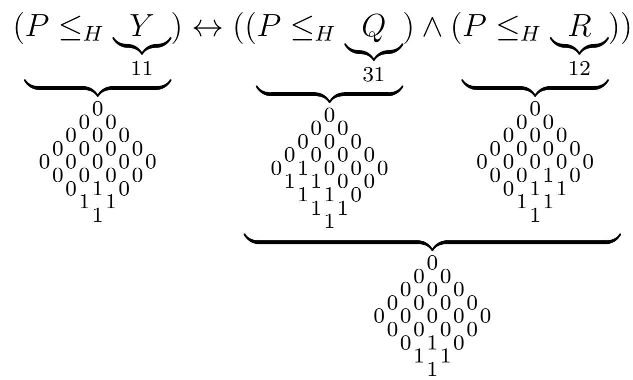
a) Let $Q = 31$ and $R = 12$. We want to see that $Q \wedge_H R = 11$, i.e., that

$$\forall P \in \Omega.\ \ (P \le_H Y) \leftrightarrow ((P \le_H Q) \wedge (P \le_H R))$$

holds for $Y = 11$ and for no other $Y \in \Omega$. We can visualize the behavior of $P \le_H Q$ for all 'P's by drawing $\lambda P{:}\Omega.(P \le_H Q)$ in the positional notation; then we do the same for $\lambda P{:}\Omega.(P \le_H R)$ and for $\lambda P{:}\Omega.((P \le_H Q) \wedge (P \le_H R))$. Suppose that the full expression, '$\forall P{:}\Omega.$_____', is true; then the behavior of the left side of the '$\leftrightarrow$', $\lambda P{:}\Omega.(P \le_H Y)$, has to be a copy of the behavior of the right side, and that lets us find the only adequate value for Y.

The order in which we calculate and draw things is below, followed by the results themselves:

$$\underbrace{(P \le_H \underbrace{Y}_{(7)})}_{(6)} \leftrightarrow \underbrace{(\underbrace{(P \le_H \underbrace{Q}_{(1)})}_{(3)} \wedge \underbrace{(P \le_H \underbrace{R}_{(2)})}_{(4)})}_{(5)}$$

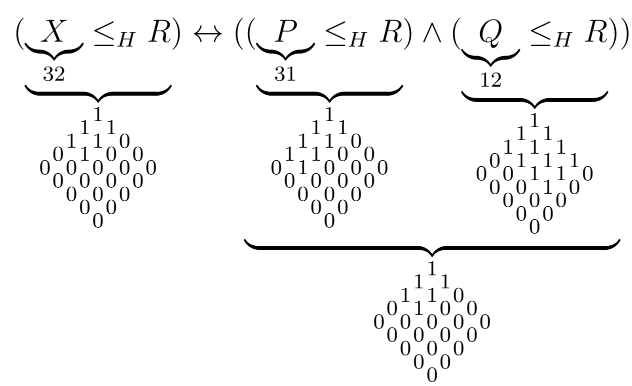


b) Let $P = 31$ and $Q = 12$. We want to see that $P \vee_H Q = 32$, i.e., that

$$\forall R{:}\Omega.\ \ (X \le_H R) \leftrightarrow ((P \le_H R) \land (Q \le_H R))$$

holds for $X = 32$ and for no other $X \in \Omega$. We do essentially the same as we did in (a), but now we calculate $\lambda R{:}\Omega.(P \le_H R)$, $\lambda R{:}\Omega.(Q \le_H R)$, and $\lambda R{:}\Omega.((P \le_H R) \land (Q \le_H R))$. The order in which we calculate and draw things is below, followed by the results themselves:

$$\underbrace{(\underbrace{X}_{(7)} \le_H R)}_{(6)} \leftrightarrow \underbrace{(\underbrace{(\underbrace{P}_{(1)} \le_H R)}_{(3)} \land \underbrace{(\underbrace{Q}_{(2)} \le_H R)}_{(4)})}_{(5)}$$

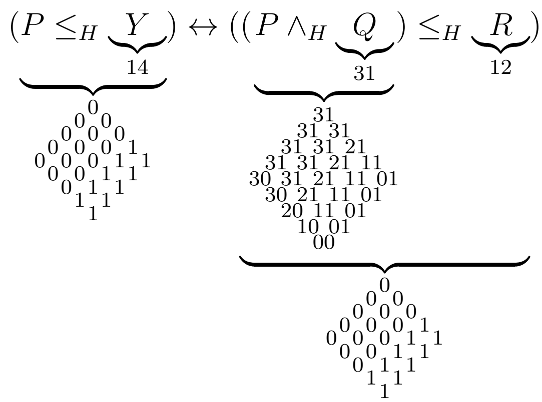


c) Let $Q = 31$ and $R = 12$. We want to see that $Q \to_H R = 14$, i.e., that

$$\forall P{:}\Omega.\ \ (P \le_H Y) \leftrightarrow ((P \land_H Q) \le_H R)$$

holds for $Y = 14$ and for no other $Y \in \Omega$. Here the strategy is slightly different. We start by visualizing $\lambda P{:}\Omega.(P \land_H Q)$, which is a function from Ω to Ω, not a function from Ω to $\{0,1\}$ like the ones we were using before. The order in which we calculate and draw things is below, followed by the results:

$$\underbrace{(P \leq_H \underbrace{Y}_{(6)})}_{(5)} \leftrightarrow \underbrace{((\underbrace{P \wedge_H \underbrace{Q}_{(1)}}_{(3)}) \leq_H \underbrace{R}_{(2)})}_{(4)}$$



10 The two implications are equivalent

In sec.8 we gave a definition of '$\to$' that is easy to calculate, and in sec.9 we saw a way to find by brute force[1] a value for $Q \to R$ that obeys

$$(P \leq (Q \to R)) \leftrightarrow (P \leq Q \wedge R)$$

for all P. In this section we will see that these two operations — called '$\overset{\text{C}}{\to}$' and '$\overset{\text{HA}}{\to}$' from here on — always give the same results.

Theorem 10.1 *We have* $(Q \overset{C}{\to} R) = (Q \overset{HA}{\to} R)$*, for any ZHA* H *and* $Q, R \in H$*.*

The proof will take the rest of this section, and our approach will be to check that for any ZHA H and $Q, R \in H$ this holds, for all $P \in H$:

$$(P \leq (Q \overset{\text{C}}{\to} R)) \leftrightarrow (P \leq Q \wedge R).$$

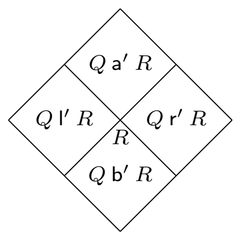
In '$\overset{\text{C}}{\to}$' the order of the cases is very important. For example, if $cd = 21$ and $ef = 23$ then both "cd below ef" and "cd leftof ef" are true, but "cd below ef" takes precedence

[1] "When in doubt use brute force" — Ken Thompson

and so $cd \stackrel{\mathrm{C}}{\to} ef = \top$. We can fix this by creating variants of below, leftof, righof and above that make the four cases disjoint. Abbreviating below, leftof, righof and above as b, l, r and a, we have:

$$\begin{array}{rcl@{\qquad}rcl}
cd\ \mathsf{b}\ ef & := & c \leq e \wedge d \leq f & cd\ \mathsf{b}'\ ef & := & c \leq e \wedge d \leq f \\
cd\ \mathsf{l}\ ef & := & c \leq e \wedge d \geq f & cd\ \mathsf{l}'\ ef & := & c \leq e \wedge d > f \\
cd\ \mathsf{r}\ ef & := & c \geq e \wedge d \leq f & cd\ \mathsf{r}'\ ef & := & c > e \wedge d \leq f \\
cd\ \mathsf{a}\ ef & := & c > e \wedge d > f & cd\ \mathsf{a}'\ ef & := & c > e \wedge d > f
\end{array}$$

visually the regions are these, for R fixed:



We clearly have:

$$Q \stackrel{\mathrm{C}}{\to} R \;=\; \left(\begin{array}{llll}
\mathsf{if} & Q\ \mathsf{b}\ R & \mathsf{then} & \top \\
\mathsf{elseif} & Q\ \mathsf{l}\ R & \mathsf{then} & \mathsf{ne}(R) \\
\mathsf{elseif} & Q\ \mathsf{r}\ R & \mathsf{then} & \mathsf{nw}(R) \\
\mathsf{elseif} & Q\ \mathsf{a}\ R & \mathsf{then} & R \\
\mathsf{end} & & &
\end{array}\right) \;=\; \left(\begin{array}{llll}
\mathsf{if} & Q\ \mathsf{b}'\ R & \mathsf{then} & \top \\
\mathsf{elseif} & Q\ \mathsf{l}'\ R & \mathsf{then} & \mathsf{ne}(R) \\
\mathsf{elseif} & Q\ \mathsf{r}'\ R & \mathsf{then} & \mathsf{nw}(R) \\
\mathsf{elseif} & Q\ \mathsf{a}'\ R & \mathsf{then} & R \\
\mathsf{end} & & &
\end{array}\right)$$

and $P \leq Q \stackrel{\mathrm{C}}{\to} R$ can be expressed as a conjunction of the four cases:

$$\begin{array}{l}
((P \leq Q \stackrel{\mathrm{C}}{\to} R) \leftrightarrow (P \wedge Q \leq R)) \\
\quad \leftrightarrow \quad \left(\begin{array}{ll}
Q\ \mathsf{b}'\ R \to ((P \leq Q \stackrel{\mathrm{C}}{\to} R) \leftrightarrow (P \wedge Q \leq R)) & \wedge \\
Q\ \mathsf{l}'\ R \to ((P \leq Q \stackrel{\mathrm{C}}{\to} R) \leftrightarrow (P \wedge Q \leq R)) & \wedge \\
Q\ \mathsf{r}'\ R \to ((P \leq Q \stackrel{\mathrm{C}}{\to} R) \leftrightarrow (P \wedge Q \leq R)) & \wedge \\
Q\ \mathsf{a}'\ R \to ((P \leq Q \stackrel{\mathrm{C}}{\to} R) \leftrightarrow (P \wedge Q \leq R)) &
\end{array}\right) \\
\quad \leftrightarrow \quad \left(\begin{array}{ll}
Q\ \mathsf{b}'\ R \to ((P \leq \top) \leftrightarrow (P \wedge Q \leq R)) & \wedge \\
Q\ \mathsf{l}'\ R \to ((P \leq \mathsf{ne}(R)) \leftrightarrow (P \wedge Q \leq R)) & \wedge \\
Q\ \mathsf{r}'\ R \to ((P \leq \mathsf{nw}(R)) \leftrightarrow (P \wedge Q \leq R)) & \wedge \\
Q\ \mathsf{a}'\ R \to ((P \leq R) \leftrightarrow (P \wedge Q \leq R)) &
\end{array}\right)
\end{array}$$

Let's introduce a notation: a "$\widehat{a}$" means "make this digit as big possible without leaving the ZHA". So,

$$
\text{in}\quad
\begin{array}{cccccc}
 & 54 & & & & \\
53 & & 44 & & & \\
 & 43 & & 34 & & \\
42 & & 33 & & 24 & \\
 & 32 & & 23 & & \\
31 & & 22 & & 13 & \\
 & 21 & & 12 & & 03 \\
20 & & 11 & & 02 & \\
 & 10 & & 01 & & \\
 & & 00 & & & \\
\end{array}
\quad\text{we have}\quad
\begin{array}{rcccl}
\widehat{1}\widehat{2} & = & 54 & = & \top, \\
1\widehat{2} & = & 13 & = & \mathsf{ne}(12), \\
\widehat{1}2 & = & 42 & = & \mathsf{nw}(12);
\end{array}
$$

This lets us rewrite $\top$ as $\widehat{e}\widehat{f}$, $\mathsf{ne}(ef)$ as $e\widehat{f}$, and $\mathsf{nw}(ef)$ as $\widehat{e}f$.
Making $P = ab$, $Q = cd$, $R = ef$, we have:

$$
\begin{array}{l}
((ab \leq cd \stackrel{\mathrm{C}}{\to} ef) \leftrightarrow (ab \wedge cd \leq ef)) \\
\leftrightarrow \quad \left(\begin{array}{ll}
cd\,\mathsf{b}'\,ef \to ((ab \leq \widehat{e}\widehat{f}) \leftrightarrow (ab \wedge cd \leq ef)) & \wedge \\
cd\,\mathsf{l}'\,ef \to ((ab \leq e\widehat{f}) \leftrightarrow (ab \wedge cd \leq ef)) & \wedge \\
cd\,\mathsf{r}'\,ef \to ((ab \leq \widehat{e}f) \leftrightarrow (ab \wedge cd \leq ef)) & \wedge \\
cd\,\mathsf{a}'\,ef \to ((ab \leq ef) \leftrightarrow (ab \wedge cd \leq ef)) &
\end{array}\right) \\
\\
\leftrightarrow \quad \left(\begin{array}{ll}
c \leq e \wedge d \leq f \to ((ab \leq \widehat{e}\widehat{f}) \leftrightarrow (ab \wedge cd \leq ef)) & \wedge \\
c > e \wedge d \leq f \to ((ab \leq e\widehat{f}) \leftrightarrow (ab \wedge cd \leq ef)) & \wedge \\
c \leq e \wedge d > f \to ((ab \leq \widehat{e}f) \leftrightarrow (ab \wedge cd \leq ef)) & \wedge \\
c > e \wedge d > f \to ((ab \leq ef) \leftrightarrow (ab \wedge cd \leq ef)) &
\end{array}\right) \\
\\
\leftrightarrow \quad \left(\begin{array}{ll}
c \leq e \wedge d \leq f \to ((ab \leq \widehat{e}\widehat{f}) \leftrightarrow (ab \wedge cd \leq cd)) & \wedge \\
c > e \wedge d \leq f \to ((ab \leq e\widehat{f}) \leftrightarrow (ab \wedge cd \leq ed)) & \wedge \\
c \leq e \wedge d > f \to ((ab \leq \widehat{e}f) \leftrightarrow (ab \wedge cd \leq cf)) & \wedge \\
c > e \wedge d > f \to ((ab \leq ef) \leftrightarrow (ab \wedge cd \leq ef)) &
\end{array}\right) \\
\\
\leftrightarrow \quad \left(\begin{array}{ll}
c \leq e \wedge d \leq f \to ((ab \leq \widehat{e}\widehat{f}) \leftrightarrow \top) & \wedge \\
c > e \wedge d \leq f \to ((ab \leq e\widehat{f}) \leftrightarrow a \leq e) & \wedge \\
c \leq e \wedge d > f \to ((ab \leq \widehat{e}f) \leftrightarrow b \leq f) & \wedge \\
c > e \wedge d > f \to ((ab \leq ef) \leftrightarrow (a \leq e \wedge b \leq f)) &
\end{array}\right)
\end{array}
$$

In the last conjunction the four cases are trivial to check.

11 Logic in a Heyting Algebra

In sec.9 we saw a set of conditions — called 1 to 8' — that characterize the "Heyting-Algebra-ness" of a PC-structure. It is easy to see that Heyting-Algebra-ness, or "HA-ness", is equivalent to this set of conditions:

1	$\forall P.$	$(P \le P)$	id
	$\forall P, Q, R.$	$(P \le R) \leftarrow (P \le Q) \land (Q \le R)$	comp
2	$\forall P.$	$(P \le \top)$	$\top_1$
3	$\forall Q.$	$(\bot \le Q)$	$\bot_1$
6	$\forall P, Q, R.$	$(P \le Q \land R) \rightarrow (P \le Q)$	$\land_1$
	$\forall P, Q, R.$	$(P \le Q \land R) \rightarrow (P \le R)$	$\land_2$
	$\forall P, Q, R.$	$(P \le Q \land R) \leftarrow (P \le Q) \land (P \le R)$	$\land_3$
7	$\forall P, Q, R.$	$(P \lor Q \le R) \rightarrow (P \le R)$	$\lor_1$
	$\forall P, Q, R.$	$(P \lor Q \le R) \rightarrow (Q \le R)$	$\lor_2$
	$\forall P, Q, R.$	$(P \lor Q \le R) \leftarrow (P \le R) \land (Q \le R)$	$\lor_3$
8	$\forall P, Q, R.$	$(P \le Q {\rightarrow} R) \rightarrow (P \land Q \le R)$	$\rightarrow_1$
	$\forall P, Q, R.$	$(P \le Q {\rightarrow} R) \leftarrow (P \land Q \le R)$	$\rightarrow_2$

We omitted the conditions 4 and 5, that defined '$\leftrightarrow$' and '$\neg$' in terms of the other operators. The last column gives a name to each of these new conditions.

These new conditions let us put (some) proofs about HAs in tree form, as we shall see soon.

Let us introduce two new notations. The first one,

$$(\mathrm{expr}) \begin{bmatrix} \scriptstyle v_1:=\mathrm{repl}_1 \\ \scriptstyle v_2:=\mathrm{repl}_2 \end{bmatrix}$$

indicates simultaneous substitution of all (free) occurrences of the variables v_1 and v_2 in expr by repl_1 and repl_2. For example,

$$((x+y)\cdot z) \begin{bmatrix} \scriptstyle x:=a+y \\ \scriptstyle y:=b+z \\ \scriptstyle z:=c+x \end{bmatrix} \quad = \quad ((a+y)+(b+z))\cdot(c+x).$$

The second is a way to write '$\rightarrow$'s as horizontal bars. In

$$\dfrac{A \quad B \quad C}{D}\,\alpha \qquad \dfrac{E \quad F}{G}\,\beta \qquad \dfrac{H}{I}\,\gamma \qquad \dfrac{}{J}\,\delta \qquad \dfrac{\dfrac{}{K}\,\epsilon \quad \dfrac{L \quad M}{N}\,\zeta \quad O}{P}\,\eta$$

the trees mean:

- if A, B, C are true then D is true (by α),

- if E, F, are true then G is true (by β),
- if H is true then I is true (by γ),
- J is true (by δ, with no hypotheses),
- K is true (by ϵ); if L and M then N (by ζ); if K, N, O, then P (by η); combining all this we get a way to prove that if L, M, O, then P,

where $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta$ are usually names of rules.

The implications in the table in the beginning of this section can be rewritten as "tree rules" as:

$$\dfrac{}{P \le P}\,\mathrm{id} \qquad \dfrac{P \le Q \quad Q \le R}{P \le R}\,\mathrm{comp} \qquad \dfrac{}{P \le \top}\,\top_1 \qquad \dfrac{}{\bot \le Q}\,\bot_1$$

$$\dfrac{P \le Q \land R}{P \le Q}\,\land_1 \qquad \dfrac{P \le Q \land R}{P \le R}\,\land_2 \qquad \dfrac{P \le Q \quad P \le R}{P \le Q \land R}\,\land_3$$

$$\dfrac{P \lor Q \le R}{P \le R}\,\lor_1 \qquad \dfrac{P \lor Q \le R}{Q \le R}\,\lor_2 \qquad \dfrac{P \le R \quad Q \le R}{P \lor Q \le R}\,\lor_3$$

$$\dfrac{P \le Q {\to} R}{P \land Q \le R}\,{\to}_1 \qquad \dfrac{P \land Q \le R}{P \le Q {\to} R}\,{\to}_2$$

Note that the '$\forall P, Q, R \in \Omega$'s are left implicit in the tree rules, which means that *every substitution instance of the tree rules hold*; sometimes — but rarely — we will indicate the substitution explicitly, like this,

$$\left(\dfrac{P \land Q \le R}{P \le Q {\to} R}\,{\to}_2\right)\left[\begin{smallmatrix} Q:=P\to\bot \\ R:=\bot \end{smallmatrix}\right] \quad \rightsquigarrow \quad \dfrac{P \land (P{\to}\bot) \le \bot}{P \le ((P{\to}\bot){\to}\bot)}\,{\to}_2$$

$$({\to}_2)\left[\begin{smallmatrix} Q:=P\to\bot \\ R:=\bot \end{smallmatrix}\right] \quad \rightsquigarrow \quad \dfrac{P \land (P{\to}\bot) \le \bot}{P \le ((P{\to}\bot){\to}\bot)}\,{\to}_2\left[\begin{smallmatrix} Q:=P\to\bot \\ R:=\bot \end{smallmatrix}\right]$$

Usually we will only say '${\to}_2$' instead of '${\to}_2\left[\begin{smallmatrix} Q:=P\to\bot \\ R:=\bot \end{smallmatrix}\right]$' at the right of a bar, and the task of discovering which substitution has been used is left to the reader.

The tree rules can be composed in a nice visual way. For example, this,

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \land Q \le P \land Q}\,\mathrm{id}}{P \land Q \le P}\,\land_1 \quad P \le R}{P \land Q \le R}\,\mathrm{comp} \qquad \dfrac{\dfrac{\dfrac{}{P \land Q \le P \land Q}\,\mathrm{id}}{P \land Q \le Q}\,\land_2 \quad Q \le S}{P \land Q \le S}\,\mathrm{comp}}{P \land Q \le R \land S}\,\land_3$$

"is" a proof for:

$$\forall P, Q, R, S \in \Omega.\ (P \leq R) \land (Q \leq S) \to ((P \land Q) \leq (R \land S)).$$

11.1 Derived rules

Note: in this section we will ignore the operators '$\leftrightarrow$' and '$\neg$' in PC-structures; we will think that every '$P \leftrightarrow Q$' is as abbreviation for '$(P{\to}Q) \land (Q{\to}P)$' and every '$\neg P$' is an abbreviation for '$P{\to}\top$'.

We'll write $[\top_1], \ldots, [{\to}_2]$ for the "linear" versions of the rules in last section — for example, $[{\to}_2]$ is $(\forall P, Q, R \in \Omega.\ (P{\land}Q \leq R) \to (P \leq Q{\to}R))$ — and if $S = \{r_1, \ldots, r_n\}$ is a set of rules, each in tree form, then $[S] = [r_1] \land \ldots \land [r_n]$, and an "$S$-tree" is a proof in tree form that only uses rules that are in the set S.

Let $\mathsf{HA\text{-}ness}_1$, $\mathsf{HA\text{-}ness}_2$, $\mathsf{HA\text{-}ness}_3$, be these sets, with the rules from sec.11:

$$\begin{array}{rcl} \mathsf{HA\text{-}ness}_1 & = & \{\mathrm{id}, \mathrm{comp}, \top_1, \bot_1, \land_3, \lor_3, {\to}_2\}, \\ \mathsf{HA\text{-}ness}_2 & = & \{\land_1, \land_2, \lor_1, \lor_2, {\to}_1\}, \\ \mathsf{HA\text{-}ness}_3 & = & \mathsf{HA\text{-}ness}_1 \cup \mathsf{HA\text{-}ness}_2 \end{array}$$

and let $\mathsf{HA\text{-}ness}_4$, $\mathsf{HA\text{-}ness}_5$ and $\mathsf{HA\text{-}ness}_7$ be these ones, where the new rules are the ones at the left column of Figure 1:

$$\begin{array}{rcl} \mathsf{HA\text{-}ness}_4 & = & \{\land_4, \land_5, \lor_4, \lor_5, \mathsf{MP}_0, \mathsf{MP}\} \\ \mathsf{HA\text{-}ness}_5 & = & \mathsf{HA\text{-}ness}_1 \cup \mathsf{HA\text{-}ness}_4 \\ \mathsf{HA\text{-}ness}_7 & = & \mathsf{HA\text{-}ness}_1 \cup \mathsf{HA\text{-}ness}_2 \cup \mathsf{HA\text{-}ness}_4 \end{array}$$

Note that the trees in the right of Figure 1 are $\mathsf{HA\text{-}ness}_3$-trees.

Figure 1 can be interpreted in two ways. The first one is that it shows that

$$\begin{array}{rcl} [\mathsf{HA\text{-}ness}_3] & \to & [\land_4], \\ [\mathsf{HA\text{-}ness}_3] & \to & [\land_5], \\ [\mathsf{HA\text{-}ness}_3] & \to & [\lor_4], \\ [\mathsf{HA\text{-}ness}_3] & \to & [\lor_5], \\ [\mathsf{HA\text{-}ness}_3] & \to & [\mathsf{MP}_0], \\ [\mathsf{HA\text{-}ness}_3] & \to & [\mathsf{MP}], \\ [\mathsf{HA\text{-}ness}_3] & \to & [\mathsf{HA\text{-}ness}_4], \\ [\mathsf{HA\text{-}ness}_3] & \to & [\mathsf{HA\text{-}ness}_7]; \end{array}$$

the second one is that it shows a way to replace occurrences of $\land_4$, $\land_5$, $\lor_4$, $\lor_5$, MP_0, MP. Take an $\mathsf{HA\text{-}ness}_7$-tree, T. Call it hypotheses $H_1, \ldots, H_n$, and its conclusion C, Replace each occurrence of $\land_4$, $\land_5$, $\lor_4$, $\lor_5$, MP_0, MP in T by the corresponding tree in the right side of Figure 1. The result is a new tree, T', which is "equivalent" to T in the sense of having the same hypotheses and conclusion as T. So,

$$\dfrac{}{Q \wedge R \leq Q}\ \wedge_4 \quad := \quad \dfrac{\dfrac{}{Q \wedge R \leq Q \wedge R}\ \mathrm{id}\left[P{:=}Q{\wedge}R\right]}{Q \wedge R \leq Q}\ \wedge_1\left[P{:=}Q{\wedge}R\right]$$

$$\dfrac{}{Q \wedge R \leq R}\ \wedge_5 \quad := \quad \dfrac{\dfrac{}{Q \wedge R \leq Q \wedge R}\ \mathrm{id}\left[P{:=}Q{\wedge}R\right]}{Q \wedge R \leq R}\ \wedge_2\left[P{:=}Q{\wedge}R\right]$$

$$\dfrac{}{P \leq P \vee Q}\ \vee_4 \quad := \quad \dfrac{\dfrac{}{P \vee Q \leq P \vee Q}\ \mathrm{id}\left[P{:=}P{\vee}Q\right]}{P \leq P \vee Q}\ \vee_1\left[R{:=}P{\vee}Q\right]$$

$$\dfrac{}{Q \leq P \vee Q}\ \vee_5 \quad := \quad \dfrac{\dfrac{}{P \vee Q \leq P \vee Q}\ \mathrm{id}\left[P{:=}P{\vee}Q\right]}{Q \leq P \vee Q}\ \vee_2\left[R{:=}P{\vee}Q\right]$$

$$\dfrac{}{Q \wedge (Q{\to}R) \leq R}\ \mathsf{MP}_0 \quad := \quad \dfrac{\dfrac{}{Q{\to}R \leq Q{\to}R}\ \mathrm{id}}{(Q{\to}R) \wedge Q \leq R}\ {\to_1}$$

$$\dfrac{P \leq Q \quad P \leq Q{\to}R}{P \leq R}\ \mathsf{MP} \quad := \quad \dfrac{\dfrac{P \leq Q \quad P \leq Q{\to}R}{P \leq Q \wedge (Q{\to}R)} \qquad \dfrac{}{Q \wedge (Q{\to}R) \leq R}\ \mathsf{MP}_0}{P \leq R}\ \mathrm{comp}$$

Figure 1: Derived rules

$$\frac{P \le Q \wedge R}{P \le Q}\ \wedge_1 \quad := \quad \frac{P \le Q \wedge R \qquad \overline{Q \wedge R \le Q}\ \wedge_4}{P \le Q}\ \text{comp}$$

$$\frac{P \le Q \wedge R}{P \le R}\ \wedge_2 \quad := \quad \frac{P \le Q \wedge R \qquad \overline{Q \wedge R \le R}\ \wedge_5}{P \le R}\ \text{comp}$$

$$\frac{P \vee Q \le R}{P \le R}\ \vee_1 \quad := \quad \frac{\overline{P \le P \vee Q}\ \vee_4 \qquad P \vee Q \le R}{P \le R}\ \text{comp}$$

$$\frac{P \vee Q \le R}{Q \le R}\ \vee_2 \quad := \quad \frac{\overline{Q \le P \vee Q}\ \vee_5 \qquad P \vee Q \le R}{Q \le R}\ \text{comp}$$

$$\frac{P \le Q{\to}R}{P \wedge Q \le R}\ {\to}_1 \quad :=$$

$$\frac{\dfrac{\overline{P \wedge Q \le Q}\ \wedge_5 \qquad \dfrac{\overline{P \wedge Q \le P}\ \wedge_4 \qquad P \le Q{\to}R}{P \wedge Q \le Q{\to}R}\ \text{comp}}{P \wedge Q \le Q \wedge (Q{\to}R)}\ \wedge_3 \qquad \overline{Q \wedge (Q{\to}R) \le R}\ \mathsf{MP}_0}{P \wedge Q \le R}\ \text{comp}$$

Figure 2: Derived rules (2)

- every $\mathsf{HA\text{-}ness}_3$-tree is an $\mathsf{HA\text{-}ness}_7$-tree,
- every $\mathsf{HA\text{-}ness}_7$-tree is "equivalent" to an $\mathsf{HA\text{-}ness}_3$-tree.

We call this trick "derived rules" — the rules in $\mathsf{HA\text{-}ness}_4$ are "derived" from $\mathsf{HA\text{-}ness}_3$, and $\mathsf{HA\text{-}ness}_3$ and $\mathsf{HA\text{-}ness}_7$ are "equivalent" in the sense that they "prove the same things".

Now look at Figure 2. It has the rules in $\mathsf{HA\text{-}ness}_2$ at the left, and $\mathsf{HA\text{-}ness}_5$-trees at the right; it shows that

$$\begin{array}{rcl}
[\mathsf{HA\text{-}ness}_5] & \to & [\wedge_1], \\
[\mathsf{HA\text{-}ness}_5] & \to & [\wedge_2], \\
[\mathsf{HA\text{-}ness}_5] & \to & [\vee_1], \\
[\mathsf{HA\text{-}ness}_5] & \to & [\vee_2], \\
[\mathsf{HA\text{-}ness}_5] & \to & [\to_2], \\
[\mathsf{HA\text{-}ness}_5] & \to & [\mathsf{HA\text{-}ness}_2], \\
[\mathsf{HA\text{-}ness}_5] & \to & [\mathsf{HA\text{-}ness}_7],
\end{array}$$

and it also shows how to take an $\mathsf{HA\text{-}ness}_7$-tree T and replace every occurrence of an $\mathsf{HA\text{-}ness}_4$-rule in it by an $\mathsf{HA\text{-}ness}_3$-tree, producing an $\mathsf{HA\text{-}ness}_3$-tree T' which is "equivalent" to T. This means that:

- every $\mathsf{HA\text{-}ness}_5$-tree is an $\mathsf{HA\text{-}ness}_7$-tree,
- every $\mathsf{HA\text{-}ness}_7$-tree is "equivalent" to an $\mathsf{HA\text{-}ness}_5$-tree,

and that $\mathsf{HA\text{-}ness}_3$, $\mathsf{HA\text{-}ness}_7$ and $\mathsf{HA\text{-}ness}_5$ are all "equivalent".

12 Topologies

The best way to connect ZHAs to several standard concepts is by seeing that ZHAs are topologies on certain finite sets — actually on 2-column acyclical graphs (sec.15). This will be done here and in the next few sections.

A *topology* on a set X is a subset $\mathcal{U}$ of $\mathcal{P}(X)$ that contains the "everything" and the "nothing" and is closed by binary unions and intersections and by arbitrary unions. Formally:

1) $\mathcal{U}$ contains X and $\varnothing$,
2) if $P, Q \in \mathcal{U}$ then $\mathcal{U}$ contains $P \cup Q$ and $P \cap Q$,
3) if $\mathcal{V} \subset \mathcal{U}$ then $\mathcal{U}$ contains $\bigcup \mathcal{V}$.

A *topological space* is a pair $(X, \mathcal{U})$ where X is a set and $\mathcal{U}$ is a topology on X.

When $(X, \mathcal{U})$ is a topological space and $U \in \mathcal{U}$ we say that U is *open* in $(X, \mathcal{U})$.

For example, let X be the ZSet $\begin{smallmatrix}\bullet&&\bullet\\&\bullet&\\\bullet&&\bullet\end{smallmatrix}$, and let's use the characteristic function notation from sec.2 to denote its subsets — we write $X = \begin{smallmatrix}1&&1\\&1&\\1&&1\end{smallmatrix}$ and $\varnothing = \begin{smallmatrix}0&&0\\&0&\\0&&0\end{smallmatrix}$ instead of $X = \begin{smallmatrix}\bullet&&\bullet\\&\bullet&\\\bullet&&\bullet\end{smallmatrix}$ and $\varnothing = \begin{smallmatrix}\cdot&&\cdot\\&\cdot&\\\cdot&&\cdot\end{smallmatrix}$.

If $\mathcal{U} = \left\{\begin{smallmatrix}1&&0\\&0&\\0&&0\end{smallmatrix}, \begin{smallmatrix}0&&1\\&0&\\0&&0\end{smallmatrix}, \begin{smallmatrix}0&&0\\&1&\\0&&0\end{smallmatrix}, \begin{smallmatrix}0&&0\\&0&\\1&&0\end{smallmatrix}, \begin{smallmatrix}0&&0\\&0&\\0&&1\end{smallmatrix}\right\}$ then $\mathcal{U} \subset \mathcal{P}(X)$ but $\mathcal{U}$ fails all the conditions in 1, 2, 3 above:

1) $X = \begin{smallmatrix}1&&1\\&1&\\1&&1\end{smallmatrix} \notin \mathcal{U}$ and $\varnothing = \begin{smallmatrix}0&&0\\&0&\\0&&0\end{smallmatrix} \notin \mathcal{U}$

2) Let $P = \begin{smallmatrix}1&&0\\&0&\\0&&0\end{smallmatrix} \in \mathcal{U}$ and $Q = \begin{smallmatrix}0&&1\\&0&\\0&&0\end{smallmatrix} \in \mathcal{U}$. Then $P \cap Q = \begin{smallmatrix}0&&0\\&0&\\0&&0\end{smallmatrix} \notin \mathcal{U}$ and $P \cup Q = \begin{smallmatrix}1&&1\\&0&\\0&&0\end{smallmatrix} \notin \mathcal{U}$.

3) Let $\mathcal{V} = \left\{\begin{smallmatrix}0&&1\\&0&\\0&&0\end{smallmatrix}, \begin{smallmatrix}0&&0\\&1&\\0&&0\end{smallmatrix}, \begin{smallmatrix}0&&0\\&0&\\1&&0\end{smallmatrix}\right\} \subset \mathcal{U}$. Then $\bigcup \mathcal{V} = \begin{smallmatrix}0&&1\\&0&\\0&&0\end{smallmatrix} \cup \begin{smallmatrix}0&&0\\&1&\\0&&0\end{smallmatrix} \cup \begin{smallmatrix}0&&0\\&0&\\1&&0\end{smallmatrix} = \begin{smallmatrix}0&&1\\&1&\\1&&0\end{smallmatrix} \notin \mathcal{U}$.

Now let $K = \begin{smallmatrix}&\bullet&\\\bullet&&\bullet\\&\bullet&\\&\bullet&\end{smallmatrix}$ and $\mathcal{U} = \left\{\begin{smallmatrix}&0&\\0&&0\\&0&\\&0&\end{smallmatrix}, \begin{smallmatrix}&0&\\0&&0\\&0&\\&1&\end{smallmatrix}, \begin{smallmatrix}&0&\\0&&0\\&1&\\&1&\end{smallmatrix}, \begin{smallmatrix}&0&\\0&&1\\&1&\\&1&\end{smallmatrix}, \begin{smallmatrix}&0&\\1&&0\\&1&\\&1&\end{smallmatrix}, \begin{smallmatrix}&0&\\1&&1\\&1&\\&1&\end{smallmatrix}, \begin{smallmatrix}&1&\\1&&1\\&1&\\&1&\end{smallmatrix}\right\}$. In this case $(K, \mathcal{U})$ is a topological space.

Some sets have "default" topologies on them, denoted with '$\mathcal{O}$'. For example, $\mathbb{R}$ is often used to mean the topological space $(\mathbb{R}, \mathcal{O}(\mathbb{R}))$, where:

$$\mathcal{O}(\mathbb{R}) = \{\, U \subset \mathbb{R} \mid U \text{ is a union of open intervals} \,\}.$$

We say that a subset $U \subset \mathbb{R}$ is "open in $\mathbb{R}$" ("in the default sense"; note that now we are saying just "open in $\mathbb{R}$", not "open in $(\mathbb{R}, \mathcal{O}(\mathbb{R}))$") when U is a union of open intervals, i.e., when $U \in \mathcal{O}(\mathbb{R})$; but note that $\mathcal{P}(\mathbb{R})$ and $\{\varnothing, \mathbb{R}\}$ are also topologies on $\mathbb{R}$, and:

$$\begin{array}{lll}
\{2,3,4\} \in \mathcal{P}(\mathbb{R}), & \text{so} & \{2,3,4\} \text{ is open in } (\mathbb{R}, \mathcal{P}(\mathbb{R})), \\
\{2,3,4\} \notin \mathcal{O}(\mathbb{R}), & \text{so} & \{2,3,4\} \text{ is not open in } (\mathbb{R}, \mathcal{O}(\mathbb{R})), \\
\{2,3,4\} \notin \{\varnothing, \mathbb{R}\}, & \text{so} & \{2,3,4\} \text{ is not open in } (\mathbb{R}, \{\varnothing, \mathbb{R}\});
\end{array}$$

when we say just "U is open in X", this means that:

1) $\mathcal{O}(X)$ is clear from the context, and

2) $U \in \mathcal{O}(X)$.

13 The default topology on a ZSet

Let's define a default topology $\mathcal{O}(D)$ for each ZSet D.

For each ZSet D we define $\mathcal{O}(D)$ as:

$$\begin{array}{rcl}
\mathcal{O}(D) & := & \{\, U \subset D \mid \forall ((x,y),(x',y')) \in \mathsf{BPM}(D). \\
& & \qquad\qquad (x,y) \in U \to (x',y') \in U \,\}
\end{array}$$

whose visual meaning is this. Turn D into a ZDAG by adding arrows for the black pawns moves (sec.3), and regard each subset $U \subset D$ as a board configuration in which the black pieces may move down to empty positions through the arrows. A subset U is "stable" when no moves are possible because all points of U "ahead" of a black piece are already occupied by black pieces; a subset U is "non-stable" when there is at least one arrow $((x,y),(x',y')) \in \mathsf{BPM}(D)$ in which (x,y) had a black piece and (x',y') is an empty position.

In our two notations for subsets (sec.2) a subset $U \subset D$ is unstable when it has an arrow like '$\bullet \to \cdot$' or '$1 \to 0$'; remember that black pawn moves arrows go down. A subset $U \subset D$ is stable when none of its '$\bullet$'s or '1's can move down to empty positions.

"Open" is the same as "stable". $\mathcal{O}(D)$ is the set of stable subsets of D.

Some examples:
$\begin{smallmatrix} &0& \\ 0&0&1 \\ &0& \\ &0& \end{smallmatrix}$ is not open because it has a 1 above a 0,
$\mathcal{O}(\begin{smallmatrix} &\bullet& \\ \bullet&&\bullet \\ &\bullet& \\ &\bullet& \end{smallmatrix}) = \left\{ \begin{smallmatrix} &0& \\ 0&&0 \\ &0& \\ &0& \end{smallmatrix}, \begin{smallmatrix} &0& \\ 0&&0 \\ &0& \\ &1& \end{smallmatrix}, \begin{smallmatrix} &0& \\ 0&&0 \\ &1& \\ &1& \end{smallmatrix}, \begin{smallmatrix} &0& \\ 0&&1 \\ &1& \\ &1& \end{smallmatrix}, \begin{smallmatrix} &0& \\ 1&&0 \\ &1& \\ &1& \end{smallmatrix}, \begin{smallmatrix} &0& \\ 1&&1 \\ &1& \\ &1& \end{smallmatrix}, \begin{smallmatrix} &1& \\ 1&&1 \\ &1& \\ &1& \end{smallmatrix} \right\}$,
$\mathcal{O}(\begin{smallmatrix} &\bullet& \\ \bullet&&\bullet \\ \bullet&&\bullet \end{smallmatrix}) = \{ \begin{smallmatrix} &0& \\ 0&&0 \\ 0&&0 \end{smallmatrix}, \begin{smallmatrix} &0& \\ 0&&0 \\ 0&&1 \end{smallmatrix}, \begin{smallmatrix} &0& \\ 0&&0 \\ 1&&0 \end{smallmatrix}, \begin{smallmatrix} &0& \\ 0&&0 \\ 1&&1 \end{smallmatrix}, \begin{smallmatrix} &0& \\ 0&&1 \\ 0&&1 \end{smallmatrix}, \begin{smallmatrix} &0& \\ 0&&1 \\ 1&&1 \end{smallmatrix}, \begin{smallmatrix} &0& \\ 1&&0 \\ 1&&0 \end{smallmatrix}, \begin{smallmatrix} &0& \\ 1&&0 \\ 1&&1 \end{smallmatrix}, \begin{smallmatrix} &0& \\ 1&&1 \\ 1&&1 \end{smallmatrix}, \begin{smallmatrix} &1& \\ 1&&1 \\ 1&&1 \end{smallmatrix} \}$.

The definition of $\mathcal{O}(D)$ above can be generalized to any directed graph. If (A,R) is a directed graph, then $(A, \mathcal{O}_R(A))$ is a topological space if we define:

$$\mathcal{O}_R(A) := \{\, U \subseteq A \mid \forall (a,b) \in R.\ (a \in U \to b \in U) \,\}$$

The two definitions are related as this: $\mathcal{O}(D) = \mathcal{O}_{\mathsf{BPM}(D)}(D)$.

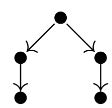
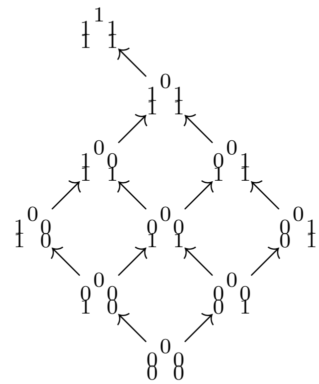
Note that we can see the arrows in $\mathsf{BPM}(D)$ or in R as *obligations* that open sets must obey; each arrow $a \to b$ says that every open set that contains a is forced to contain b too.

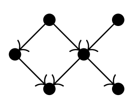
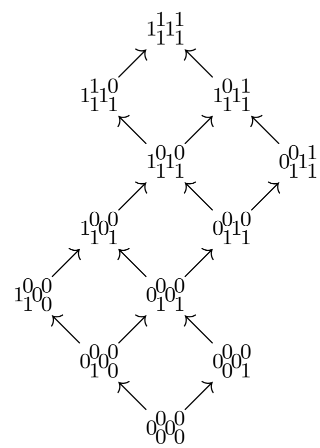
14 Topologies as partial orders

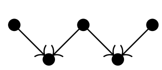
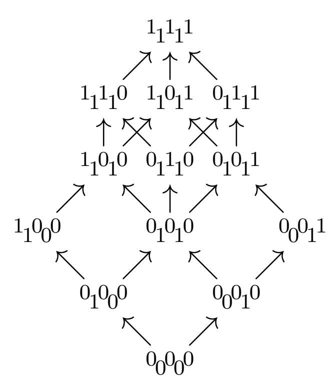
For any topological space $(X, \mathcal{O}(X))$ we can regard $\mathcal{O}(X)$ as a partial order, ordered by inclusion, with $\varnothing$ as its minimal element and X as its maximal element; we denote that partial order by $(\mathcal{O}(X), \subseteq)$.

Take any ZSet D. The partial order $(\mathcal{O}(D), \subseteq)$ will *sometimes* be a ZHA when we draw it with $\varnothing$ at the bottom, D at the top, and inclusions pointing up, as can be seen in the three figures below; when $D = \begin{smallmatrix} &\bullet& \\ \bullet&&\bullet \\ \bullet&&\bullet \end{smallmatrix}$ or $D = \begin{smallmatrix} &\bullet&&\bullet \\ \bullet&&\bullet& \\ &\bullet&&\bullet \end{smallmatrix}$ the result is a ZHA, but when $D = \begin{smallmatrix} \bullet&&\bullet&&\bullet \\ &\bullet&&\bullet& \end{smallmatrix}$ it not.

Let's write "$V \subset_1 U$" for "$V \subseteq U$ and V and U differ in exactly one point". When D is a ZSet the relation $\subseteq$ on $\mathcal{O}(D)$ is the transitive-reflexive closure of $\subset_1$, and $(\mathcal{O}(D), \subset_1)$ is easier to draw than $(\mathcal{O}(D), \subseteq)$.

$(H, \mathsf{BPM}(H)) =$  $(\mathcal{O}(H), \subset_1) =$ 

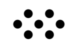
$(G, \mathsf{BPM}(G)) =$  $(\mathcal{O}(G), \subset_1) =$ 

$(W, \mathsf{BPM}(W)) =$  $(\mathcal{O}(W), \subset_1) =$ 

We can formalize a "way to draw $\mathcal{O}(D)$ as a ZHA" (or "...as a ZDAG") as a bijective function f from a ZHA (or from a ZSet) S to $\mathcal{O}(D)$ that creates a perfect correspondence between the white moves in S and the "$V \subset_1 U$-arrows"; more precisely, an f such that this holds: if $a, b \in S$ then $(a, b) \in \mathsf{WPM}(S)$ iff $f(a) \subset_1 f(b)$.

Note that the *number of elements* in an open set corresponds to the *height* where it is drawn; if $f : S \to \mathcal{O}(D)$ is a way to draw $\mathcal{O}(D)$ as a ZHA or a ZDAG then f takes points of the form $(\_\_\_, y)$ to open sets with y elements, and if $f : S \to \mathcal{O}(D)$ is a way to draw $\mathcal{O}(D)$ as a ZHA (not a ZDAG!) then we also have that $f((0,0)) = \varnothing \in \mathcal{O}(D)$.

The diagram for $(\mathcal{O}(H), \subset_1)$ above is a way to draw $\mathcal{O}(H)$ as a ZHA.
The diagram for $(\mathcal{O}(G), \subset_1)$ above is a way to draw $\mathcal{O}(G)$ as a ZHA.
The diagram for $(\mathcal{O}(W), \subset_1)$ above is *not* a way to draw $\mathcal{O}(W)$ as a ZSet. Look at ${}^{0}{}_{1}{}^{1}{}_{1}{}^{0}$ and ${}^{1}{}_{1}{}^{0}{}_{1}{}^{1}$ in the middle of the cube formed by all open sets of the form ${}^{a}{}_{1}{}^{b}{}_{1}{}^{c}$. We don't have ${}^{0}{}_{1}{}^{1}{}_{1}{}^{0} \subset_1 {}^{1}{}_{1}{}^{0}{}_{1}{}^{1}$, but we do have a white pawn move (not draw in the diagram!) from $f^{-1}({}^{0}{}_{1}{}^{1}{}_{1}{}^{0})$ to $f^{-1}({}^{1}{}_{1}{}^{0}{}_{1}{}^{1})$. We say that a ZSet is *thin* when it doesn't have three independent points.

Every time that a ZSet D has three independent points, as in W, we will have a situation like in $(\mathcal{O}(W), \subset_1)$; for example, if $B =$  then the open sets of B of the form ${}_{a}{}^{0}_{1}{}_{b}{}^{0}_{1}{}_{c}$ form a cube.

15 2-Column Graphs

Note: in this section we will manipulate objects with names like $1\_\_, 2\_\_, 3\_\_, \ldots, \_\_1, \_\_2, \_\_3, \ldots$; here are two good ways to formalize them:

$$\begin{array}{ll} \vdots & \vdots \\ 4\_\_ = (0,4) & \_\_4 = (1,4) \\ 3\_\_ = (0,3) & \_\_3 = (1,3) \\ 2\_\_ = (0,2) & \_\_2 = (1,2) \\ 1\_\_ = (0,1) & \_\_1 = (1,1) \end{array} \quad \text{or} \quad \begin{array}{ll} \vdots & \vdots \\ 4\_\_ = \texttt{"4\_"} & \_\_4 = \texttt{"\_4"} \\ 3\_\_ = \texttt{"3\_"} & \_\_3 = \texttt{"\_3"} \\ 2\_\_ = \texttt{"2\_"} & \_\_2 = \texttt{"\_2"} \\ 1\_\_ = \texttt{"1\_"} & \_\_1 = \texttt{"\_1"} \end{array},$$

where `"1_"`, `"_2"`, `""`, `"Hello!"`, etc are strings.

We define:

$$\begin{array}{rcl} LC(l) & := & \{1\_\_, 2\_\_, \ldots, l\_\_\} \\ RC(r) & := & \{\_\_1, \_\_2, \ldots, \_\_r\}, \end{array}$$

which generate a "left column" of height l and a "right column" of height r.

A *description for a 2-column graph* (a "D2CG") is a 4-tuple (l, r, R, L), where $l, r \in \mathbb{N}$, $R \subset \mathsf{LC}(l) \times \mathsf{RC}(r)$, $L \subset \mathsf{RC}(r) \times \mathsf{LC}(l)$; l is the height of the left column, r is the height of the right column, and R and L are set of intercolumn arrows (going right and left respectively).

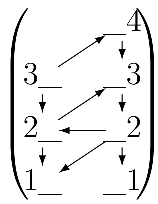
The operation $\mathsf{2CG}$ (in a sans-serif font) generates a directed graph from a D2CG:

$$\mathsf{2CG}(l, r, R, L) \quad := \quad \left(\mathsf{LC}(l) \cup \mathsf{RC}(r), \left\{ \begin{array}{c} \{l\_\_{\to}(l{-}1)\_\_,\ \ldots,\ 2\_\_{\to}1\_\_\} \cup \\ \{\_\_r{\to}\_\_(r{-}1),\ \ldots,\ \_\_2{\to}\_\_1\} \cup \\ R \cup L \end{array} \right\} \right)$$

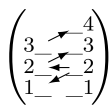
For example,

$$\mathsf{2CG}(3,4,\left\{\begin{smallmatrix}3\_\!\to\!\_4,\\ 2\_\!\to\!\_3\end{smallmatrix}\right\},\left\{\begin{smallmatrix}2\_\!\leftarrow\!\_2,\\ 1\_\!\leftarrow\!\_2\end{smallmatrix}\right\}) \quad := \quad \left(\left\{\begin{smallmatrix}3\_,\ 2\_,\ 1\_,\\ \_4,\ \_3,\ \_2,\ \_1\end{smallmatrix}\right\},\left\{\begin{smallmatrix}3\_\!\to\!2\_,\ 2\_\!\to\!1\_,\\ \_4\!\to\!\_3,\ \_3\!\to\!\_2,\ \_2\!\to\!\_1,\\ 3\_\!\to\!\_4,\ 2\_\!\to\!\_3,\\ 2\_\!\leftarrow\!\_2,\ 1\_\!\leftarrow\!\_2\end{smallmatrix}\right\}\right)$$

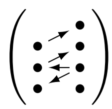
which is:



we will usually draw that more compactly, by omitting the intracolumn (i.e., vertical) arrows:



or



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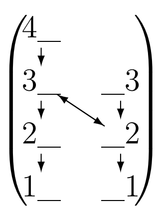
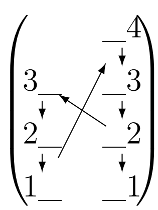
A *2-column graph* (a "2CG") is a directed graph that is of the form $\mathsf{2CG}(l,r,R,L)$. We will often say $(P,A) = \mathsf{2CG}(l,r,R,L)$, where the P stand for "points" and A for "arrows".

A *2-column acyclical graph* (a "2CAG") is a 2CG that doesn't have cycles. If L has an arrow that is the opposite of an arrow in R, this generates a cycle of length 2; if R has an arrow $l\_\!\!\to\!\!\_r'$ and L has an arrow $l'\_\!\!\leftarrow\!\!\_r$, where $l \le l'$ and $r \le r'$, this generates a cycle that can have a more complex shape — a triangle or a bowtie. For example,



and



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16 Topologies on 2CGs

In this section we will see that ZHAs are topologies on 2CAGs.

Let $(P,A) = \mathsf{2CG}(l,r,R,L)$ be a 2-column graph.
What happens if we look at the open sets of (P,A), i.e., at $\mathcal{O}_A(P)$? Two things:

1) every open set $U \in \mathcal{O}_A(P)$ is of the form $\mathsf{LC}(a) \cup \mathsf{RC}(b)$,

2) arrows in R and L forbids some '$\mathsf{LC}(a) \cup \mathsf{RC}(b)$'s from being open sets.

In order to understand that we need to introduce some notations for "piles".

The function

$$\mathsf{pile}(\langle a,b\rangle) := \mathsf{LC}(a) \cup \mathsf{RC}(b)$$

converts an element $\langle a,b\rangle \in \mathbb{LR}$ into a pile of elements in the left column of height a and a pile of elements in the right column of height b. We will write subsets of the points of a 2CG using a positional notation with arrows. So, for example, if $(P,A) = \mathsf{2CG}(3,4,\{2\_{\to}\_3\},\{2\_{\leftarrow}\_2\})$ then

$$(P,A) = \begin{pmatrix} & & \_4 \\ 3\_ & & \_3 \\ & \nearrow & \\ 2\_ & \leftarrow & \_2 \\ 1\_ & & \_1 \end{pmatrix} \qquad \text{and} \qquad \mathsf{pile}(21) = \begin{pmatrix} & & 0 \\ 0 & & 0 \\ & \nearrow & \\ 1 & \leftarrow & 0 \\ 1 & & 1 \end{pmatrix} \quad \text{(as a subset of } P\text{).}$$

Note that $\mathsf{pile}(21)$ is not open in $(P,\mathcal{O}_A(P))$, as it has an arrow '$1 \to 0$'. In fact, the presence of the arrow $\{2\_{\to}\_3\}$ in A means that all piles of the form

$$\begin{pmatrix} & & 0 \\ ? & & 0 \\ & \nearrow & \\ 1 & \leftarrow & ? \\ 1 & & ? \end{pmatrix}$$

are not open, the presence of the arrow $\{2\_{\leftarrow}\_2\}$ means that the piles of the form

$$\begin{pmatrix} & & ? \\ 0 & & ? \\ & \nearrow & \\ 0 & \leftarrow & 1 \\ ? & & 1 \end{pmatrix}$$

are not open sets.

The effect of these prohibitions can be expressed nicely with implications. If

$$(P,A) = \mathsf{2CG}(l,r,\left\{ \begin{smallmatrix} c\_{\to}\_d, \\ e\_{\to}\_f \end{smallmatrix} \right\},\left\{ \begin{smallmatrix} g\_{\leftarrow}\_h, \\ i\_{\leftarrow}\_j \end{smallmatrix} \right\})$$

then

$$\mathcal{O}_A(P) = \{\, \mathsf{pile}(ab) \mid a \in \{0,\ldots,l\}, b \in \{0,\ldots,r\}, \begin{pmatrix} a{\geq}c{\to}b{\geq}d \,\land \\ a{\geq}e{\to}b{\geq}f \,\land \\ a{\geq}g{\leftarrow}b{\geq}h \,\land \\ a{\geq}i{\leftarrow}b{\geq}j \end{pmatrix} \}$$

Let's use a shorter notation for comparing 2CGs and their topologies:

$$\mathcal{O}\left(\begin{tikzcd}[row sep=small] & \_5 \arrow[d] \\ 4\_ \arrow[d] & \_4 \arrow[d] \arrow[dl] \\ 3\_ \arrow[d] \arrow[dr] & \_3 \arrow[d] \arrow[dl] \\ 2\_ \arrow[d] & \_2 \arrow[d] \\ 1\_ & \_1 \end{tikzcd} \right) = \begin{array}{ccccccc} & & & 45 & & & \\ & & 44 & & 35 & & \\ & 43 & & 34 & & 25 & \\ 42 & & 33 & & 24 & & \\ & 32 & & 23 & & 14 & \\ & & 22 & & 13 & & \\ & 21 & & 12 & & 03 & \\ 20 & & 11 & & 02 & & \\ & 10 & & 01 & & & \\ & & 00 & & & & \end{array}$$

the arrows in R and L and the values of l and r are easy to read from the 2CG at the left, and we omit the 'pile's at the right.

In a situation like the above we say that the 2CG in the '$\mathcal{O}(\ldots)$' *generates* the ZHA at the right. There is an easy way to draw the ZHA generated by a 2CG, and a simple way to find the 2CG that generates a given ZHA. To describe them we need two new concepts.

If (A, R) is a directed graph and $S \subset A$ then $\downarrow S$ is the smallest open set in $\mathcal{O}_R(A)$ that contains S. If (A, R) is a ZDAG with black pawns moves as its arrows, think that the '1's in S are painted with a black paint that is very wet, and that that paint flows into the '0's below; the result of $\downarrow S$ is what we get when all the '0's below '1's get painted black. For example: $\downarrow 0^{0}_{0}0^{1}_{0} = 0^{0}_{1}1^{1}_{1}$. When (P, A) is a 2CG and $S \subseteq P$, we have to think that the paint flows along the arrows, even if some of the intercolumn arrows point upward. For example:

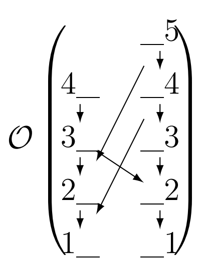
$$\downarrow\begin{pmatrix} & 0 \\ 0 & 0 \\ 0 \nearrow\!\leftarrow & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} & 0 \\ 0 & 1 \\ 1 \nearrow\!\leftarrow & 1 \\ 1 & 1 \end{pmatrix}$$

and if S consists of a single point, $S = \{s\}$, then we may write $\downarrow s$ instead of $\downarrow\{s\} = \downarrow S$. In the 2CG above, we have (omitting the 'pile's):

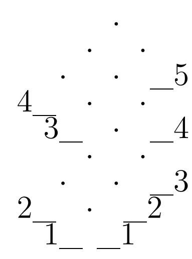
$$\downarrow\_2 \;=\; \downarrow\{\_2\} = \downarrow\begin{pmatrix} & 0 \\ 0 & 0 \\ 0 \nearrow\!\leftarrow & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} & 0 \\ 0 & 1 \\ 1 \nearrow\!\leftarrow & 1 \\ 1 & 1 \end{pmatrix} = 23, \quad \text{and} \quad \begin{array}{ll} & \downarrow\_4{=}24, \\ \downarrow 3\_{=}33, & \downarrow\_3{=}23, \\ \downarrow 2\_{=}23, & \downarrow\_2{=}23, \\ \downarrow 1\_{=}10, & \downarrow\_1{=}01, \end{array}$$

The second concept is this: the "generators" of a ZDAG D with white pawns moves as its arrows — or of a ZHA D — are the points of D that have exactly one white pawn move pointing *to* them (not *going out of* them).

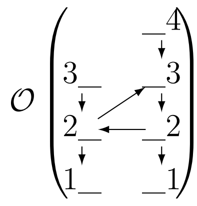
If (P, A) is a 2CAG, then $\mathcal{O}_A(P)$ is a ZHA, and '$\downarrow$' is a bijection from P to the generators of $\mathcal{O}_A(P)$; for example:



$$= \begin{array}{ccccccc} & & & 45 & & & \\ & & 44 & & 35 & & \\ & 43 & & 34 & & 25 & \\ 42 & & 33 & & 24 & & \\ & 32 & & 23 & & 14 & \\ & & 22 & & 13 & & \\ & 21 & & 12 & & 03 & \\ 20 & & 11 & & 02 & & \\ & 10 & & 01 & & & \\ & & 00 & & & & \end{array}$$



but if (P, A) is a 2CG with cycles, then $\mathcal{O}_A(P)$ is not a ZHA because each cycle generates a "gap" that disconnects the points of $\mathcal{O}_A(P)$. We just saw an example of a 2CG with a cycle in which $\downarrow 2\_ = 23 = \downarrow\_3 = \downarrow\_2$; look at its topology:



$$= \begin{array}{ccc} & 34 & \\ 33 & & 24 \\ & 23 & \\ \\ \\ & 11 & \\ 10 & & 01 \\ & 00 & \end{array}$$

17 Topologies as Heyting Algebras

The *open-set semantics* for Intuitionistic Propositional Logic is based on this idea: choose any topological space $(X, \mathcal{O}(X))$; the opens sets of $\mathcal{O}(X)$ will play the role of truth-values, and we define the components of a Heyting Algebra (sec.9) as this:

$$\begin{array}{rcllcl}
\Omega & := & \mathcal{O}(X) \\
P \le Q & := & P \subseteq Q \\
\top & := & \{\, x \in X \mid \top \,\} & = & X \\
\bot & := & \{\, x \in X \mid \bot \,\} & = & \emptyset \\
P \land Q & := & \{\, x \in X \mid x \in P \land x \in Q \,\} & = & P \cap Q \\
P \lor Q & := & \{\, x \in X \mid x \in P \lor x \in Q \,\} & = & P \cup Q \\
P \overset{\mathrm{M}}{\to} Q & := & \{\, x \in X \mid x \in P \to x \in Q \,\} \\
 & = & \{\, x \in X \mid x \notin P \lor x \in Q \,\} & = & (X \backslash P) \cup Q
\end{array}$$

However, this '$\overset{\mathrm{M}}{\to}$' *may* return a non-open result even when given open inputs,

$$\begin{smallmatrix} & 0 & \\ 1 & & 0 \\ 1 & & 0 \end{smallmatrix} \overset{\mathrm{M}}{\to} \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ 1 & & 1 \end{smallmatrix} \;=\; \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ 1 & & 1 \end{smallmatrix}$$

so our definition is broken; we can fix it by taking the interior (see sec.38):

$$P \to Q \quad := \quad \mathsf{int}(P \overset{\mathrm{M}}{\to} Q) \quad = \quad \mathsf{int}((X \backslash P) \cup Q)$$

Theorem 17.1 *For any topological space* $(X, \mathcal{O}(X))$ *the structure* $(\Omega, \le, \top, \bot, \land, \lor, \to)$ *defined as above is a Heyting Algebra. In particular, this holds for any* $P, Q, R \in \Omega$*:* $P \le (Q \to R)$ *iff* $(P \land Q) \le R$*.*

Proof. Standard; see for example [Awo06] (section 6.3). ∎

Note that Theorem 17.1 gives us another way to calculate the connectives in 2CGs. In sec.8 we saw how to calculate $\neg\neg P \to P$ in a certain ZHA when $P = 10$; the topological version of that is:

$$\underbrace{\underbrace{\neg \underbrace{\neg \underbrace{P}_{10}}_{02}}_{20} \to \underbrace{P}_{10}}_{12} \qquad\qquad \underbrace{\underbrace{\neg \underbrace{\neg \underbrace{P}_{\begin{smallmatrix} & 0 & \\ 0 & & 0 \\ 1 & & 0 \end{smallmatrix}}}_{\begin{smallmatrix} & 0 & \\ 0 & & 1 \\ 0 & & 1 \end{smallmatrix}}}_{\begin{smallmatrix} & 0 & \\ 1 & & 0 \\ 1 & & 0 \end{smallmatrix}} \to \underbrace{P}_{\begin{smallmatrix} & 0 & \\ 0 & & 0 \\ 1 & & 0 \end{smallmatrix}}}_{\begin{smallmatrix} & 0 & \\ 0 & & 1 \\ 1 & & 1 \end{smallmatrix}}$$

18 ZHA Logic is between IPL and CPL

We saw in sec.7 a figure that shows that $P \vee Q \to P \wedge Q$ is not a tautology in Classical Logic, and in sec.8 we saw a figure that shows that $\neg(P \wedge Q) \to (\neg P \vee \neg Q)$ is not a tautology in a certain ZHA; it reappered in sec.17, translated to a topological setting. We saw very little about deductive systems — only a bit in sec.11.

There is an easy argument that shows that "ZHA Logic" lies between Classical Propositional Logical and Intuitionistic Propositional Logic, and is distinct from both. We will work on the sets of tautologies. Let:

$$\begin{array}{rcl} S_P & := & P \to (Q \vee R) \\ S_Q & := & Q \to (P \vee R) \\ S_R & := & R \to (P \vee Q) \\ S & := & S_P \vee S_Q \vee S_R \end{array}$$

We will try to find a countermodel for S, and in the process we will discover that $\mathsf{Taut}(\mathsf{IPL}) \subsetneq \mathsf{Taut}(\mathsf{ZHAL}) \subsetneq \mathsf{Taut}(\mathsf{CPL})$.

If E is a PC-expression (sec.7) on a set $\mathbf{V}$ of variables — say, $\mathbf{V} = \{P, Q, R\}$ — then a *valuation* for E if a triple (W, A, v), where W is a finite set of "worlds", $A \subseteq W \times W$ is an "accessibility relation" on W, and $v : \mathbf{V} \to \mathcal{O}_A(W)$ is a function that assigns an open set to each variable in $\mathbf{V}$. Our examples will only need cases where W is a ZSet and $A = \mathsf{BPM}(W)$, and this lets us use a very compact notation for a triple (W, A, v) in which only v is shown and W and A are left implicit.

19 Converting between ZHAs and 2CAGs

Let's now see how to start from a 2CAG and produce its topology (a ZHA) quickly, and how to find quickly the 2CAG that generates a given ZHA.

From 2CAGs to ZHAs. Let $(P, A) = \mathsf{2CG}(l, r, R, L)$ be a 2CAG, and call the ZHA generated by it H. Then the top point of H is lr, its bottom point is 00. Let $C := \{00, {\downarrow}1\_, {\downarrow}2\_, \ldots, {\downarrow}l\_, lr\}$; then C has some of the points of the left wall (sec.5) of H, but usually not all. To "complete" C, apply this operation repeatedly: if $ab \in C$ and $ab \neq lr$, then test if either $(a+1)b$ or $a(b+1)$ are in C; if none of them are, add $a(b+1)$, which is northeast of ab. When there is nothing else to add, then C is the whole of the left wall of H. For the right wall, start with $D := \{00, {\downarrow}\_1, {\downarrow}\_2, \ldots, {\downarrow}\_r, lr\}$, and for each $ab \in C$ with $ab \neq lr$, test if either $(a+1)b$ or $a(b+1)$ are in D; if none of them are, add $(a+1)b$, which is northwest of ab. When there is nothing else to add, then D is the whole of the right wall of H.

In the acyclic example of the last section this yields:

$$\begin{array}{rcl}
C & = & \{00, \downarrow 1_, \downarrow 2_, \downarrow 3_, \downarrow 4_, lr\} \\
 & = & \{00, 10, 20, 32, 42, 45\} \\
 & \rightsquigarrow & \{00, 10, 20, 21, 22, 32, 42, 43, 44, 45\}, \\
D & = & \{00, \downarrow_1, \downarrow_2, \downarrow_3, \downarrow_4, \downarrow_5, lr\} \\
 & = & \{00, 01, 02, 03, 14, 25, 45\} \\
 & \rightsquigarrow & \{00, 01, 02, 03, 13, 14, 24, 25, 35, 45\}.
\end{array}$$

and the ZHA is everything between the "left wall" C and the "right wall" D.

From ZHAs to 2CAGs. Let H be a ZHA and let lr be its top point. Form the sequence of its left wall generators (the generators of H in which the arrow pointing to them points northwest) and the sequence of its right wall generators (the generators of H in which the arrow pointing to them points northeast). Look at where there are "gaps" in these sequences; each gap in the left wall generators becomes an intercolumn arrow going right, and each gap in the right wall generators becomes an intercolun arrow going left. In the acyclic example of the last section, this yields:

$$\begin{array}{llll}
 & & _5 = 25 & \\
 & & & \text{(gap becomes } 2_{\leftarrow}_5) \\
4_ = 42 & & _4 = 14 & \\
 & \text{(no gap)} & & \text{(gap becomes } 1_{\leftarrow}_4) \\
3_ = 32 & & _3 = 03 & \\
 & \text{(gap becomes } 3_{\rightarrow}_2) & & \text{(no gap)} \\
2_ = 20 & & _2 = 02 & \\
 & \text{(no gap)} & & \text{(no gap)} \\
1_ = 10 & & _1 = 01 &
\end{array}$$

We know l and r from the top point of the ZHA, and from the gaps we get R and L; the 2CAG that generates this ZHA is:

$$(4, 5, \{3\_{\rightarrow}\_2\}, \left\{\begin{matrix} 2\_{\leftarrow}\_5, \\ 1\_{\leftarrow}\_4 \end{matrix}\right\}).$$

Theorem 19.1 *The two operations above are inverse to one another in the following sense. If we start with a ZHA H, produce its 2CAG, and produce a ZHA H' from that, we get the same ZHA: $H' = H$. In the other direction, if we start with a 2CAG $(P, A) = \mathsf{2CG}(l, r, R, L)$, produce its ZHA, H, and then obtain a 2CAG $(P', A') = \mathsf{2CG}(l', r', R', L')$ from H, we get back the original 2CAG* if and only if it didn't have *any superfluous arrows; if the original 2CAG had superflous arrows then then new 2CAG will have $l' = l$, $r' = r$, and R' and L' will be R and L minus these "superfluous arrows", that are the ones that can be deleted without changing which 2-piles are forbidden. For example:*

$$\left(\begin{tikzcd}[row sep=small] 4\_ \arrow[r] \arrow[d] \arrow[dr] & \_4 \arrow[d] \\ 3\_ \arrow[d] \arrow[dr] & \_3 \arrow[d] \\ 2\_ \arrow[r] \arrow[d] & \_2 \arrow[d] \\ 1\_ & \_1 \end{tikzcd}\right) \rightsquigarrow \begin{array}{cccccc} & 44 & & & & \\ & & 34 & & & \\ & 33 & & 24 & & \\ 32 & & 23 & & 14 & \\ & 22 & & 13 & & 04 \\ & & 12 & & 03 & \\ & 11 & & 02 & & \\ 10 & & 01 & & & \\ & 00 & & & & \end{array} \rightsquigarrow \left(\begin{tikzcd}[row sep=small] 4\_ \arrow[r] \arrow[d] & \_4 \arrow[d] \\ 3\_ \arrow[d] & \_3 \arrow[d] \\ 2\_ \arrow[r] \arrow[d] & \_2 \arrow[d] \\ 1\_ & \_1 \end{tikzcd}\right)$$

In this case we have $R = \left\{ \begin{smallmatrix} 4\_\to\_4, \\ 4\_\to\_3, \\ 3\_\to\_2, \\ 2\_\to\_2 \end{smallmatrix} \right\}$ *and* $R' = \left\{ \begin{smallmatrix} 4\_\to\_4, \\ 2\_\to\_2 \end{smallmatrix} \right\}$.

20 Piccs and slashings

A picc ("partition into contiguous classes") of an interval $I = \{0, \ldots, n\}$ is a partition P of I that obeys this condition ("picc-ness"):

$$\forall a, b, c \in \{0, \ldots, n\}. \; (a < b < c \;\; \& \;\; a \sim_P c) \to (a \sim_P b \sim_P c).$$

So $P = \{\{0\}, \{1, 2, 3\}, \{4, 5\}\}$ is a picc of $\{0, \ldots, 5\}$, and $P' = \{\{0\}, \{1, 2, 4, 5\}, \{3\}\}$ is a partition of $\{0, \ldots, 5\}$ that is not a picc.

A short notation for piccs is this:

$$0|123|45 \equiv \{\{0\}, \{1, 2, 3\}, \{4, 5\}\}$$

we list all digits in the "interval" in order, and we put bars to indicate where we change from one equivalence class to another.

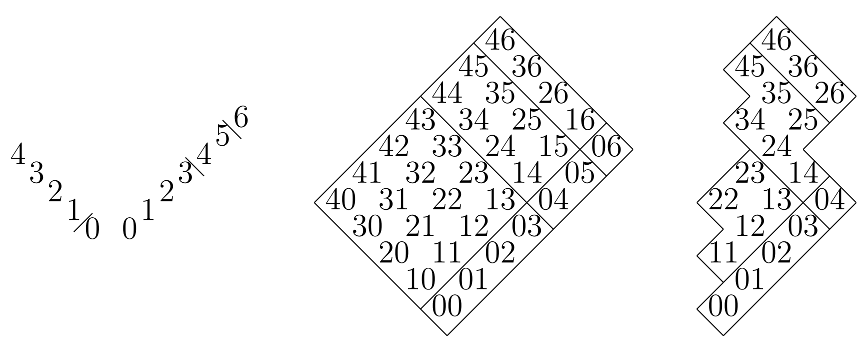
Let's define a notation for "intervals" in $\mathbb{LR}$,

$$[ab, ef] := [\langle a, b\rangle, \langle e, f\rangle] := \{\, \langle c, d\rangle \in \mathbb{LR} \mid a \leq c \leq e \;\; \& \;\; b \leq d \leq f \,\},$$

Note that it can be adapted to define "intervals" in a ZHAs H:

$$\begin{array}{rcl} [ab, ef] \cap H & := & \{\, \langle c, d\rangle \in \mathbb{LR} \mid a \leq c \leq e \;\; \& \;\; b \leq d \leq f \,\} \cap H \\ & = & \{\, \langle c, d\rangle \in H \mid a \leq c \leq e \;\; \& \;\; b \leq d \leq f \,\}. \end{array}$$

A *slashing* S on a ZHA H with top element ab is a pair of piccs, $S = (L, R)$, where L is a picc on $\{0, \ldots, a\}$ and R is a picc on $\{0, \ldots, b\}$; for example, $S = (4321/0,\ 0123\backslash 45\backslash 6)$ is a slashing on $[00, 46]$. We write the bars in L as '/'s and the bars in R as '\' as a reminder that they are to be interpreted as northeast and northwest "cuts" respectively; $S = (4321/0,\ 0123\backslash 45\backslash 6)$ is interpreted as the diagram at the left below, and it "slashes" $[00, 46]$ and the ZHA at the right below as:



A slashing $S = (L, R)$ on a ZHA H with top element ab induces an equivalence relation '$\sim_S$' on H that works like this: $\langle c, d\rangle \sim_S \langle e, f\rangle$ iff $c \sim_L e$ and $d \sim_R f$. We write

$$\begin{array}{rcl} [c]_L & := & \{\, e \in \{0, \ldots, a\} \mid c \sim_L a \,\} \\ [d]_R & := & \{\, f \in \{0, \ldots, b\} \mid d \sim_L f \,\} \\ [cd]_S & := & \{\, ef \in H \mid cd \sim_S ef \,\} \end{array}$$

for the equivalence classes, and note that

$$\begin{array}{lrcl} \text{if} & [c]_L & = & \{c', \ldots, c''\} \\ \text{and} & [d]_L & = & \{d', \ldots, d''\} \\ \text{then} & [cd]_S & = & [c'd', c''d''] \cap H; \end{array}$$

for example, in the ZHA at the right at the example above we have:

$$\begin{array}{rcl} [1]_L & = & \{1,2,3,4\}, \\ [2]_R & = & \{0,1,2,3\}, \\ [12]_S & = & [10,43] \cap H \;=\; \{11,12,13,22,23\}. \end{array}$$

We say that a slashing S on a ZHA H partitions H into *slash-regions*; later (sec.26) we will see that a J-operator J also partitions H, and we will refer to its equivalence classes as *J-regions*.

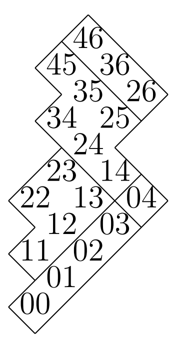
Slash-regions are intervals, but note that neither 10 or 43 belong to the slash-region $[12]_S = [10,43] \cap H$ above.

A *slash-partition* is a partition on a ZHA induced by a slashing, and a *slash-equivalence* is an equivalence relation on a ZHA induced by a slashing. Formally, a slash-partition on H is a set of subsets of H, and a slash-equivalence is subset of $H \times H$, but it is so easy to convert between partitions and equivalence relations that we will often use both terms interchangeably. Our visual representation for slash-partitions and slash-equivalences on a ZHA H will be the same: H slashed by diagonal cuts.

21 From slash-partitions back to slashings

We saw how to go from a slashing $S = (L, R)$ on H to an equivalence relation $\sim_S$ on H; let's see now how to recover L and R from $\sim_S$.

Let LW_H be the left wall of H, and RW_H the right wall of H. For example,

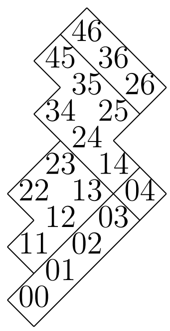
$H =$ 

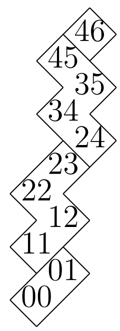
$$LW_H = \{00,01,11,12,22,23,24,34,35,45,46\}$$
$$RW_H = \{00,01,02,03,04,14,24,25,26,36,46\}$$

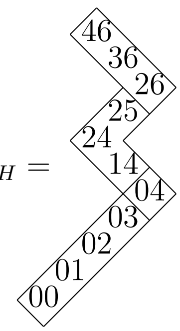
To recover the picc L — which is a picc on $\{0,1,2,3,4\}$ — we need to find where we change from an L-equivalence class to another when we go from one digit to the next;

and to recover the picc R — which is a picc on $\{0,1,2,3,4,5,6\}$ — we need to find where we change from an R-equivalence class to another when we go from one digit to the next.

We can recover L and R by walking LW_H (or RW_H) from bottom to top in a series of white pawns moves, and checking when we change from one S-equivalence class to another. Northwest moves give information about L, and northeast moves give information about R. Look at the example below, in which we walk on RW_H:

$H =$ 

$LW_H =$ 

$RW_H =$ 

$$
\begin{array}{llllll}
{}_{25}\nearrow^{26} & : 25\not\sim_S 26 & \Rightarrow & 5\not\sim_R 6 & \Rightarrow & 5\backslash 6 \\
{}_{24}\nearrow^{25} & : 24\sim_S 25 & \Rightarrow & 4\sim_R 5 & \Rightarrow & 45 \\
{}_{03}\nearrow^{04} & : 03\not\sim_S 04 & \Rightarrow & 3\not\sim_R 4 & \Rightarrow & 3\backslash 4 \\
{}_{02}\nearrow^{03} & : 02\sim_S 03 & \Rightarrow & 2\sim_R 3 & \Rightarrow & 23 \\
{}_{01}\nearrow^{02} & : 01\sim_S 02 & \Rightarrow & 1\sim_R 2 & \Rightarrow & 12 \\
{}_{00}\nearrow^{01} & : 00\sim_S 01 & \Rightarrow & 0\sim_R 1 & \Rightarrow & 01
\end{array}
$$

$$
\begin{array}{llllll}
{}^{46}\nwarrow_{36} & : 36\sim_S 46 & \Rightarrow & 3\sim_L 4 & \Rightarrow & 43 \\
{}^{36}\nwarrow_{26} & : 26\sim_S 36 & \Rightarrow & 2\sim_L 3 & \Rightarrow & 32 \\
{}^{24}\nwarrow_{14} & : 14\sim_S 24 & \Rightarrow & 1\sim_L 2 & \Rightarrow & 21 \\
{}^{14}\nwarrow_{04} & : 04\not\sim_S 14 & \Rightarrow & 0\not\sim_L 1 & \Rightarrow & 1/0
\end{array}
$$

$$(L,R) \;=\; (4321/0, 0123\backslash 45\backslash 6)$$

22 Slash-regions have maximal elements

...here is how our argument will work, in a particular case:

$$
\begin{array}{rcl}
[1]_L & = & \{1,2,3,4\}, \\
[2]_R & = & \{0,1,2,3\}, \\
I & = & [10,43], \\
[12]_S & = & I\cap H \;=\; \{11,12,13,22,23\}.
\end{array}
$$

$$\underbrace{\underbrace{\underbrace{(\underbrace{((11 \vee 12}_{=12\in I}) \vee 13}_{=13\in I}) \vee 22}_{=23\in I}) \vee 23}_{=23\in I} \qquad \underbrace{\underbrace{\underbrace{(\underbrace{((11 \vee 12}_{=12\in H}) \vee 13}_{=13\in H}) \vee 22}_{=23\in H}) \vee 23}_{=23\in H}$$

$$\bigvee[12]_S = \bigvee\{11, 12, 13, 22, 23\} = 11 \vee 12 \vee 13 \vee 22 \vee 23 \in I \cap H$$

$$11 \leq \bigvee[12]_S,\ 12 \leq \bigvee[12]_S,\ \ldots,\ 23 \leq \bigvee[12]_S$$

We have $[12]_S = I \cap H$, and $\bigvee[12]_S$ belongs to $I \cap H$ and is greter-or-equal than all elements of $I \cap H$, so $\bigvee[12]_S$ is the maximal element of $[12]_S$.

Here is how we can do that in the general case. Let $S = (L, R)$ be a slashing on a ZHA H. Let P be a point of H. The equivalence class $[P]_S$ is a finite set $\{P_1, \ldots, P_n\}$, and we know that $[P]_S = H \cap I$ for some interval I. Look at the elements P_1, $P_1 \vee P_2$, $(P_1 \vee P_2) \vee P_3$, …, $((P_1 \vee P_2) \vee \ldots) \vee P_n$ We can see that all of them belong to both H and I, so we conclude that $\bigvee[P]_S = ((P_1 \vee P_2) \vee \ldots) \vee P_n$ belongs to $H \cap I$, and it is easy to see that it is greater-or-equal that all elements in $H \cap I$, so it is the maximal element of $H \cap I$.

A similar argument shows that $\bigwedge[P]_S = ((P_1 \wedge P_2) \wedge \ldots) \wedge P_n$ is the smallest element of $[P]_S$.

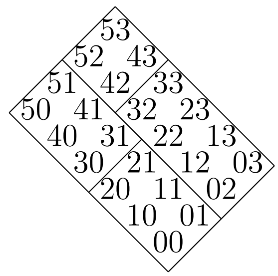
The same argument shows that if C is any non-empty set of the form $I \cap H$, where I is an interval, then $\bigvee C \in C$, $\bigwedge C \in C$, $[\bigwedge C, \bigvee C] \cap H = C$.

Remember that an *interval* in a ZHA H is any set of the form $[P, Q] \cap H$. Let's introduce a new definition: a *closed interval* in a ZHA H is a non-empty set $C \subset H$, with $\bigvee C \in C$, $\bigwedge C \in C$, $[\bigwedge C, \bigvee C] \cap H = C$; informally, a closed interval in a ZHA has a lowest and highest element, and it "is" everything between them.

23 Cuts stopping midway

We saw in the last section that every slash-region is a closed interval. A *partition into closed intervals* of a ZHA H is, as its name says, a partition of H whose equivalence classes are all closed intervals in H.

Some partitions into closed intervals of a ZHA are not slashings — for example, take the partition P with these equivalence classes:



Here is an easy way to prove formally that the partition above does not come from a slashing $S = (L, R)$. We will adapt the idea from sec.21, where we recovered L and R from northwest and northeast steps.

$$\begin{array}{ccccc}
\underbrace{21 \sim_P 31}_{\text{false}} & \leftrightarrow & \underbrace{2 \sim_L 3}_{=(} & \leftrightarrow & \underbrace{22 \sim_P 32}_{\text{true}} \\
\underbrace{31 \sim_P 41}_{\text{true}} & \leftrightarrow & \underbrace{3 \sim_L 4}_{=(} & \leftrightarrow & \underbrace{32 \sim_P 42}_{\text{false}}
\end{array}$$

The problem is that the figure above has "cuts stopping midway"... if its cuts all crossed the ZHA all the way through, we would have this for L and northeast cuts,

$$\begin{array}{ccccccccc}
0 \sim_L 1 & \leftrightarrow & 00 \sim_P 10 & \leftrightarrow & 01 \sim_P 11 & \leftrightarrow & 02 \sim_P 12 & \leftrightarrow & 03 \sim_P 13 \\
1 \sim_L 2 & \leftrightarrow & 10 \sim_P 20 & \leftrightarrow & 11 \sim_P 21 & \leftrightarrow & 12 \sim_P 22 & \leftrightarrow & 13 \sim_P 23 \\
2 \sim_L 3 & \leftrightarrow & 20 \sim_P 30 & \leftrightarrow & 21 \sim_P 31 & \leftrightarrow & 22 \sim_P 32 & \leftrightarrow & 23 \sim_P 33 \\
3 \sim_L 4 & \leftrightarrow & 30 \sim_P 40 & \leftrightarrow & 31 \sim_P 41 & \leftrightarrow & 32 \sim_P 42 & \leftrightarrow & 33 \sim_P 43 \\
4 \sim_L 5 & \leftrightarrow & 40 \sim_P 50 & \leftrightarrow & 41 \sim_P 51 & \leftrightarrow & 42 \sim_P 52 & \leftrightarrow & 43 \sim_P 53 \\
5 \sim_L 6 & \leftrightarrow & 50 \sim_P 60 & \leftrightarrow & 51 \sim_P 61 & \leftrightarrow & 52 \sim_P 62 & \leftrightarrow & 53 \sim_P 63
\end{array}$$

and something similar for R and northwest cuts.

Formally, a partition P on H has an "L-cut between c and c^+ stopping midway" if $cd \sim_P c^+d \;\not\leftrightarrow\; cd \sim_P c^+d$ for some d, and it has an "R-cut between d and d^+ stopping midway" if $cd \sim_P cd^+ \;\not\leftrightarrow\; c^+d \sim_P c^+d^+$ for some c; here we are writing x^+ for $x + 1$.

Theorem 23.1 *A partition of H into closed intervals is a slash-partition if and only if it doesn't have any cuts stopping midway.*

Proof. Use the ideas above to recover L and R from $\sim_P$, and then check that $S = (L, R)$ induces an equivalence relation $\sim_S$ that coincides with $\sim_P$. ■

24 Slash-operators

We can define operations that take each each $P \in H$ to the maximal and to the minimal element of its S-equivalent class, now that we know that these maximal and minimal elements exist:

$$\begin{array}{rcll} P^S & := & \bigvee [P]_S & \text{(maximal element)}, \\ P^{\mathrm{co}S} & := & \bigwedge [P]_S & \text{(minimal element)}. \end{array}$$

Note that $[P]_S = [P^{\mathrm{co}S}, P^S] \cap H$.

We will use the operation $\cdot^S$ a lot and $\cdot^{\mathrm{co}S}$ very little. The 'co' in 'co$S$' means that $\cdot^{\mathrm{co}S}$ is dual to $\cdot^S$, in a sense that will be made precise later.

A *slash-operator* on a ZHA H is a function $\cdot^S : H \to H$ induced by a slashing $S = (L, R)$ on H. It is easy to see that $P \leq P^S$ ("$\cdot^S$ is non-decreasing") and that $P^S = (P^S)^S$ ("$\cdot^S$ is idempotent").

Any idempotent function $\cdot^F : H \to H$ induces an equivalence relation on H: $P \sim_F Q$ iff $P^F = Q^F$. We can use that to test if a given $\cdot^F : H \to H$ is a slash-operator: $\cdot^F$ is a slash-operator iff it obeys all this:

1) $\cdot^F$ is idempotent,
2) $\cdot^F$ is non-decreasing,
3) $\sim_F$ partitions H into closed intervals,
4) $\sim_F$ doesn't have cuts stopping midway.

25 Slash-operators: a property

Slash-operators obey a certain property that will be very important later. Let's state that property in five equivalent ways:

1) If $cd \sim_S c'd'$ and $ef \sim_S e'f'$ then $cd \wedge ef \sim_S c'd' \wedge e'f'$.
2) If $P \sim_S P'$ and $Q \sim_S Q'$ then $P \wedge Q \sim_S P' \wedge Q'$.
3) If $P \sim_S P'$ and $Q \sim_S Q'$ then $(P \wedge Q)^S = (P' \wedge Q')^S$.
4) If $P \sim_S P'$ and $Q \sim_S Q'$ then

$$\begin{array}{rcll} (P \wedge Q)^S & = & (P^S \wedge Q^S)^S & \text{(a)} \\ & = & ((P')^S \wedge (Q')^S)^S & \text{(b)} \\ & = & (P' \wedge Q')^S & \text{(c)} \end{array}$$

5) $(P \wedge Q)^S = (P^S \wedge Q^S)^S$.

Here's a proof of $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4 \leftrightarrow 5$.

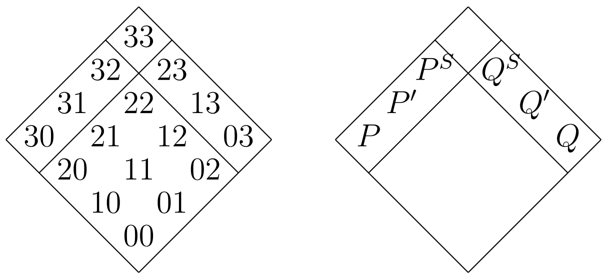
$1 \leftrightarrow 2$: we just changed notation,
$2 \leftrightarrow 3$: because $A \sim_S B$ iff $A^S = B^S$,

$3 \to 5$: make the substitution $\left[\begin{smallmatrix} P':=P^S \\ Q':=Q^S \end{smallmatrix}\right]$ in 3,

$5 \to 4$: 4a is just a copy of 5, and 4c is a copy of 5 with $\left[\begin{smallmatrix} P:=P' \\ Q:=Q' \end{smallmatrix}\right]$. For 4b, note that $P \sim_P P'$ implies $P^S = (P')^S$ and $Q \sim_P Q'$ implies $Q^S = (Q')^S$,

$4 \to 3$: 4 is an equality between more expressions than 3,

...and here is a way to visualize what is going on:



$$\underbrace{(\underbrace{\underbrace{P}_{30} \land \underbrace{Q}_{03}}_{00})^S}_{22} = \underbrace{\underbrace{(\underbrace{\underbrace{P}_{30}{}^S}_{32} \land \underbrace{\underbrace{Q}_{03}{}^S}_{23})}_{22}{}^S}_{22} = \underbrace{\underbrace{(\underbrace{\underbrace{P'}_{31}{}^S}_{32} \land \underbrace{\underbrace{Q'}_{13}{}^S}_{23})}_{22}{}^S}_{22} = \underbrace{(\underbrace{\underbrace{P'}_{31} \land \underbrace{Q'}_{13}}_{11})^S}_{22}$$

Note that all subexpressions belong to three S-regions: a region with P, P', $P^S = P'^S$, another with Q, Q', $Q^S = Q'^S$, and one with all the '$\land$'s. If we had cuts stopping midway then some of the '$\land$'s could be in different regions.

I think that the clearest way to show (1) is by putting its proof in tree form:

$$\dfrac{\dfrac{\dfrac{\dfrac{cd \sim_S c'd'}{c \sim_L c'} \quad \dfrac{ef \sim_S e'f'}{e \sim_L e'}}{\min(c,e) \sim_L \min(c',e')} \quad \dfrac{\dfrac{cd \sim_S c'd'}{d \sim_R d'} \quad \dfrac{ef \sim_S e'f'}{f \sim_R f'}}{\min(d,f) \sim_R \min(d',f')}}{\min(c,e)\min(d,f) \sim_S \min(c',e')\min(d',f')}}{cd \land ef \sim_S c'd' \land e'f'}$$

26 J-operators and J-regions

A *J-operator* on a Heyting Algebra $H = (\Omega, \leq, \top, \bot, \land, \lor, \to, \leftrightarrow, \neg)$ is a function $J : \Omega \to \Omega$ that obeys the axioms J1, J2, J3 below; we usually write J as $\cdot^* : \Omega \to \Omega$, and write the axioms as rules.

$$\dfrac{}{P \leq P^*}\ \mathsf{J1} \qquad \dfrac{}{P^* = P^{**}}\ \mathsf{J2} \qquad \dfrac{}{(P\&Q)^* = P^*\&Q^*}\ \mathsf{J3}$$

J1 says that the operation $\cdot^*$ is non-decreasing.

J2 says that the operation $\cdot^*$ is idempotent.
J3 is a bit mysterious but will have interesting consequences.

Note that when H is a ZHA then any slash-operator on H is a J-operator on it; see secs.24 and 25.

A J-operator induces an equivalence relation and equivalence classes on Ω, like slashings do:

$$\begin{array}{rcl} P \sim_J Q & \text{iff} & P^* = Q^* \\ [P]^J & := & \{\, Q \in \Omega \mid P^* = Q^* \,\} \end{array}$$

The axioms J1, J2, J3 have many consequences. The first ones are listed in Figure 3 as derived rules, whose names mean:
Mop (monotonicity for products): a lemma used to prove Mo,
Mo (monotonicity): $P \leq Q$ implies $P^* \leq Q^*$,
Sand (sandwiching): all truth values between P and P^* are equivalent,
EC&: equivalence classes are closed by '&',
EC$\vee$: equivalence classes are closed by '$\vee$',
ECS: equivalence classes are closed by sandwiching,

Take a J-equivalence class, $[P]^J$, and list its elements: $[P]^J = \{P_1, \ldots, P_n\}$. Let $P_\wedge := ((P_1 \wedge P_2) \wedge \ldots) \wedge P_n$ and Let $P_\vee := ((P_1 \vee P_2) \vee \ldots) \vee P_n$. It turns out that $[P]^J = [P_\wedge, P_\vee] \cap \Omega$; let's prove that by doing '$\subseteq$' first, then '$\supseteq$'.
Using EC& and EC$\vee$ several times we see that

$$\begin{array}{rr} P_1 \wedge P_2 \sim_J P & P_1 \vee P_2 \sim_J P \\ (P_1 \wedge P_2) \wedge P_3 \sim_J P & (P_1 \vee P_2) \vee P_3 \sim_J P \\ \vdots & \vdots \\ ((P_1 \wedge P_2) \wedge \ldots) \wedge P_n \sim_J P & ((P_1 \vee P_2) \vee \ldots) \vee P_n \sim_J P \end{array}$$

so $P_\wedge \sim_J P_\vee \sim_J P$, and by the sandwich lemma $([P_\wedge, P_\vee] \cap \Omega) \subseteq [P]^J$.
For any $P_i \in [P]^J$ we have $P_\wedge \leq P_i \leq P_\vee$, which means that:

$$\begin{array}{rcl} [P]^J & = & \{P_1, \ldots, P_n\} \\ & \subseteq & \{\, Q \in \Omega \mid P_\wedge \leq Q \leq P_\vee \,\} \\ & = & [P_\wedge, P_\vee] \cap \Omega, \end{array}$$

so $[P]^J \subseteq [P_\wedge, P_\vee] \cap \Omega$.

As the operation '$\cdot^*$' is increasing and idempotent, each equivalence class $[P]^J$ has exactly one maximal element, which is P^*; but $P_\vee$ is also the maximal element of $[P]^J$, so $P_\vee = P^*$, and we can interpret the operation '$\cdot *$' as "take each P to the top element

$$\dfrac{}{(P\&Q)^* \le Q^*}\,\mathsf{Mop} \quad := \quad \dfrac{\dfrac{}{(P\&Q)^* = P^*\&Q^*}\,\mathsf{J3} \qquad \dfrac{}{P^*\&Q^* \le Q^*}}{(P\&Q)^* \le Q^*}$$

$$\dfrac{P \le Q}{P^* \le Q^*}\,\mathsf{Mo} \quad := \quad \dfrac{\dfrac{\dfrac{P \le Q}{P = P\&Q}}{P^* = (P\&Q)^*} \qquad \dfrac{}{(P\&Q)^* \le Q^*}\,\mathsf{Mop}}{P^* \le Q^*}$$

$$\dfrac{P \le Q \le P^*}{P^* = Q^*}\,\mathsf{Sand} \quad := \quad \dfrac{\dfrac{P \le Q}{P^* \le Q^*}\,Mo \qquad \dfrac{\dfrac{Q \le P^*}{Q^* \le P^{**}}\,\mathsf{Mo} \qquad \dfrac{}{P^{**} = P^*}\,\mathsf{J2}}{Q^* \le P^*}}{P^* = Q^*}$$

$$\dfrac{P^* = Q^*}{P^* = Q^* = (P\&Q)^*}\,\mathsf{EC\&} \quad := \quad \dfrac{\dfrac{P^* = Q^*}{P^* = Q^* = P^*\&Q^*} \qquad \dfrac{}{P^*\&Q^* = (P\&Q)^*}\,\mathsf{J3}}{P^* = Q^* = (P\&Q)^*}$$

$$\dfrac{P^* = Q^*}{P^* = Q^* = (P \vee Q)^*}\,\mathsf{EC}\vee \quad := \quad \dfrac{P^* = Q^* \qquad \dfrac{\dfrac{\dfrac{}{P \le P \vee Q} \qquad \dfrac{\dfrac{}{P \le P^*}\,\mathsf{J1} \qquad \dfrac{\dfrac{}{Q \le Q^*}\,\mathsf{J1} \qquad \dfrac{P^* = Q^*}{Q^* = P^*}}{Q \le P^*}}{P \vee Q \le P^*}}{P \le P \vee Q \le P^*}}{P^* = (P \vee Q)^*}\,\mathsf{Sand}}{P^* = Q^* = (P \vee Q)^*}$$

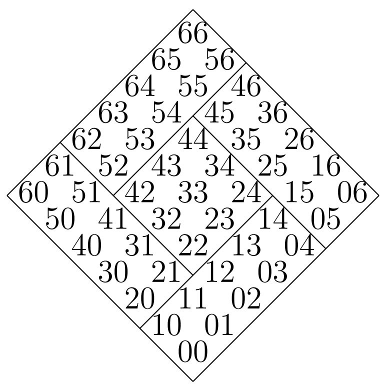
$$\dfrac{P \le Q \le R \quad P^* = R^*}{P^* = Q^* = R^*}\,\mathsf{ECS} \quad := \quad \dfrac{\dfrac{\dfrac{P \le Q \le R \qquad \dfrac{}{R \le R^*}\,\mathsf{J1} \qquad \dfrac{P^* = R^*}{R^* = P^*}}{P \le Q \le P^*}}{P^* = Q^*}\,\mathsf{Sand} \qquad P^* = R^*}{P^* = Q^* = R^*}$$

Figure 3: J-operators: basic derived rules

in its equivalence class", which is similar to how we defined an(other) operation '$\cdot^*$' on slashings in the previous section.

The operation "take each P to the bottom element in its equivalence class" will be useful in a few occasions; we will call it '$\cdot^{\mathrm{co}*}$' to indicate that it is dual to '$\cdot^*$' in some sense. Note that $P^{\mathrm{co}*} = P_\wedge$.

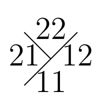
Look at the figure below, that shows a partition of a ZHA $A = [00, 66]$ into five regions, each region being an interval; this partition does not come from a slashing, as it has cuts that stop midway. Define an operation '$\cdot^*$' on A, that works by taking each truth-value P in it to the top element of its region; for example, $30^* = 61$.



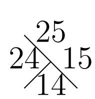
It is easy to see that '$\cdot^*$' obeys J1 and J2; however, it does *not* obey J3 — we will prove that in sec.27. As we will see, *the partitons of a ZHA into intervals that obey* J1, J2, J3 *ae exactly the slashings;* or, in other words, *every J-operator comes from a slashing*.

27 The are no Y-cuts and no λ-cuts

We want to see that if a partition of a ZHA H into intervals has "Y-cuts" or "λ-cuts" like these parts of the last diagram in the last section,



$\Leftarrow$ this is a Y-cut



$\Leftarrow$ this is a λ-cut

then it operation J that takes each element to the top of its equivalence class cannot obey J1, J2 and J3 at the same time. We will prove that by deriving rules that say that if $11 \sim_J 12$ then $21 \sim_J 22$, and that if $15 \sim_J 25$ then $14 \sim_J 24$; actually, our rules will say that if $11^* = 12^*$ then $(11 \lor 21)^* = (12 \lor 21)^*$, and that if $15^* = 25^*$ then $(15 \land 24)^* = (25 \land 24)^*$. The rules are:

$$\dfrac{P^* = Q^*}{(P \lor R)^* = (Q \lor R)^*}\ \mathsf{NoYcuts} \quad := \quad \dfrac{\dfrac{\dfrac{P^* = Q^*}{P \lor R^* = Q \lor R^*}}{(P \lor R^*)^* = (Q \lor R^*)^*}}{(P \lor R)^* = (Q \lor R)^*}\ \lor^*\mathsf{Cube}$$

$$\dfrac{P^* = Q^*}{(P\&R)^* = (Q\&R)^*}\ \mathsf{No\lambda cuts} \quad := \quad \dfrac{\dfrac{P^* = Q^*}{P^*\&R^* = Q^*\&R^*}}{(P\&R)^* = (Q\&R)^*}\ \mathsf{J3}$$

The top derivation mentions a rule called '$\lor^*\mathsf{Cube}$', which will be proved in the next section.

28 How J-operators interact with connectives

Let's start by proving another three derived rules:

$$\dfrac{}{(P^*\&Q^*)^* = P^*\&Q^* = (P\&Q)^*}\ \&^*\mathsf{C}_0 \quad := \quad \dfrac{\dfrac{\dfrac{}{P^{**} = P^*}\ \mathsf{J2} \quad \dfrac{}{Q^{**} = Q^*}\ \mathsf{J2}}{(P^*\&Q^*)^* = P^{**}\&Q^{**} = P^*\&Q^* = (P\&Q)^*}\ \mathsf{J3}}{(P^*\&Q^*)^* = P^*\&Q^* = (P\&Q)^*}$$

$$\dfrac{}{(P^* \lor Q^*)^* \leq (P \lor Q)^*}\ \lor^*\mathsf{C}_0 \quad := \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \leq P \lor Q}}{P^* \leq (P \lor Q)^*}\ \mathsf{Mo} \quad \dfrac{\dfrac{}{Q \leq P \lor Q}}{Q^* \leq (P \lor Q)^*}\ \mathsf{Mo}}{P^* \lor Q^* \leq (P \lor Q)^*}}{(P^* \lor Q^*)^* \leq (P \lor Q)^{**}}\ \mathsf{Mo}}{(P^* \lor Q^*)^* \leq (P \lor Q)^*}\ \mathsf{J2}$$

$$\dfrac{}{(P \to Q^*)^* \leq P^* \to Q^*}\ \to^*\mathsf{C}_0 \quad := \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \to Q^* \leq P \to Q^*}}{(P \to Q^*)\&P \leq Q^*}}{((P \to Q^*)\&P)^* \leq Q^{**}}\ \mathsf{Mo}}{((P \to Q^*)\&P)^* \leq Q^*}\ \mathsf{J2}}{(P \to Q^*)^*\&P^* \leq Q^*}\ \mathsf{J3}}{(P \to Q^*)^* \leq P^* \to Q^*}$$

It is easy to prove each one of the arrows below ($A \longrightarrow B$ means $A \leq B$):

$$\begin{tikzcd}[column sep=tiny]
& (P^*\&Q^*)^* & \\
(P^*\&Q)^* \arrow[ur] & P^*\&Q^* \arrow[u] & (P\&Q^*)^* \arrow[ul] \\
P^*\&Q \arrow[u] \arrow[ur] & (P\&Q)^* \arrow[ul] \arrow[ur] & P\&Q^* \arrow[ul] \arrow[u] \\
& P\&Q \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& (P^*\vee Q^*)^* & \\
(P^*\vee Q)^* \arrow[ur] & P^*\vee Q^* \arrow[u] & (P\vee Q^*)^* \arrow[ul] \\
P^*\vee Q \arrow[u] \arrow[ur] & (P\vee Q)^* \arrow[ul] \arrow[ur] & P\vee Q^* \arrow[ul] \arrow[u] \\
& P\vee Q \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& (P^*\to Q^*)^* \arrow[dr] & \\
(P^*\to Q)^* \arrow[ur] \arrow[dr] & P^*\to Q^* \arrow[u] \arrow[dr] & (P\to Q^*)^* \\
P^*\to Q \arrow[u] \arrow[ur] \arrow[dr] & (P\to Q)^* \arrow[ur] & P\to Q^* \arrow[u] \\
& P\to Q \arrow[u] \arrow[ur] &
\end{tikzcd}$$

The cubes above will be called the “obvious and-cube”, the “obvious or-cube”, and the “obvious implication-cube”, and they show partial orders between expressions of the form $(P^? \odot Q^?)^?$, where the ‘$\odot$’ stands for one of the connectives ‘$\wedge$’, ‘$\vee$’ or ‘$\to$’, and each ‘?’ marks a place where we can put either a ‘*’ or nothing.

The rules $\&^*\mathsf{C}_0$, $\vee^*\mathsf{C}_0$ and $\to^*\mathsf{C}_0$ that we proved in the beginning of the section can be used to add more information to the partial orders given by the three “obvious” cubes above; adding them yields the cubes below, that will be called the “full and-cube”, the “full or-cube”, and the “full implication-cube”.

$$\begin{tikzcd}[column sep=tiny]
& (P^*\&Q^*)^* & \\
(P^*\&Q)^* \arrow[ur, equal] \arrow[dr, equal] & P^*\&Q^* \arrow[u, equal] & (P\&Q^*)^* \arrow[ul, equal] \\
P^*\&Q \arrow[u] \arrow[ur] & (P\&Q)^* \arrow[ur, equal] & P\&Q^* \arrow[ul] \arrow[u] \\
& P\&Q \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& (P^*\vee Q^*)^* & \\
(P^*\vee Q)^* \arrow[ur, equal] \arrow[dr, equal] & P^*\vee Q^* \arrow[u] & (P\vee Q^*)^* \arrow[ul, equal] \\
P^*\vee Q \arrow[u] \arrow[ur] & (P\vee Q)^* \arrow[ur, equal] & P\vee Q^* \arrow[ul] \arrow[u] \\
& P\vee Q \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& (P^*\to Q^*)^* \arrow[dr, equal] & \\
(P^*\to Q)^* \arrow[ur] \arrow[dr] & P^*\to Q^* \arrow[u, equal] \arrow[dr, equal] & (P\to Q^*)^* \\
P^*\to Q \arrow[u] \arrow[ur] \arrow[dr] & (P\to Q)^* \arrow[ur] & P\to Q^* \arrow[u, equal] \\
& P\to Q \arrow[u] \arrow[ur] &
\end{tikzcd}$$

We say that $\mathrm{expr}_1 \leq \mathrm{expr}_2$ is true “by the full and-cube” when $\mathrm{expr}_1 \leq \mathrm{expr}_2$ can be read from the (non-strict!) partial order in the the full and-cube; for example, $P \wedge Q^* \leq (P^* \wedge Q)^*$ is true “by the full and-cube”, and similary $P^* \vee Q^* \leq (P \vee Q)^*$ is true by the full or-cube and $(P \to Q)^* \leq P \to Q^*$ is true by the full implication-cube.

We write

$$\frac{}{\mathrm{expr}_1 \leq \mathrm{expr}_2}\ \&^*\mathsf{Cube} \qquad \frac{}{\mathrm{expr}_1 \leq \mathrm{expr}_2}\ \vee^*\mathsf{Cube} \qquad \frac{}{\mathrm{expr}_1 \leq \mathrm{expr}_2}\ \to^*\mathsf{Cube}$$

to indicate that the expression below the bar is a consequence (a substitution instance) of the partial order in the full and-cubes, the full or-cube, or the full implication-cube.

The six cubes will be discussed in more detail in the section 31.

29 J-cubes as partial orders

If we number the vertices of the cubes of sec.28 like ths,

$$\begin{array}{ccc} & 7 & \\ 5 & 3 & 6 \\ 1 & 4 & 2 \\ & 0 & \end{array}$$

then we can refer to their nodes as $(\land)_0, \ldots, (\land)_7$, $(\lor)_0, \ldots, (\lor)_7$, $(\to)_0, \ldots, (\to)_7$; note that

$$\begin{array}{rcl@{\qquad}rcl}
(\land)_0 & = & P \land Q, & (\land)_4 & = & (P \land Q)^*, \\
(\land)_1 & = & P^* \land Q, & (\land)_{1+4} & = & (P^* \land Q)^*, \\
(\land)_2 & = & P \land Q^*, & (\land)_{2+4} & = & (P \land Q^*)^*, \\
(\land)_{1+2} & = & P^* \land Q^*, & (\land)_{1+2+4} & = & (P^* \land Q^*)^*,
\end{array}$$

and the same for $(\lor)_k$ and $(\to)_k$.

With this convention we can interpret s set of arrows in a cube as a subset of $\{0, \ldots, 7\}^2$, and use the positional notation for subsets from sec.2 to represent that as a grid of '0's and '1's:

$$\begin{tikzcd}
& 7 & \\
5 \arrow[ur] & 3 \arrow[u] & 6 \arrow[ul] \\
1 \arrow[u] \arrow[ur] & 4 \arrow[ul] \arrow[ur] & 2 \arrow[ul] \arrow[u] \\
& 0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
= \left\{ \begin{array}{c} (0,1), (2,3), (4,5), (6,7), \\ (0,2), (1,3), (4,6), (5,7), \\ (0,4), (1,5), (2,6), (3,7) \end{array} \right\}
= \begin{array}{c}
0\,0\,0\,1\,0\,1\,1\,0 \\
0\,0\,1\,0\,1\,0\,0\,0 \\
0\,1\,0\,0\,1\,0\,0\,0 \\
1\,0\,0\,0\,0\,0\,0\,0 \\
0\,1\,1\,0\,0\,0\,0\,0 \\
1\,0\,0\,0\,0\,0\,0\,0 \\
1\,0\,0\,0\,0\,0\,0\,0 \\
0\,0\,0\,0\,0\,0\,0\,0
\end{array}$$

This gives us a way to represent explictly the transitive-reflexive closure of a set of arrows:

$$\left(\begin{tikzcd}
& 7 & \\
5 \arrow[ur] & 3 \arrow[u] & 6 \arrow[ul] \\
1 \arrow[u] \arrow[ur] & 4 \arrow[ul] \arrow[ur] & 2 \arrow[ul] \arrow[u] \\
& 0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd} \right)^*
= \begin{array}{c}
1\,1\,1\,1\,1\,1\,1\,1 \\
1\,0\,1\,0\,1\,0\,1\,0 \\
1\,0\,0\,0\,1\,1\,0\,0 \\
1\,1\,0\,0\,1\,0\,0\,0 \\
1\,1\,1\,1\,0\,0\,0\,0 \\
1\,0\,1\,0\,0\,0\,0\,0 \\
1\,1\,0\,0\,0\,0\,0\,0 \\
1\,0\,0\,0\,0\,0\,0\,0
\end{array}$$

The derived rule $\&^*\mathsf{C}_0$ from sec.28 proves

$$(P^* \land Q^*)^* = P^* \land Q^* = (P \land Q)^*,$$

that corresponds to arrows $7 \rightleftarrows 3 \rightleftarrows 4$; if we add these arrows to the cube above we get this,

$$\begin{tikzcd}[row sep=small, column sep=small]
& 7 \arrow[d, equal] & \\
5 \arrow[ur] & 3 & 6 \arrow[ul] \\
1 \arrow[u] \arrow[ur] & 4 \arrow[u, equal] \arrow[ul] \arrow[ur] & 2 \arrow[u] \arrow[ul] \\
& 0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd} \quad ;$$

We have

$$\begin{tikzcd}[row sep=small, column sep=small]
& 7 \arrow[d, equal] & \\
5 \arrow[ur] & 3 & 6 \arrow[ul] \\
1 \arrow[u] \arrow[ur] & 4 \arrow[u, equal] \arrow[ul] \arrow[ur] & 2 \arrow[u] \arrow[ul] \\
& 0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
\quad \neq \quad
\begin{tikzcd}[row sep=small, column sep=small]
& 7 \arrow[d, equal] \arrow[dl, equal] \arrow[dr, equal] & \\
5 \arrow[dr, equal] & 3 & 6 \arrow[dl, equal] \\
1 \arrow[u] \arrow[ur] & 4 & 2 \arrow[u] \arrow[ul] \\
& 0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

but:

$$\left(\begin{tikzcd}[row sep=small, column sep=small]
& 7 \arrow[d, equal] & \\
5 \arrow[ur] & 3 & 6 \arrow[ul] \\
1 \arrow[u] \arrow[ur] & 4 \arrow[u, equal] \arrow[ul] \arrow[ur] & 2 \arrow[u] \arrow[ul] \\
& 0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}\right)^*
=
\left(\begin{tikzcd}[row sep=small, column sep=small]
& 7 \arrow[d, equal] \arrow[dl, equal] \arrow[dr, equal] & \\
5 \arrow[dr, equal] & 3 & 6 \arrow[dl, equal] \\
1 \arrow[u] \arrow[ur] & 4 & 2 \arrow[u] \arrow[ul] \\
& 0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}\right)^*
=
\begin{matrix}
1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1\\
1&0&1&0&0&0&0&0\\
1&1&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0
\end{matrix}$$

Let's give a name to this (non-strict) partial order: "$\&^*\mathsf{Cube_n}$", the "numerical version" of the full and-cube. Now we can see more clearly the extent of the rule $\&^*\mathsf{Cube}$ defined in the end of sec.28: we have

$$\frac{}{(\wedge)_i \leq (\wedge)_j} \; \&^*\mathsf{Cube}$$

whenever $(i, j) \in \&^*\mathsf{Cube_n}$.

We have something similar for the or-cube and the implication-cube:

$$\vee^*\mathsf{Cube_n} \quad = \quad
\left(\begin{tikzcd}[row sep=small, column sep=small]
& 7 \arrow[dl, equal] \arrow[dr, equal] & \\
5 \arrow[dr, equal] & 3 \arrow[u] & 6 \arrow[dl, equal] \\
1 \arrow[u] \arrow[ur] & 4 & 2 \arrow[u] \arrow[ul] \\
& 0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}\right)^*
=
\begin{matrix}
1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1\\
1&1&1&1&0&0&0&0\\
1&0&1&0&0&0&0&0\\
1&1&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0
\end{matrix}$$

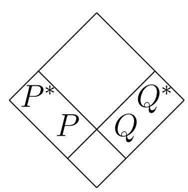
$$
\to^*\mathsf{Cube_n} \quad = \quad \left(\begin{tikzcd}
 & 7 & \\
5 \arrow[ur, equal] \arrow[d, equal] & 3 \arrow[u, equal] \arrow[dl, equal] & 6 \arrow[ul] \arrow[dl] \\
1 & 4 \arrow[ul] & 2 \arrow[u] \arrow[ul] \arrow[dl] \\
 & 0 \arrow[ul] \arrow[u] &
\end{tikzcd}\right)^* \quad = \quad
\begin{array}{cccccccc}
1&1&1&1&1&1&1&1\\
0&0&1&0&0&0&1&0\\
1&1&1&1&1&1&1&1\\
0&0&1&0&1&0&1&0\\
1&1&1&1&1&1&1&1\\
0&0&1&0&0&0&0&0\\
1&1&1&1&1&1&1&1\\
1&0&1&0&0&0&0&0
\end{array}
$$

Note that the arrows $2 \to 0$ and $6 \to 4$ in the version for the implication-cube above are not mistakes — they correspond to $P^*{\to}Q \le P{\to}Q$ and $(P^*{\to}Q)^* \le (P{\to}Q)^*$.

30 Valuations induce partial orders

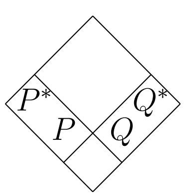
Let H be a ZHA, J be a J-operator on H, and v be a "valuation" that assigns to the variables P and Q values in H; v is a function from $\{P, Q\}$ to H, where P and Q are seen as names. Once we have (H, J, v) we have a natural way to extend v to make it assign values in H for P^*, Q^*, and for the expressions in the nodes of the and-cube, the or-cube and the implication-cube.

We will represent a triple (H, J, v) graphically by something like this,



that shows the ZHA H, the slashing on H corresponding to J, and *at least* $v(P)$ and $v(Q)$; sometimes the diagram will show also $v(P^*)$ and $v(Q^*)$, for convenience. With this information is it easy to calculate $v(\text{expr})$ for all 'expr's of the form $(P^? \odot Q^?)^?$, i.e., all the expressions in the nodes of the and-cube, the or-cube and the implication-cube.

Let's restrict our attention to '$\lor$' at this moment. We have:



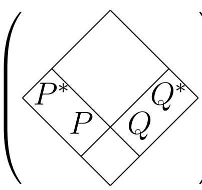
$$
\begin{array}{rcl}
v(P) &=& 10 \\
v(Q) &=& 01 \\
v(P^*) &=& 20 \\
v(Q^*) &=& 02
\end{array}
\qquad
\begin{array}{rcccl}
v(P \lor Q) &=& 11 &=& v((\lor)_0) \\
v(P^* \lor Q) &=& 21 &=& v((\lor)_1) \\
v(P \lor Q^*) &=& 12 &=& v((\lor)_2) \\
v(P^* \lor Q^*) &=& 22 &=& v((\lor)_3) \\
v((P \lor Q)^*) &=& 22 &=& v((\lor)_4) \\
v((P^* \lor Q)^*) &=& 22 &=& v((\lor)_5) \\
v((P \lor Q^*)^*) &=& 22 &=& v((\lor)_6) \\
v((P^* \lor Q^*)^*) &=& 22 &=& v((\lor)_7)
\end{array}
$$

This induces a partial order $\vee^*\mathsf{Cube_v}(v) \subseteq \{0,\dots,7\}^2$ in the following way: $i \leq_v j$ iff $v((\vee)_i) \leq_H v((\vee)_j)$. One easy way to calculate this '$\leq_v$' is to replace each number from 0 to 7 in the cube by $v((\vee)_i)$, and then draw arrows on that to represent the partial order in H, and then bring these arrows "back":

$$\begin{array}{ccc} & 7 & \\ 5 & 3 & 6 \\ 1 & 4 & 2 \\ & 0 & \end{array} \rightsquigarrow \begin{array}{ccc} & 22 & \\ 22 & 22 & 22 \\ 21 & 22 & 12 \\ & 11 & \end{array} \rightsquigarrow \left(\begin{tikzcd} & 22 & \\ 22 \arrow[ur, equal] & 22 \arrow[u, equal] & 22 \arrow[ul, equal] \\ 21 \arrow[u] \arrow[ur] & 22 \arrow[ul, equal] \arrow[ur, equal] & 12 \arrow[u] \arrow[ul] \\ & 11 \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}\right)^* \rightsquigarrow \left(\begin{tikzcd} & 7 & \\ 5 \arrow[ur, equal] & 3 \arrow[u, equal] & 6 \arrow[ul, equal] \\ 1 \arrow[u] \arrow[ur] & 4 \arrow[ul, equal] \arrow[ur, equal] & 2 \arrow[u] \arrow[ul] \\ & 0 \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}\right)^*$$

We can do this more compactly, as:

$\vee^*\mathsf{Cube_v}$ (



)

$$= \left(\begin{tikzcd} & 7 & \\ 5 \arrow[ur, equal] & 3 \arrow[u, equal] & 6 \arrow[ul, equal] \\ 1 \arrow[u] \arrow[ur] & 4 \arrow[ul, equal] \arrow[ur, equal] & 2 \arrow[u] \arrow[ul] \\ & 0 \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}\right)^*$$

$$= \left(\begin{tikzcd} & (P^*\vee Q^*)^* & \\ (P\vee Q^*)^* \arrow[ur, equal] & P^*\vee Q^* \arrow[u, equal] & (P^*\vee Q)^* \arrow[ul, equal] \\ P\vee Q^* \arrow[u] \arrow[ur] & (P\vee Q)^* \arrow[ul, equal] \arrow[ur, equal] & P^*\vee Q \arrow[u] \arrow[ul] \\ & P\vee Q \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}\right)^*$$

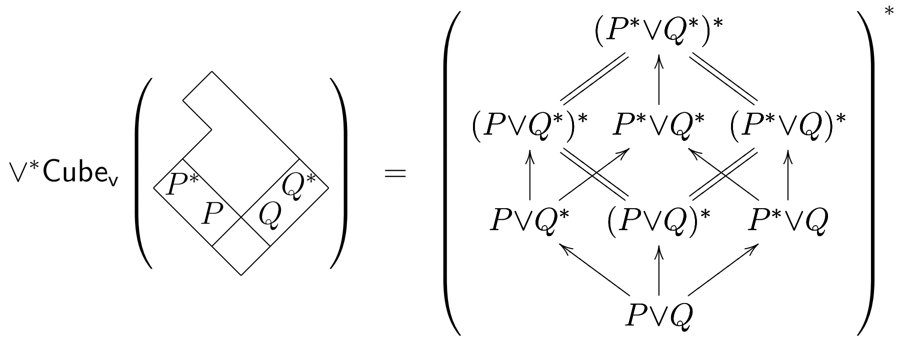
which shows that *in this valuation* we have, for example, $v((\vee)_3) = v((\vee)_7)$, i.e., $P^*\vee Q^* = (P^*\vee Q^*)^*$. The important information that a valuation gives, though, is in its '$\not\leq$'s. For example, here we have

$$\begin{array}{rcl} v((\vee)_1) & < & v((\vee)_5) \\ v((\vee)_5) & > & v((\vee)_1) \\ v((\vee)_5) & \not\leq & v((\vee)_1) \end{array} \qquad \begin{array}{rcl} P\vee Q^* & < & (P\vee Q^*)^* \\ (P\vee Q^*)^* & > & P\vee Q^* \\ (P\vee Q^*)^* & \not\leq & P\vee Q^* \end{array}$$

If it were possible to prove — as in sec.28 — that $(P\vee Q^*)^* \leq P\vee Q^*$, then that would be true in all valuations; by showing a valuation where $(P\vee Q^*)^* \not\leq P\vee Q^*$ we show that

$(P{\vee}Q^*)^* \le P{\vee}Q^*$ cannot be a theorem, and that all attempts to find a tree-like proof for $(P{\vee}Q^*)^* \le P{\vee}Q^*$ are doomed to fail.

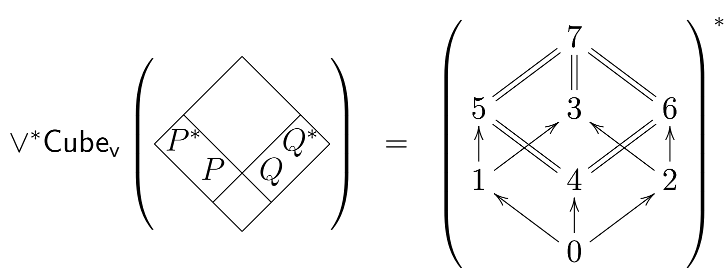
Note that



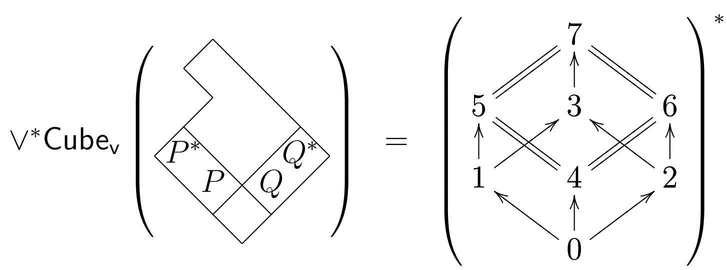
This new valuation tells us something that the previous one didn't: that $P^*{\vee}Q^* < (P^*{\vee}Q^*)^*$ in some valuation, and so $(P^*{\vee}Q^*)^* \le P^*{\vee}Q^*$ cannot be a theorem.

31 Comparing partial orders

If we represent the partial orders of the examples of the last section as subsets of $\{0,\dots,7\}^2$ we get:



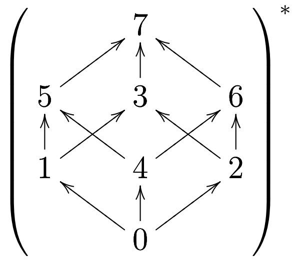
$$= \begin{matrix} 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&0&1&0&0&0&0&0\\ 1&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{matrix}$$



$$= \begin{matrix} 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 1&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{matrix}$$

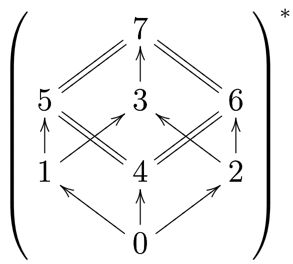
If we represent the transitive-reflexive closures of the obvious or-cube and the full or-cube of sec.29 as subsets of $\{0, \ldots, 7\}^2$, we get:

$$\left(\begin{matrix} \text{obvious} \\ \text{or-cube} \end{matrix} \right)^* =$$



$$= \begin{matrix} 1&1&1&1&1&1&1&1 \\ 1&0&1&0&1&0&1&0 \\ 1&1&0&0&1&1&0&0 \\ 1&0&0&0&1&0&0&0 \\ 1&1&1&1&0&0&0&0 \\ 1&0&1&0&0&0&0&0 \\ 1&1&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \end{matrix}$$

$$\left(\begin{matrix} \text{full} \\ \text{or-cube} \end{matrix} \right)^* =$$

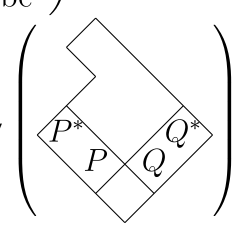


$$= \begin{matrix} 1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1 \\ 1&1&1&1&0&0&0&0 \\ 1&0&1&0&0&0&0&0 \\ 1&1&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \end{matrix}$$

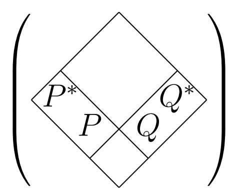
If we compare these four partial orders we get:

$$\left(\begin{matrix} \text{obvious} \\ \text{or-cube} \end{matrix} \right)^* \subsetneq \left(\begin{matrix} \text{full} \\ \text{or-cube} \end{matrix} \right)^*$$

$$= \lor^* \mathsf{Cube_v}$$



$$\subsetneq \lor^* \mathsf{Cube_v}$$



Note that each '1' in the grid of the obvious or-cube tells us something that we know how to prove; the same for the full or-cube, and the full or-cube has more '1's in its grid, so it has "more information" — about the existence of tree-like theorems — than the obvious or-cube. For example, the obvious or-cube tells us that we know how prove $(P \lor Q)^* \leq (P^* \lor Q^*)^*$, and the full or-cube tells us that we know how to prove $(P \lor Q)^* = (P^* \lor Q^*)^*$.

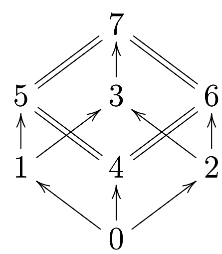
Each '0' in the grid of a valuation-cube tells us about something that cannot be be proved as a theorem, because that valuation is a "countermodel" for it. The first valuation in the beginning of this section is on a ZHA with 9 elements, and the second one is on a ZHA with 10 elements; let's refer to them as (H_9, J_9, v_9) and (H_{10}, J_{10}, v_{10}), or just as v_9 and v_{10}. Note that the grid for v_{10} has more '0's; and $\lor^* \mathsf{Cube_v}(v_{10}) \subsetneq$

$\vee^*\mathsf{Cube}_\mathsf{v}(v_9)$; for example, we have $(7,3) \in \vee^*\mathsf{Cube}_\mathsf{v}(v_9)$ but

$$\begin{aligned}
(7,3) \notin \vee^*\mathsf{Cube}_\mathsf{v}(v_{10}) \quad &\Rightarrow \quad v_{10}(v((\vee)_7)) \not\leq_{H_{10}} v_{10}(v((\vee)_3)) \\
&\Rightarrow \quad v_{10}((P^*\vee Q^*)^*) \not\leq_{H_{10}} v_{10}(P^*\vee Q^*) \\
&\Rightarrow \quad v_{10} \text{ is a countermodel for } (P^*\vee Q^*)^* \leq P^*\vee Q^* \\
&\Rightarrow \quad v_{10} \text{ shows that } (P^*\vee Q^*)^* \leq P^*\vee Q^* \\
& \quad \text{cannot be a theorem,}
\end{aligned}$$

so v_{10} has "more information" — now about the *non-existence* of tree-like theorems — than v_9.

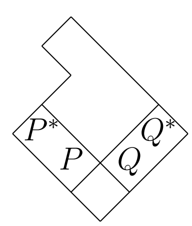
The full or-cube is "better" than the obvious or-cube, and the v_{10}-cube is "better" than the v_9-cube. Moreover, the full or-cube and the v_{10}-cube coincide, and this means that the status of every statement of the form $v((\vee)_i) \leq v((\vee)_j)$ can be determined in the following way: if $v((\vee)_i) \leq v((\vee)_j)$ is true in this partial order



then $v((\vee)_i) \leq v((\vee)_j)$ is a consequence of the obvious or-cube plus $\vee^*\mathsf{C}_0$ (sec.28); if $v((\vee)_i) \leq v((\vee)_j)$ is not true in the partial order, then it cannot be proved as a theorem, and the valuation v_{10} is a countermodel for it.

We can do even better, and extract all information from well-chosen valuations.

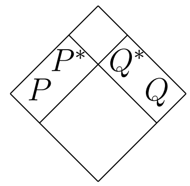
Theorem 31.1 *Take any statement of the form $v((\vee)_i) \leq v((\vee)_j)$. If it is true in the valuation below,*

$v_{(\vee)} \quad = \quad v_{10} \quad =$ 

then it is a theorem and can be proved using the obvious or-cube and $\vee^*\mathsf{C}_0$; if the statement is false in the valuation $v_{(\vee)}$, then it cannot be a theorem and $v_{(\vee)}$ is a countermodel that shows that.

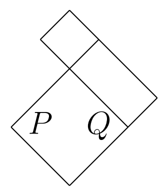
We also have:

Theorem 31.2 *Take any statement of the form* $(P^?{\wedge}Q^?)^? \leq (P^?{\wedge}Q^?)^?$. *If it is true in the valuation below,*

$v_{(\wedge)} \quad =$ 

then it is a theorem and can be proved using the obvious and-cube and $\&^*\mathsf{C}_0$; *if the statement is false in the valuation* $v_{(\wedge)}$, *then it cannot be a theorem and* $v_{(\wedge)}$ *is a countermodel that shows that.*

Theorem 31.3 *Take any statement of the form* $(P^?{\to}Q^?)^? \leq (P^?{\to}Q^?)^?$. *If it is true in the valuation below,*

$v_{(\to)} \quad =$ 

then it is a theorem and can be proved using the obvious implication-cube and $\to^*\mathsf{C}_0$; *if the statement is false in the valuation* $v_{(\to)}$, *then it cannot be a theorem and* $v_{(\to)}$ *is a countermodel that shows that.*

32 Fragments of Lindenbaum Algebras

33 Polynomial J-operators

It is not hard to check that for any Heyting Algebra H and any $Q, R \in H$ the operations $(\neg\neg)$, …, $(\vee Q \wedge {\to} R)$ below are J-operators:

$$\begin{array}{rcl}
(\neg\neg)(P) &=& \neg\neg P \\
(\to\to R)(P) &=& (P{\to}R){\to}R \\
(\vee Q)(P) &=& P \vee Q \\
(\to R)(P) &=& P{\to}R \\
(\vee Q \wedge {\to}R)(P) &=& (P{\vee}Q) \wedge (P{\to}R)
\end{array}$$

Checking that they are J-operators means checking that each of them obeys J1, J2, J3. Let's define formally what are J1, J2 and J3 "for a given $F : H \to H$":

$$\begin{array}{rcc}
\mathsf{J1}_F &:=& (P \leq F(P)) \\
\mathsf{J2}_F &:=& (F(P) = F(F(P)) \\
\mathsf{J3}_F &:=& (F(P \wedge P') = F(P) \wedge F(P'))
\end{array}$$

and:

$$\mathsf{J123}_F \quad := \quad \mathsf{J1}_F \land \mathsf{J2}_F \land \mathsf{J3}_F.$$

Checking that $(\neg\neg)$ obeys $\mathsf{J1}$, $\mathsf{J2}$, $\mathsf{J3}$ means proving $\mathsf{J123}_{(\neg\neg)}$ using only the rules from intuitionist logic from sec.11; we will leave the proof of this, of and $\mathsf{J123}_{(\to\to R)}$, $\mathsf{J123}_{(\lor Q)}$, and so on, to the reader.

The J-operator $(\lor Q \land {\to}R)$ is a particular case of building more complex J-operators from simpler ones. If $J, K : H \to H$, we define:

$$(J \land K) := \lambda P{:}H.(J(P) \land K(P))$$

it not hard to prove $\mathsf{J123}_{(J\land K)}$ from $\mathsf{J123}_J$ and $\mathsf{J123}_K$ using only the rules from intuitionistic logic.

The J-operators above are the first examples of J-operators in Fourman and Scott's "Sheaves and Logic" ([FS79]); they appear in pages 329–331, but with these names (our notation for them is at the right):

(i) *The closed quotient,*

$$J_a p = a \lor p \qquad J_Q = (\lor Q).$$

(ii) *The open quotient,*

$$J^a p = a \to p \qquad J^R = (\to R).$$

(iii) *The Boolean quotient.*

$$B_a p = (p \to a) \to a \qquad B_R = (\to\to R).$$

(iv) *The forcing quotient.*

$$(J_a \land J^b)p = (a \lor p) \land (b \to p) \qquad (J_Q \land J^R) = (\lor Q \land {\to}R).$$

(vi) *A mixed quotient.*

$$(B_a \land J^a)p = (p \to a) \to p \qquad (B_Q \land J^Q) = (\to\to Q \land {\to}Q).$$

The last one is tricky. From the definition of B_a and J^a what we have is

$$(B_a \land J^a)p = ((p \to a) \to a) \land (a \to p),$$

but it is possible to prove

$$((p \to a) \to a) \land (a \to p) \quad \leftrightarrow \quad ((p \to a) \to p)$$

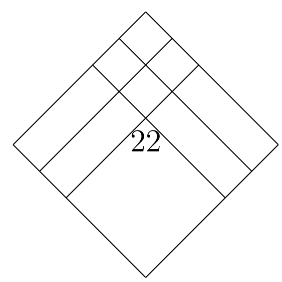
intuitionistically.

The operators above are "polynomials on $P, Q, R, \to, \land, \lor, \bot$" in the terminology of Fourman/Scott: "If we take a polynomial in $\to, \land, \lor, \bot$, say, $f(p, a, b, \ldots)$, it is a decidable question whether for all $a, b, \ldots$ it defines a J-operator" (p.331).

When I started studying sheaves I spent several years without any visual intuition about the J-operators above. I was saved by ZHAs and brute force — and the brute force method also helps in testing if a polynomial (in the sense above) is a J-operator in a particular case. For example, take the operators $\lambda P{:}H.(P \land 22)$ and $(\lor 22)$ on $H = [00, 44]$:

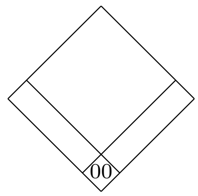
$$\lambda P{:}H.(P \land 22) \quad = \quad \begin{array}{c} 22 \\ 22 \;\; 22 \\ 22 \;\; 22 \;\; 22 \\ 21 \;\; 22 \;\; 22 \;\; 12 \\ 20 \;\; 21 \;\; 22 \;\; 12 \;\; 02 \\ 20 \;\; 21 \;\; 12 \;\; 02 \\ 20 \;\; 11 \;\; 02 \\ 10 \;\; 01 \\ 00 \end{array}$$

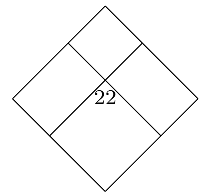
$$(\lor 22) \quad = \quad \begin{array}{c} 44 \\ 43 \;\; 34 \\ 42 \;\; 33 \;\; 24 \\ 42 \;\; 32 \;\; 23 \;\; 24 \\ 42 \;\; 32 \;\; 22 \;\; 23 \;\; 24 \\ 32 \;\; 22 \;\; 22 \;\; 23 \\ 22 \;\; 22 \;\; 22 \\ 22 \;\; 22 \\ 22 \end{array} \quad =$$

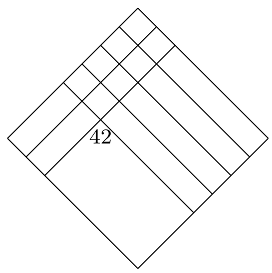


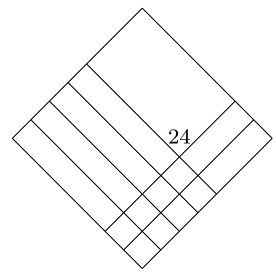
The first one, $\lambda P{:}H.(P \land 22)$, is not a J-operator; one easy way to see that is to look at the region in which the result is 22 — its top element is 44, and this violates the conditions on slash-operators in sec.24. The second operator, $(\lor 22)$, is a slash operator and a J-operator; at the right we introduce a convenient notation for visualizing the action of a polynomial slash-operator, in which we draw only the contours of the equivalence classes and the constants that appear in the polynomial.

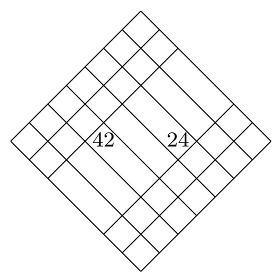
Using this new notation, we have:

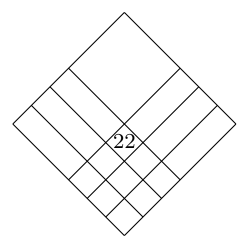
$(\neg\neg) \;=\; (\to\to 00) \;=$ 

$(\to\to 22) \;=$ 

$(\vee 42) \;=$ 

$(\to 24) \;=$ 

$(\vee 42 \wedge \to 24) \;=$ 

$(\to\to 22 \wedge \to 22) \;=$ 

Note that the slashing for $(\vee 42 \wedge \to 24)$ has all the cuts for $(\vee 42)$ plus all the cuts for $(\to 24)$, and $(\vee 42 \wedge \to 24)$ "forces $42 \leq 24$" in the following sense: if $P^* = (\vee 42 \wedge \to 24)(P)$ then $42^* \leq 24^*$.

34 An algebra of piccs

We saw in the last section a case in which $(J \wedge K)$ has all the cuts from J plus all the cuts from K; this suggests that we *may* have an operation dual to that, that behaves as this: $(J \vee K)$ has exactly the cuts that are both in J and in K:

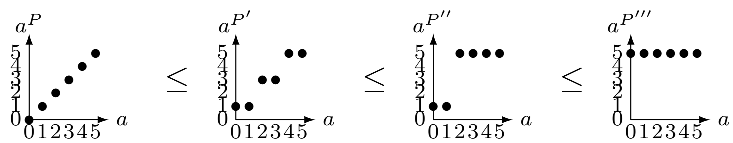
$$\begin{array}{rcl} \mathsf{Cuts}(J \wedge K) & = & \mathsf{Cuts}(J) \cup \mathsf{Cuts}(K) \\ \mathsf{Cuts}(J \vee K) & = & \mathsf{Cuts}(J) \cap \mathsf{Cuts}(K) \end{array}$$

And it $J_1, \ldots, J_n$ are all the slash-operators on a given ZHA, then

$$\begin{array}{rcl} \mathsf{Cuts}(J_1 \wedge \ldots \wedge J_n) & = & \mathsf{Cuts}(J_1) \cup \ldots \cup \mathsf{Cuts}(J_k) & = & \text{(all cuts)} \\ \mathsf{Cuts}(J_1 \vee \ldots \vee J_n) & = & \mathsf{Cuts}(J_1) \cap \ldots \cap \mathsf{Cuts}(J_k) & = & \text{(no cuts)} \end{array}$$

yield the minimal element and the maximal element, respectively, of an algebra of slash-operators; note that the slash-operator with "all cuts" is the identity map $\lambda P\colon H.P$, and the slash-operator with "no cuts" is the one that takes all elements to $\top$: $\lambda P\colon H.\top$. This yields a lattice of slash-operators, in which the partial order is $J \leq K$ iff $\mathsf{Cuts}(J) \supseteq \mathsf{Cuts}(K)$. This is somewhat counterintuitive if we think in terms of cuts — the order seems to be reversed — but it makes a lot of sense if we think in terms of piccs (sec.20) instead.

Each picc P on $\{0, \ldots, n\}$ has an associated function $\cdot^P$ that takes each element to the top element of its equivalence class. If we define $P \leq P'$ to mean $\forall a \in \{0, \ldots, n\}.\, a^P \leq a^{P'}$, then we have this:



$$\begin{array}{ccccccc} 0|1|2|3|4|5 & \leq & 01|23|45 & \leq & 01|2345 & \leq & 012345 \\ P & \leq & P' & \leq & P'' & \leq & P''' \end{array}$$

This yields a partial order on piccs, whose bottom element is the identity function $0|1|2|\ldots|n$, and the top element is $012\ldots n$, that takes all elements to n.

The piccs on $\{0, \ldots, n\}$ form a Heyting Algebra, where $\top = 01\ldots n$, $\bot = 0|1|\ldots|n$, and '$\wedge$' and '$\vee$' are the operations that we have discussed above; it is possible to define a '$\to$' there, but this '$\to$' is not going to be useful for us and we are mentioning it just as a curiosity. We have, for example:

$$\begin{tikzcd} & 01234 & \\ & 01|234 \arrow[u] & \\ 0|1|234 \arrow[ur] & & 01|2|34 \arrow[ul] \\ & 0|1|2|34 \arrow[ul] \arrow[ur] & \\ & 0|1|2|3|4 \arrow[u] & \end{tikzcd} \qquad \begin{tikzcd} & \top & \\ & P \vee Q \arrow[u] & \\ P \arrow[ur] & & Q \arrow[ul] \\ & P \wedge Q \arrow[ul] \arrow[ur] & \\ & \bot \arrow[u] & \end{tikzcd}$$

35 An algebra of J-operators

Fourman and Scott define the operations $\wedge$ and $\vee$ on J-operators in pages 325 and 329 ([FS79]), and in page 331 they list ten properties of the algebra of J-operators:

$$
\begin{array}{rlcll}
\text{(i)} & J_a \vee J_b & = & J_{a\vee b} & (\vee 21) \vee (\vee 12) = (\vee 22) \\
\text{(ii)} & J^a \vee J^b & = & J^{a\wedge b} & (\to 32) \vee (\to 23) = (\to 22) \\
\text{(iii)} & J_a \wedge J_b & = & J_{a\wedge b} & (\vee 21) \wedge (\vee 12) = (\vee 11) \\
\text{(iv)} & J^a \wedge J^b & = & J^{a\vee b} & (\to 32) \wedge (\to 23) = (\to 33) \\
\text{(v)} & J_a \wedge J^a & = & \bot & (\vee 22) \wedge (\to 22) = (\bot) \\
\text{(vi)} & J_a \vee J^a & = & \top & (\vee 22) \vee (\to 22) = (\top) \\
\text{(vii)} & J_a \vee K & = & K \circ J_a & \\
\text{(viii)} & J^a \vee K & = & J^a \circ K & \\
\text{(ix)} & J_a \vee B_a & = & B_a & \\
\text{(x)} & J^a \vee B_b & = & B_{a\to b} &
\end{array}
$$

The first six are easy to visualize; we won't treat the four last ones. In the right column of the table above we've put a particular case of (i), ..., (vi) in our notation, and the figures below put all together.

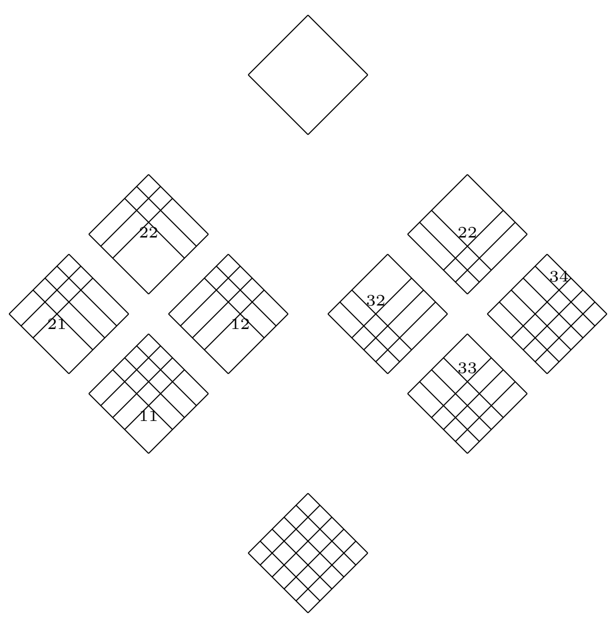
In Fourman and Scott's notation,

$$
\begin{tikzcd}
 & & & J_\top = \top = J^\bot & & & \\
 & J_{22} \arrow[urr] & & & & J^{22} \arrow[ull] & \\
J_{21} \arrow[ur] & & J_{12} \arrow[ul] & & J^{32} \arrow[ur] & & J^{23} \arrow[ul] \\
 & J_{11} \arrow[ul] \arrow[ur] & & & & J^{11} \arrow[ul] \arrow[ur] & \\
 & & & J_\bot = \bot = J^\top \arrow[ull] \arrow[urr] & & &
\end{tikzcd}
$$

in our notation,

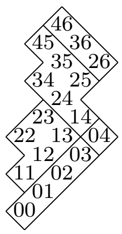
$$\begin{tikzcd}[column sep=tiny]
 & & & (\top\vee) = (\lambda P.\top) = (\bot\to) & & & \\
 & (22\vee) \arrow[urr] & & & & (22\to) \arrow[ull] & \\
(21\vee) \arrow[ur] & & (12\vee) \arrow[ul] & & (32\to) \arrow[ur] & & (23\to) \arrow[ul] \\
 & (11\vee) \arrow[ul] \arrow[ur] & & & & (33\to) \arrow[ul] \arrow[ur] & \\
 & & & (\bot\vee) = (\lambda P.P) = (\top\to) \arrow[ull] \arrow[urr] & & &
\end{tikzcd}$$

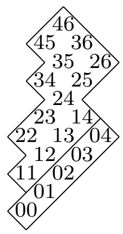
and drawing the polynomial J-operators as in sec.33:

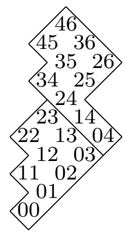


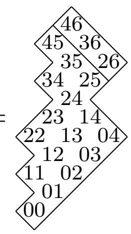
36 All slash-operators are polynomial

Here is an easy way to see that all slashings — i.e., J-operators on ZHAs — are polynomial. Every slashing J has only a finite number of cuts; call them $J_1, \ldots, J_n$. For example:

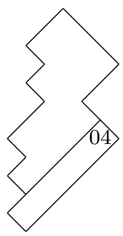
$J =$ 

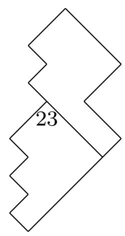
$J_1 =$ 

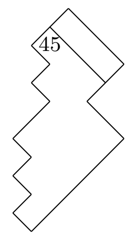
$J_2 =$ 

$J_3 =$ 

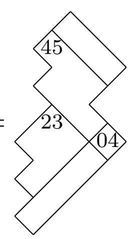
Each cut J_i divides the ZHA into an upper region and a lower region, and $J_i(00)$ yields the top element of the lower region. Also, $(\to\to J_i(00))$ is a polynomial way of expressing that cut:

$J_1 =$
$(\to\to 04) =$ 

$J_2 =$
$(\to\to 23) =$ 

$J_3 =$
$(\to\to 45) =$ 

The conjunction of these '$(\to\to J_i(00))$'s yields the original slashing:

$(\to\to 04) \land (\to\to 23) \land (\to\to 45) =$  $= J$

37 Open sets of a certain form

A 2-column graph with question marks (a "2CGQ") is a triple $((P,A),B,D)$, where (P,A) is a 2CG and $B \subseteq D \subseteq P$; let $G = ((P,A),B,D)$. We represent G graphically like (P,A), but with '0's, '?'s and '1's on the points of P, and the expression "C is of the form G" means $B \subseteq C \subseteq D$. For example:

$$\left(\begin{tikzcd}[row sep=small] & 0 \arrow[d] \\ 1 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 & 1 \end{tikzcd}\right) \quad \text{is of the form} \quad \left(\begin{tikzcd}[row sep=small] & 0 \arrow[d] \\ ? \arrow[r] \arrow[d] & ? \arrow[d] \\ 0 & 1 \end{tikzcd}\right)$$

Informally, a '0' in the graphical representation of a 2CGQ Q means "all 'C's of the form G have a '0' here", a '1' means "all 'C's of the form G have a '1' here", and a '?' means "some 'C's of the form G have '0's there and some have '1's". More formally, a 2CGQ G corresponds to a partition of P into P_0, P_1 and $P_?$ — the sets of '0's, '1's and '?'s of the graphical representation of G — and we have $P_1 = B$, $P_? = D \backslash B$, $P_0 = P \backslash D$, $D = P_1 \cup P_?$.

Our main use for 2CGQs will be for giving us a nice notation for "the set of open sets of (P,A) between B and D":

$$\mathsf{Opens}((P,A),B,D) \;=\; \{\, U \subseteq P \mid B \subseteq U \subseteq D \text{ and } U \in \mathcal{O}_A(P) \,\}$$

Note that adding intercolumn arrows reduce sets of opens sets,

$$\mathsf{Opens}\begin{pmatrix} ? & ? \\ 0 & 0 \\ ? & ? \\ ? & ? \\ 1 & 1 \\ ? & ? \end{pmatrix} \supseteq \mathsf{Opens}\left(\begin{tikzcd}[row sep=tiny, column sep=small] ? & ? \\ 0 & 0 \\ ? \arrow[dr] & ? \\ ? & ? \\ 1 & 1 \\ ? & ? \end{tikzcd}\right) \supseteq \mathsf{Opens}\left(\begin{tikzcd}[row sep=tiny, column sep=small] ? & ? \\ 0 & 0 \\ ? \arrow[dr] & ? \arrow[ul] \\ ? & ? \\ 1 & 1 \\ ? & ? \end{tikzcd}\right)$$

because each arrow is a "restriction" (sec.16) on what is considered an open set. We can propagate '1's forward along arrows like '$1 \to ?$' and '0's backward along arrows like '$? \to 0$' without changing the result of '$\mathsf{Opens}(\ldots)$':

$$\mathsf{Opens}\begin{pmatrix} ? & ? \\ 0 & 0 \\ ? & ? \\ ? & ? \\ 1 & 1 \\ ? & ? \end{pmatrix} = \mathsf{Opens}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ ? & ? \\ ? & ? \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \mathsf{Opens}\left(\begin{tikzcd}[row sep=tiny, column sep=small] & 0 \\ & ? \\ ? \arrow[ur] & 0 \arrow[dl] \\ ? \arrow[ur] & ? \\ ? \arrow[r] & ? \\ 1 \arrow[r] & ? \end{tikzcd}\right) = \mathsf{Opens}\left(\begin{tikzcd}[row sep=tiny, column sep=small] & 0 \\ & 0 \\ 0 \arrow[ur] & 0 \arrow[dl] \\ 0 \arrow[ur] & ? \\ ? \arrow[r] & ? \\ 1 \arrow[r] & 1 \end{tikzcd}\right)$$

38 Propagation

Fix a 2CG (P,A). There are two good, natural ways to get rid of all arrows '$1 \to 0$' in a subset $C \subseteq P$: one, called '$\mathsf{prp}_{1,(P,A)}$' or 'prp_1', "propagates the '1's forward", and the other one, called 'prp_0' or '$\mathsf{prp}_{1,(P,A)}$', "propagates the '0's backward". An example:

$$\mathsf{prp}_0 \left(\begin{array}{ccc} & & 0 \\ & 1 \to & 0 \\ & 0 & 1 \end{array} \right) = \left(\begin{array}{ccc} & & 0 \\ & 0 \to & 0 \\ & 0 & 1 \end{array} \right) \qquad \mathsf{prp}_1 \left(\begin{array}{ccc} & & 0 \\ & 1 \to & 0 \\ & 0 & 1 \end{array} \right) = \left(\begin{array}{ccc} & & 0 \\ & 1 \to & 1 \\ & 1 & 1 \end{array} \right)$$

It easy to see that $\mathsf{prp}_1(C)$ returns the smallest open set containing C, and $\mathsf{prp}_0(C)$ returns the largest open set contained in C,

The *interior* of a set S in a topology $\mathcal{U}$ on P is the biggest open set in $\mathcal{U}$ contained in S, and, dually, the *cointerior* of a set S is the smallest open set in $\mathcal{U}$ containing S. In finite topologies cointeriors always exist.

Theorem 38.1 *For any 2CG (P, A) and $S \subseteq P$ we have*

$$\mathsf{int}(S) = \mathsf{prp}_0(S) \subseteq S \subseteq \mathsf{prp}_1(S) = \mathsf{coint}(S).$$

We can define propagations for 2CGQs in a way that changes only the '?'s. If $G = ((P, A), B, D)$ is a 2CGQ, then $\mathsf{prp}_1(G)$ propagates forward only the '1's in arrows like '$1 \to ?$', and $\mathsf{prp}_0(G)$ propagates backward only the '0's in arrows like '$? \to 0$'.

The operations 'prp_1' and 'prp_0' on 2CGQs need not commute:

$$\mathsf{prp}_1 \left(\mathsf{prp}_0 \left(\begin{array}{cc} 1 & ? \\ ? \nearrow & 0 \end{array} \right) \right) \;=\; \left(\begin{array}{cc} 1 & 0 \\ 0 \nearrow & 0 \end{array} \right)$$

$$\mathsf{prp}_0 \left(\mathsf{prp}_1 \left(\begin{array}{cc} 1 & ? \\ ? \nearrow & 0 \end{array} \right) \right) \;=\; \left(\begin{array}{cc} 1 & 1 \\ 1 \nearrow & 0 \end{array} \right)$$

but they can only fail to commute when $\mathsf{Opens}(G) = \emptyset$. When they commute we will write their composite as 'prp'.

Theorem 38.2 *Let $G = ((P, A), B, D)$ be a 2CGQ with $\mathsf{Opens}(G) \neq \emptyset$ and let $G' = \mathsf{prp}(G) = \mathsf{Opens}((P, A), B', D')$, $P'_1 = B'$, $P'_? = D' \backslash B'$, $P'_1 = P \backslash D'$. Then:*

a) In G' everything below a '1' is also '1',
b) In G' everything above a '0' is also '0',
c) $B' = P'_1$ is an open set,
d) $D' = P'_1 \cup P'_? = P \backslash P'_0$ is an open set,
e) $B' = \mathsf{prp}_1(B) = \mathsf{coint}(B)$,
f) $D' = \mathsf{prp}_0(D) = \mathsf{int}(D)$,
g) $B' = \mathsf{pile}(ab)$ for some ab,
h) $D' = \mathsf{pile}(ef)$ for some ef,
i) $B' \in \mathsf{Opens}(G) = \mathsf{Opens}(G')$,
j) $D' \in \mathsf{Opens}(G) = \mathsf{Opens}(G')$.

An example:

$$G = \begin{pmatrix} & & 0 \\ & & ? \\ ? & \nearrow & 0 \\ ? & \swarrow & ? \\ ? & \to & ? \\ 1 & \to & ? \end{pmatrix} \qquad G' = \mathsf{prp}(G) = \begin{pmatrix} & & 0 \\ & & 0 \\ 0 & \nearrow & 0 \\ 0 & \swarrow & ? \\ ? & \to & ? \\ 1 & \to & 1 \end{pmatrix} = ((P,A), \mathsf{pile}(11), \mathsf{pile}(23))$$

The next theorem translates this to ZHAs, and shows that when $\mathsf{Opens}(G) \neq \emptyset$ then it returns an interval in a ZHA (in the sense of sec.20),

Theorem 38.3 *Let $G = ((P,A), B, D)$ be a 2CGQ with $\mathsf{Opens}(G) \neq \emptyset$ and let $G' = \mathsf{prp}(G) = \mathsf{Opens}((P,A), B', D')$, $ab = \mathsf{pile}^{-1}(B')$, $ef = \mathsf{pile}^{-1}(D')$, $I = \mathsf{pile}^{-1}(\mathsf{Opens}(G)) = \mathsf{pile}^{-1}(\mathsf{Opens}(G'))$, and let H be the ZHA generated by (P,A), i.e., $H = \mathsf{pile}^{-1}(\mathcal{O}_A(P))$. Then:*
a) ab is the minimal point of I,
b) ef is the maximal point of I,
c) $I \subseteq H$,
d) $I = [ab, ef] \cap H$,
e) if A has no intercolumn arrows then $I = [ab, ef]$.

With Theorem 38.3 we can extend the last example to:

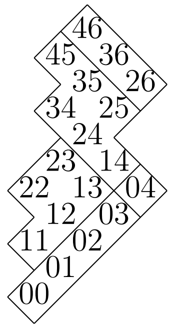
$$G = \begin{pmatrix} & & 0 \\ & & ? \\ ? & \nearrow & 0 \\ ? & \swarrow & ? \\ ? & \to & ? \\ 1 & \to & ? \end{pmatrix} \qquad G' = \mathsf{prp}(G) = \begin{pmatrix} & & 0 \\ & & 0 \\ 0 & \nearrow & 0 \\ 0 & \swarrow & ? \\ ? & \to & ? \\ 1 & \to & 1 \end{pmatrix} = ((P,A), \mathsf{pile}(11), \mathsf{pile}(23))$$

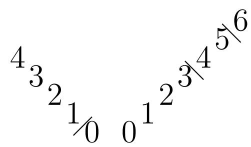
$$\mathsf{Opens}(G) = \mathsf{Opens}(G') = I \quad \underset{\mathsf{pile}^{-1}}{\overset{\mathsf{pile}}{\rightleftarrows}} \quad [11,23] \cap \begin{array}{ccccc} & & 46 & & \\ & 45 & & 36 & \\ & & 35 & & 26 \\ & 34 & & 25 & \\ & & 24 & & \\ & 23 & & 14 & \\ 22 & & 13 & & 04 \\ & 12 & & 03 & \\ 11 & & 02 & & \\ & 01 & & & \\ 00 & & & & \end{array}$$

In the next sections we will see that in some important cases the results of $\mathsf{Opens}(\ldots)$ coincide with J-equivalence classes.

39 The set of relevant points of a slashing

We saw in sec.20 that a slashing on a ZHA H can be represented a pair (L,R) of piccs, that we drew in a V-shaped diagram; let's write S for the set of numbers above the cuts in the V-shaped diagram, converting them to the notation for elements of the left and the right columns of 2-column graphs:

$J =$ 

$(L, R) =$ 

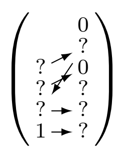
$S = \{1\_,\ \_4, \_6\}$

We also saw (sec.26) that on ZHAs there is a bijection between slashings and J-operators. Let $\mathsf{relev}(J)$ be the operation that obtains the set S for a J-operator J: $\mathsf{relev}(J) = \{1\_,\ \_4, \_6\}$ for the J above. We will call $S \subseteq P$ the *set of relevant points* of the J-operator J, and $Q = \mathsf{qmarks}(J) = P \backslash S$ will be the *set of (points that will be replaced by) question marks* by J. Note that we can also go from a set $Q \subseteq P$ to a slashing and a J-operator, but we will not need a notation for that.

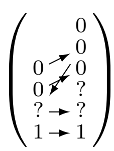
We can define the operation that receives a $C \subseteq P$ and "forgets the information on the points of Q" on C, returning a 2CGQ, as:

$$\mathsf{forget}_{(P,A),Q}(C) \quad = \quad ((P, A), C\backslash Q, C \cup Q)$$

for example:

$$\mathsf{forget}_{(P,A),Q}(\mathsf{pile}(12)) =$$ 

Note that

$$\mathsf{prp}(\mathsf{forget}_{(P,A),Q}(\mathsf{pile}(12))) \quad =$$ 

$$= \quad ((P, A), \mathsf{pile}(11), \mathsf{pile}(23))$$

and that:

$$\begin{array}{rcl} \mathsf{pile}^{-1}(\mathsf{Opens}(\mathsf{prp}(\mathsf{forget}_{(P,A),Q}(\mathsf{pile}(12))))) & = & [11, 23] \cap H \\ & = & [\mathrm{co}J(12), J(12)] \cap H \\ & = & [12]^J \end{array}$$

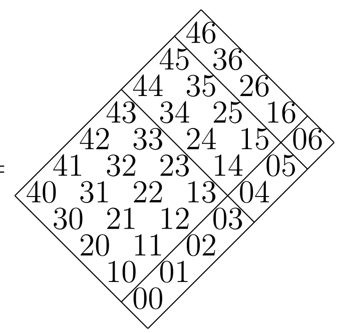
this holds in general, as we will see soon.

40 Rectangular versions

The "rectangular version" of a 2CG, a ZHA and a J-operator are defined as this. Let (P, A) be a 2CG and H its associated ZHA, and $J : H \to H$ a J-operator on H; then A' is A minus its intercolumn arrows, H' is the (rectangular) ZHA associated to (P, A'), and $J' : H' \to H'$ is J-operator on H' that has the same cuts as J. The primes on A', H' and J' will always mean from here on that we are on the rectangular versions. Let $Q = \mathsf{qmarks}(J) = \mathsf{qmarks}(J')$.

The rectangular versions for the (P, A) and the J that we are using in our examples are:

$$(P, A') = \begin{pmatrix} & \_6 \\ & \downarrow \\ & \_5 \\ & \downarrow \\ 4\_ & \_4 \\ \downarrow & \downarrow \\ 3\_ & \_3 \\ \downarrow & \downarrow \\ 2\_ & \_2 \\ \downarrow & \downarrow \\ 1\_ & \_1 \end{pmatrix} \qquad J' =$$



.

Take any $C \subseteq P$, The result of $\mathsf{forget}_{(P,A'),Q}(C)$ is always of this form,

$$\mathsf{forget}_{(P,A'),Q}(C) = \begin{pmatrix} & c \\ & ? \\ ? & b \\ ? & ? \\ ? & ? \\ a & ? \end{pmatrix}$$

for some $a, b, c \in \{0, 1\}$; moreover, if C is open then $\mathsf{forget}_{(P,A'),Q}(C)$ doesn't have '1's above '0's. Take any $C \subseteq P$ open in (P, A); C will be of the form $\mathsf{pile}(cd)$ for some $cd \in H'$. Let $G = \mathsf{forget}_{(P,A'),Q}(C)$. The action of prp on 'G's of this form is particularly simple: each column of G is made of blocks of consecutive '?'s separated by '0's or '1's, and prp acts homogeneously on each block, leaving '?'s in at most one block in each column. For example, if $a = b = 1$ and $c = 0$ then

$$\mathsf{prp}(\mathsf{forget}_{(P,A'),Q}(C)) = \begin{pmatrix} & 0 \\ & ? \\ ? & 1 \\ ? & 1 \\ ? & 1 \\ 1 & 1 \end{pmatrix}$$

It is easy to see that:

Theorem 40.1 *If* $C = \mathsf{pile}(cd)$ *then* $\mathsf{pile}^{-1}(\mathsf{Opens}(\mathsf{prp}(\mathsf{forget}_{(P,A'),Q}(C))))$ *is a* J'*-equivalence class.*

Theorem 40.2 *If* $C = \mathsf{pile}(cd)$ *then* $\mathsf{pile}^{-1}(\mathsf{Opens}(\mathsf{prp}(\mathsf{forget}_{(P,A'),Q}(C))))$ *is* $[\mathrm{co}J'(cd), J'(cd)]$.

Theorem 40.3 *Suppose that $cd \in H$ (instead of $cd \in H'$) and let:*

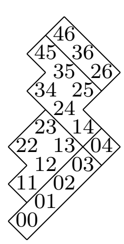
$$\begin{array}{rcl} C & = & \mathsf{pile}(cd) \\ G & = & \mathsf{forget}_{(P,A'),Q}(C) \\ G' & = & \mathsf{prp}(\mathsf{forget}_{(P,A'),Q}(C)) \\ G'' & = & \mathsf{prp}(\mathsf{forget}_{(P,A),Q}(C)) \\ I' & = & \mathsf{pile}^{-1}(\mathsf{Opens}(G')) \\ I'' & = & \mathsf{pile}^{-1}(\mathsf{Opens}(G'')) \end{array}$$

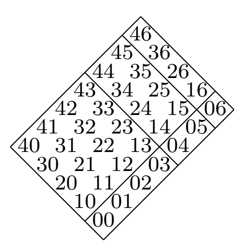
then G' is a "rectangular" (and "propagated") 2CGQ, and $I' = [\mathrm{co}J'(cd), J'(cd)]$ is a "rectangular interval"; G'' is G' plus the intercolumn arrows, and with the propagations having been done through the intercolumn arrows too. It is not hard to see that:
a) $\mathsf{Opens}(G) = \mathsf{Opens}(G') \supseteq \mathsf{Opens}(G'')$
b) $I'' = I' \cap H$
c) $cd \in I''$
d) $I'' = [\mathrm{co}J(cd), J(cd)] \cap H$
e) $\mathsf{pile}(\mathrm{co}J(cd)), \mathsf{pile}(J(cd)) \in I''$
f) $G'' = ((P,A), \mathsf{pile}(\mathrm{co}J(cd)), \mathsf{pile}(J(cd)))$
g) $G'' = ((P,A), \mathsf{coint}(C\backslash Q), \mathsf{int}(C \cup Q))$, *so:*
h) $\mathsf{pile}(\mathrm{co}J(cd)) = \mathsf{coint}(C\backslash Q) = \mathsf{prp}_1(C\backslash Q)$ *and*
i) $\mathsf{pile}(J(cd)) = \mathsf{int}(C \cup Q) = \mathsf{prp}_0(C \cup Q)$,
j) $\mathrm{co}J(cd)) = \mathsf{pile}^{-1}(\mathsf{coint}(C\backslash Q)) = \mathsf{pile}^{-1}(\mathsf{prp}_1(C\backslash Q))$ *and*
k) $J(cd) = \mathsf{pile}^{-1}(\mathsf{int}(C \cup Q)) = \mathsf{pile}^{-1}(\mathsf{prp}_0(C \cup Q))$.

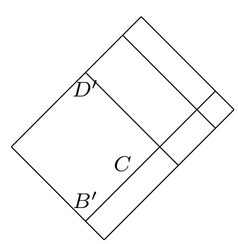
A way to visualize what Theorem 40.3 means is to define $B, B', B'', D, D'D''$ like this:

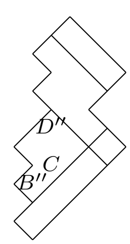
$$\begin{array}{rcl} (B,D) & = & (C\backslash Q, C \cup Q) \\ G' & = & ((P,A'), B', D') \\ G'' & = & ((P,A), B'', D'') \end{array}$$

then, in the example we are using, omitting some 'pile's and 'pile^{-1}'s, we have:

$J =$ 

$J' =$ 





Theorem 40.3 shows several ways to calculate B', C', B'', C''.

41 Sub-2-column graphs

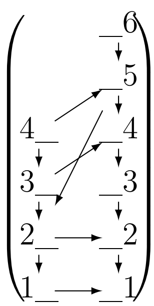
Another way to understand the properties of the operation $\mathsf{forget}_{(P,A),Q}$ is to think that it relates two topologies, $\mathcal{O}_A(P)$ and $\mathcal{O}_{A|_S}(S)$ (mnemonic: S is a "smaller set", and $S = \mathsf{relev}(J) = P \backslash Q$). We will sometimes denote $\mathcal{O}_A(P)$ and $\mathcal{O}_{A|_S}(S)$ as just $\mathcal{O}(P)$ and $\mathcal{O}(S)$; $\mathcal{O}(S)$ is a restriction of $\mathcal{O}(P)$ to S in the following sense: the open sets of $\mathcal{O}(S)$ are exactly the sets of the form $U \cap S$, where $U \in \mathcal{O}_A(P)$.

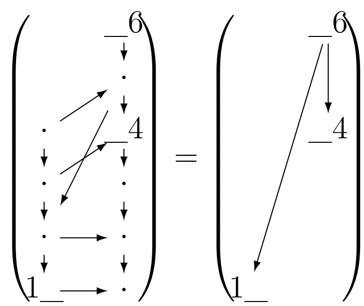
The topology $\mathcal{O}(S) = \mathcal{O}_{A|_S}(S)$ comes from a "sub-2-column graph" $(S, A|_S)$ of (P, A), where the set of arrows $A|_S$ can be obtained from A and S by

$$A|_S \quad := \quad (A^* \cap (S \times S))^{\mathsf{ess}},$$

which means: take the transitive-reflexive closure A^* of A, which yields a partial order on P, and restrict that order to S by taking $A^* \cap (S \times S)$; then (note: this last step is optional!) drop the redundant arrows in $A^* \cap (S \times S)$ and keep only the "essential" ones, which are the ones that can't be deleted without changing the order.

For clarity, we will draw the arrows in $(S, A|_S)$ as in the original 2CG (P, A), even though some arrows may look as coming from or going to nonexistent points; a really honest drawing of $(S, A|_S)$ in the example below would be the one at the right, that has only one intercolumn arrow, 1_ $\leftarrow$ _6, and only one vertical arrow, _6 $\to$ _4.

$(P, A) =$ 

$(S, A|_S) =$ 

A *sub-2-column graph* is a graph $(S, A|_S)$ generated by a 2CG (P, A) and an $S \subseteq P$ by the procedure above: $A|_S = (A^* \cap (S \times S))^{\mathsf{ess}}$.

Theorem 41.1 *Fix a ZHA H and a J-operator J on it, and from that produce (P, A), $\mathcal{U} = \mathcal{O}_A(P)$, S, and Q. We clearly have bijections between:*

1) the set of fixed points of J, $\{\, ef \in H \mid J(ef) = ef \,\}$

2) the set of fixed points of $\mathrm{co}J$, $\{\, ab \in H \mid \mathrm{co}J(ab) = ab \,\}$,

3) the image of J, $J(H) = \{\, J(cd) \mid cd \in H \,\}$,

4) the image of $\mathrm{co}J$, $\mathrm{co}J(H) = \{\, \mathrm{co}J(cd) \mid cd \in H \,\}$,

5) the set of J-equivalence classes of H, $H/J = \{\, [cd]^J \mid cd \in H \,\}$,

6) the elements $ef \in H$ such that $\mathsf{pile}(ef) = \mathsf{int}(\mathsf{pile}(ef) \cup Q)$,

7) the elements $ab \in H$ such that $\mathsf{pile}(ab) = \mathsf{coint}(\mathsf{pile}(ab) \backslash Q)$,

8) the sets $U \subseteq \mathcal{O}(P)$ such that $U = \mathsf{int}(U \cup Q)$,
9) the sets $W \subseteq \mathcal{O}(P)$ such that $W = \mathsf{coint}(W \backslash Q)$,
10) the sets $U \subseteq P$ such that $U = \mathsf{int}(U \cup Q)$,
11) the sets $W \subseteq P$ such that $W = \mathsf{coint}(W \backslash Q)$,
12) the opens sets in $\mathcal{O}(S)$.

The partial order on $\mathcal{O}(S)$ is given by inclusion; some of the corresponding partial orders on the other sets of Theorem 1 are not so obvious.

Theorem 41.2 *Let $ab, cd \in H$, $A = \mathsf{pile}(ab)$, $B = \mathsf{pile}(cd)$, $A' = A \cap S$, $B' = B \cap S$. The following are all equivalent:*
1) $A' \subseteq B'$,
2) $A \backslash Q \subseteq B \backslash Q$,
2') $A \cup Q \subseteq B \cup Q$,
3) $\mathsf{coint}(A \backslash Q) \subseteq \mathsf{coint}(B \backslash Q)$,
3') $\mathsf{int}(A \cup Q) \subseteq \mathsf{int}(B \cup Q)$,
4) $\mathsf{prp}_1(A \backslash Q) \subseteq \mathsf{prp}_1(B \backslash Q)$
4') $\mathsf{prp}_0(A \cup Q) \subseteq \mathsf{prp}_0(B \cup Q)$
5) $\mathrm{co}J(ab) \leq \mathrm{co}J(cd)$,
5') $J(ab) \leq J(cd)$,
6) $\inf([ab]^J) \leq \inf([cd]^J)$,
6') $\sup([ab]^J) \leq \sup([cd]^J)$.

Items 6 and 6' give us a way to endow H/J with a partial order. Remember that $\sup([ab]^J) = J(ab)$ and $\inf([ab]^J) = \mathrm{co}J(ab)$; we say that $[ab]^J \leq [cd]^J$ when $J(ab) \leq J(cd)$, or, equivalently, $\mathrm{co}J(ab) \leq \mathrm{co}J(cd)$.

Theorem 41.3 *For any $ab, cd, ef \in H$ we have:*
1) $[cd]^J \leq [ef]^J$ iff $cd \leq J(ef)$,
2) $[ab]^J \leq [cd]^J$ iff $\mathrm{co}J(ab) \leq cd$.

We can put that in a diagram,

$$\begin{tikzcd}
{[ef]^J} \arrow[r, mapsto, "\sup"] & J(ef) \\
{[cd]^J} \arrow[u] \arrow[ur, phantom, "\longleftrightarrow"] & cd \arrow[l, mapsto] \arrow[u] \\
{[ab]^J} \arrow[u] \arrow[ur, phantom, "\longleftrightarrow"] \arrow[r, mapsto, "\inf"'] & \mathrm{co}J(ab) \arrow[u]
\end{tikzcd}$$

that can be read as a categorical statement: the functor $[\cdot]^J : H \to H/J$ has a left adjoint $\inf : H/J \to H$ and a right adjoint $\sup : H/J \to H$, where inf returns the smallest element of a J-equivalence class, and sup returns the biggest.

42 J-operators as adjunctions

The last diagram of the last section can be translated to topological language:

$$
\begin{tikzcd}
\mathcal{O}(S) \arrow[r, shift left=2, "f_*"] \arrow[r, shift right=2, "f^!"'] & \mathcal{O}(P) \arrow[l, "f^*" description] \\
\\
\\
S \arrow[r, "f"] & P
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, mapsto, "f_*"] & \mathsf{int}(U \cup Q) \\
 & \\
V \cap S \arrow[u] \arrow[r, phantom, shift left=4, "\longleftrightarrow"] \arrow[r, phantom, shift right=4, "\longleftrightarrow"] & V \arrow[l, mapsto, "f^*"'] \arrow[uu, phantom] \arrow[u] \\
W \arrow[u] \arrow[r, mapsto, "f^!"] & \mathsf{coint}(U \backslash Q) \arrow[u] \\
S \arrow[r, "f"] & P
\end{tikzcd}
$$

The notation used in the diagram above is essentially the one from figures 6 and 7 in [Och13]; the "external view" is at the left,"internal view" is at the right, the adjunction is $f^! \dashv f^* \dashv f_*$, and the diagram shows that $f_*(U) = \mathsf{int}(U \cup Q)$, $f^*(V) = V \cap S$ and $f^!(W) = \mathsf{coint}(U \backslash Q)$ (where int and coint use the topology $\mathcal{O}(P)$).

The order in which things are constructed in the diagram above is different from last section, though. Now we start with a finite set P, a topology $\mathcal{O}(P)$, and a subset $S \subseteq P$, and we define $\mathcal{O}(S)$ by restriction:

$$\mathcal{O}(S) \quad = \quad \{\, V \cap S \mid V \in \mathcal{O}(P) \,\}$$

we define Q as $P \backslash S$, we let $f : S \to P$ be the inclusion and $f^*(V)$ be $V \cap S$; then *it turns out* (theorem!) that the $f^!$ and f_* as defined above are the left and the right adjoints of f^* — and J and co$J$ are built from $f^!$, f^* and f_*: the definitions

$$
\begin{array}{rcl}
J(V) & = & f_*(f^*(V)) \\
\mathrm{co}J(V) & = & f^!(f^*(V))
\end{array}
$$

yield a J-operator $J : \mathcal{O}(P) \to \mathcal{O}(P)$ and its 'co' version, that returns the smallest element in each equivalence class; and if $\mathcal{O}(P) = \mathcal{O}_A(P)$ for some 2CG (P, A), then we can define J and co$J$ in this other way,

$$
\begin{array}{rcl}
J(cd) & = & \mathsf{pile}^{-1}(f_*(f^*(\mathsf{pile}(cd)))) \\
\mathrm{co}J(cd) & = & \mathsf{pile}^{-1}(f^!(f^*(\mathsf{pile}(cd))))
\end{array}
$$

that yields a J-operator (and its 'co' version) on the ZHA H generated by the 2CG (P, A).

This "topological version" of the adjunction is a nice concrete starting point for understanding toposes and geometric morphisms between them — or, actually, for introducing geometric morphisms to "children" who prefer to start with finite examples in which everything can be calculated explicitly. The toposes involved are $\mathbf{Set}^{\mathcal{O}(S)^{\mathrm{op}}}$ and $\mathbf{Set}^{\mathcal{O}(P)^{\mathrm{op}}}$, and the adjunction $f^! \dashv f^* \dashv f_*$ presented above acts only on the subobjects of the terminal of each topos — it needs to be extended to an (essential) geometric morphism between these toposes. This, and several concepts from section A4 of [Joh02], will be treated in a sequel of this paper, in a joint work with Peter Arndt.

[Awo06] [Joh02] [DP02] [Och13] [FS79] [Bel88]

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