Introduction

This is *part* of the material that I've prepared for a very introductory course on λ-calculus, types, intuitionistic propositional logic, Curry-Howard, Categories, Lisp and Lua... the course is 2hs/week, has no prerequisites at all, has no homework, and is usually attended by 2nd/3rd semester CompSci students; they spend most of their time in class discussing and doing exercises together in groups on a whiteboard.

I still need to: a) typeset lots of exercises that I wrote directly on the whiteboard, and some of their solutions — see:

`http://angg.twu.net/2017.2-LA/2017.2-LA.pdf` (whiteboards, PDFized)
`http://angg.twu.net/2017.2-LA.html` (course page)

and b) write a decent introduction, and c) make this self-contained.

Outline of the course

Evaluation
Directed graphs
Basic mathematical objects
Variables (and simultaneous substitution)
Functions as their graphs (i.e., as sets of pairs)
Operations like 'Σ' (including 'λ')
(and lots more)
Btw:
`http://angg.twu.net/LATEX/idarct-preprint.pdf`
`http://angg.twu.net/LATEX/2017planar-has-1.pdf`
`http://angg.twu.net/math-b.html#idarct`
`http://angg.twu.net/math-b.html#zhas-for-children-2`
`http://angg.twu.net/logic-for-children-2018.html`

Eduardo Ochs, 2017dez20

Expressions (and reductions)

The usual way to calculate an expression, one step at a time, with '='s:

$$\begin{array}{rcl} 2\cdot 3+4\cdot 5 & = & 2\cdot 3+20 \\ & = & 6+20 \\ & = & 26 \end{array}$$

$$\begin{array}{rcl} 2\cdot 3+4\cdot 5 & = & 6+4\cdot 5 \\ & = & 6+20 \\ & = & 26 \end{array}$$

Each '=' corresponds to a '$\longrightarrow$' in the reduction diagram below.

A notation for calculating the value of an expression by calculating the values of all its subexpressions:

$$\underbrace{\underbrace{2\cdot 3}_{6}+\underbrace{4\cdot 5}_{20}}_{26}$$

Each '=' in the previous diagram corresponds to applying one '$\underbrace{\quad\quad}$'.

A *reduction diagram* for $2\cdot 3+4\cdot 5$:
(See Hindley/Seldin, pages 14 and 17)

$$\begin{array}{ccccc} 2\cdot 3+4\cdot 5 & \longrightarrow & 2\cdot 3+20 & & \\ \downarrow & & \downarrow & & \\ 6+4\cdot 5 & \longrightarrow & 6+20 & \longrightarrow & 26 \end{array}$$

Note that when we can choose two subexpressions to calculate the '↓' evaluatess the leftmost one, and the '→' evaluates the rightmost one.

The *subexpressions* of $2\cdot 3+4\cdot 5$:

$$\underbrace{\underbrace{\underbrace{2}\cdot\underbrace{3}}+\underbrace{\underbrace{4}\cdot\underbrace{5}}}$$

Exercise:
Do the same as above for these expressions:
a) $2\cdot(3+4)+5\cdot 6$
b) $2+3+4$
c) $2+3+4+5$
(Improvise when needed)

Expressions with variables

If $a = 5$ and $b = 2$, then:

$$\underbrace{(\underbrace{\underbrace{a}_{5} + \underbrace{b}_{2}}_{7}) \cdot (\underbrace{\underbrace{a}_{5} - \underbrace{b}_{2}}_{3})}_{21}$$

If $a = 10$ and $b = 1$, then:

$$\underbrace{(\underbrace{\underbrace{a}_{10} + \underbrace{b}_{1}}_{11}) \cdot (\underbrace{\underbrace{a}_{10} - \underbrace{b}_{1}}_{9})}_{99}$$

We know – by algebra, which is not for (tiny) children –
that $(a + b) \cdot (a - b) = a \cdot a - b \cdot b$ is true for all $a, b \in \mathbb{R}$
We know – without algebra – how to test
"$(a + b) \cdot (a - b) = a \cdot a - b \cdot b$"
for specific values of a and b...

If $a = 5$ and $b = 2$, then:

$$\underbrace{\underbrace{(\underbrace{\underbrace{a}_{5} + \underbrace{b}_{2}}_{7}) \cdot (\underbrace{\underbrace{a}_{5} - \underbrace{b}_{2}}_{3})}_{21} = \underbrace{\underbrace{\underbrace{a}_{5} \cdot \underbrace{a}_{5}}_{25} - \underbrace{\underbrace{b}_{2} \cdot \underbrace{b}_{2}}_{4}}_{21}}_{\text{true}}$$

If $a = 10$ and $b = 1$, then:

$$\underbrace{\underbrace{(\underbrace{\underbrace{a}_{10} + \underbrace{b}_{1}}_{11}) \cdot (\underbrace{\underbrace{a}_{10} - \underbrace{b}_{1}}_{9})}_{99} = \underbrace{\underbrace{\underbrace{a}_{10} \cdot \underbrace{a}_{10}}_{100} - \underbrace{\underbrace{b}_{1} \cdot \underbrace{b}_{1}}_{1}}_{99}}_{\text{true}}$$

A notation for (simultaneous) substitution:

$$((x + y) \cdot z) \begin{bmatrix} x:=a+y \\ y:=b+z \\ z:=c+x \end{bmatrix} \quad = \quad ((a + y) + (b + z)) \cdot (c + x).$$

Note that $((a + b) \cdot (a - b)) \left[\begin{smallmatrix} a:=5 \\ b:=2 \end{smallmatrix}\right] = (5 + 2) \cdot (5 - 2)$.

Operations with substitution and copying

We know that $\Sigma_{i=2}^{5} i^3 = 2^3 + 3^3 + 4^3 + 5^3$.
If we introduce some intermediate steps we get:

$\Sigma_{i=2}^{5} i^3$
$\rightsquigarrow \;\; \Sigma_{i \text{ in } (2,3,4,5)} i^3$
$\rightsquigarrow \;\; (i^3)[i := 2] + (i^3)[i := 3] + (i^3)[i := 4] + (i^3)[i := 5]$
$\rightsquigarrow \;\; 2^3 + 3^3 + 4^3 + 5^3$

$\forall a \in \{2,3,5\}.\, a < 4$
$\rightsquigarrow \;\; \forall a \text{ in } (2,3,5).\, a < 4$
$\rightsquigarrow \;\; (a < 4)[a := 2]\ \&\ (a < 4)[a := 3]\ \&\ (a < 4)[a := 5]$
$\rightsquigarrow \;\; (2 < 4)\ \&\ (3 < 4)\ \&\ (5 < 4)$
$\rightsquigarrow$ false

$\forall a \in \{2,3,3,5\}.\, a < 4$
$\rightsquigarrow \;\; \forall a \text{ in } (2,3,3,5).\, a < 4$
$\rightsquigarrow \;\; (a < 4)[a := 2]\ \&\ (a < 4)[a := 3]\ \&\ (a < 4)[a := 3]\ \&\ (a < 4)[a := 5]$
$\rightsquigarrow \;\; (2 < 4)\ \&\ (3 < 4)\ \&\ (3 < 4)\ \&\ (5 < 4)$
$\rightsquigarrow$ false

$\{\, a^3 \mid a \in \{2,3,5\} \,\}$
$\rightsquigarrow \;\; \{(a^3)[a := 2], (a^3)[a := 3], (a^3)[a := 5]\}$
$\rightsquigarrow \;\; \{2^3, 3^3, 5^3\}$

$\{\, (a, a^3) \mid a \in \{2,3,5\} \,\}$
$\rightsquigarrow \;\; \{(a, a^3)[a := 2], (a, a^3)[a := 3], (a, a^3)[a := 5]\}$
$\rightsquigarrow \;\; \{(2, 2^3), (3, 3^3), (5, 5^3)\}$
$\rightsquigarrow \;\; \{(2, 8), (3, 27), (5, 125)\}$

One way to understand the 'λ' operator is using the idea — from Calculus 1 and Discrete Mathematics – that a function is a set of pairs (its "graph")...

$\lambda a{:}\{2,3,5\}.\, a^3$
$\rightsquigarrow \;\; \{\, (a, a^3) \mid a \in \{2,3,5\} \,\}$
$\rightsquigarrow \;\; \{(a, a^3)[a := 2], (a, a^3)[a := 3], (a, a^3)[a := 5]\}$
$\rightsquigarrow \;\; \{(2, 2^3), (3, 3^3), (5, 5^3)\}$
$\rightsquigarrow \;\; \{(2, 8), (3, 27), (5, 125)\}$

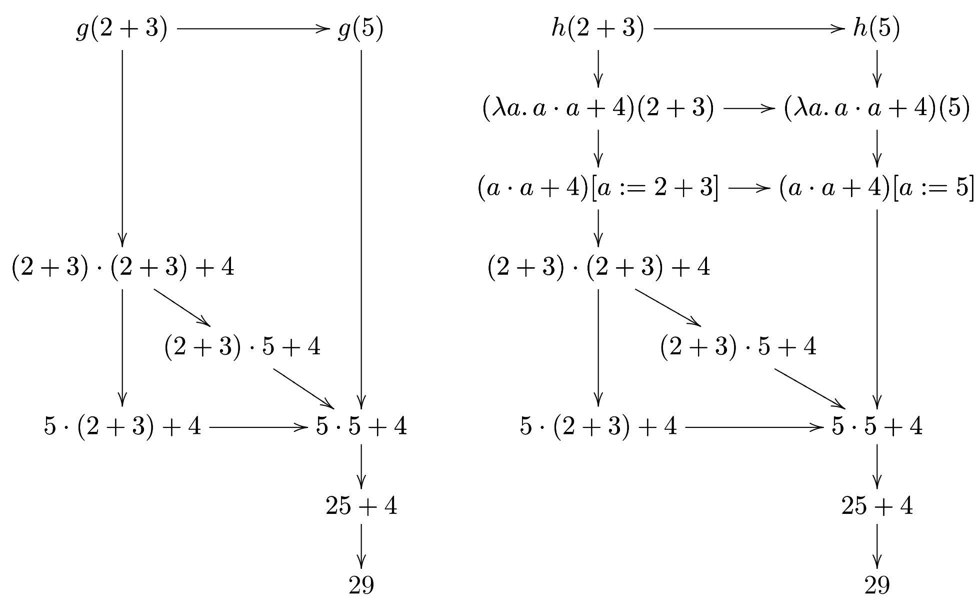
Note that

$(\lambda a{:}\{2,3,5\}.\, a^3)(5)$
$\rightsquigarrow \;\; (\{(2, 2^3), (3, 3^3), (5, 5^3)\})(5)$
$\rightsquigarrow \;\; 5^3$
$\rightsquigarrow \;\; 125$

$(\lambda a{:}\{2,3,5\}.\, a^3)(4)$
$\rightsquigarrow \;\; (\{(2, 2^3), (3, 3^3), (5, 5^3)\})(4)$
$\rightsquigarrow$ error

Lambda

A named function: $g(a) = a \cdot a + 4$
An unnamed function: $\lambda a.\, a \cdot a + 4$
Let $h = \lambda a.\, a \cdot a + 4$.
Then:



The usual notation for defining functions is like this:

$$\begin{array}{rrcl} f: & \mathbb{N} & \to & \mathbb{R} \\ & n & \mapsto & 2 + \sqrt{n} \end{array}$$

$$\begin{array}{rrcl} \text{(name)}: & \text{(domain)} & \to & \text{(codomain)} \\ & \text{(variable)} & \mapsto & \text{(expression)} \end{array}$$

It creates *named* functions
(with domains and codomains).

The usual notation for creating named functions
without specifying their domains and codomains
is just $f(n) = 2 + \sqrt{n}$.

Note that this is:

$$\begin{array}{cccc} f & (\,n\,) & = & 2 + \sqrt{n} \\ \text{(name)} & (\,\text{(variable)}\,) & = & \text{(expression)} \end{array}$$

Lambda notation: exercises

a) $(\lambda a.10a)(2+3)$
b) $(\lambda a.10a)((\lambda b.b+4)(3))$

Hint: use the speed you're most comfortable with. For example:

$$\underbrace{\underbrace{\underbrace{(\lambda a.10a)(\underbrace{\underbrace{\underbrace{(\lambda b.b+4)(3)}_{(b+4)[b:=3]}}_{3+4}}_{7})}_{(10a)[a:=7]}}_{10\cdot 7}}_{70} \qquad \underbrace{\underbrace{(\lambda a.10a)(\underbrace{\underbrace{(\lambda b.b+4)(3)}_{3+4}}_{7})}_{10\cdot 7}}_{70} \qquad \underbrace{(\lambda a.10a)(\underbrace{(\lambda b.b+4)(3)}_{7})}_{70}$$

c) $((\lambda a.(\lambda b.10a+b))(3))(4)$
d) $((\lambda f.(\lambda a.f(f(a))))(\lambda x.10x))(7)$

Hint 2: give names to subexpressions.

$$\underbrace{(\lambda a.10a)}_{\alpha}(\underbrace{\underbrace{(\lambda b.b+4)}_{\beta}(3)}_{\gamma})$$

$$\begin{array}{rcl}
\alpha(\gamma) & = & \alpha(\beta(3)) \\
 & = & \alpha((\lambda b.b+4)(3)) \\
 & = & \alpha((b+4)[b:=3]) \\
 & = & \alpha(3+4) \\
 & = & \alpha(7) \\
 & = & (\lambda a.10\text{a})(7) \\
 & = & 70
\end{array}$$

Functions as their graphs

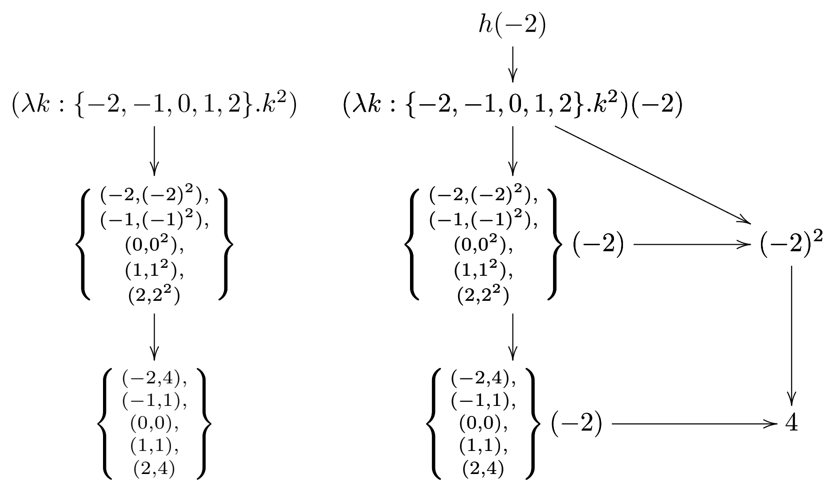
The *graph* of

$$\begin{array}{rccc} h: & \{-2,-1,0,1,2\} & \to & \{0,1,2,3,4\} \\ & k & \mapsto & k^2 \end{array}$$

is $\{(-2,4),(-1,1),(0,0),(1,1),(2,4)\}$.

We can think that a function *is* its graph,
and that a lambda-expression (with domain) reduces to a graph.
Then $h = \{(-2,4),(-1,1),(0,0),(1,1),(2,4)\}$
and $h(-2) = \{(-2,4),(-1,1),(0,0),(1,1),(2,4)\}(-2) = 4$.

Let $h := (\lambda k : \{-2,-1,0,1,2\}.k^2)$.
We have:



Note:
the graph of $(\lambda n : \mathbb{N}.n^2)$ has infinite points,
the graph of $(\lambda n : \mathbb{N}.n^2)$ is an infinite set,
the graph of $(\lambda n : \mathbb{N}.n^2)$ can't be written down *explicitly* without '...'s...

Mathematicians love infinite sets.
Computers hate infinite sets.
For mathematicians a function *is* its graph
(↑ remember Discrete Mathematics!)
For computer scientists a function *is* is a finite program.
Computer scientists love 'λ's!

I love things like this: $\left\{ \begin{smallmatrix} (3,30), \\ (4,40) \end{smallmatrix} \right\}(3) = 30$

Types (introduction)

Let:
$A = \{1, 2\}$
$B = \{30, 40\}$.

If $f : A \to B$, then f is one of these four functions:

$$\begin{smallmatrix}1\mapsto 30\\2\mapsto 30\end{smallmatrix}, \begin{smallmatrix}1\mapsto 30\\2\mapsto 40\end{smallmatrix}, \begin{smallmatrix}1\mapsto 40\\2\mapsto 30\end{smallmatrix}, \begin{smallmatrix}1\mapsto 40\\2\mapsto 40\end{smallmatrix}$$

or, in other notation,

$$\left\{\begin{smallmatrix}(1,30)\\(2,30)\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}(1,30)\\(2,40)\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}(1,40)\\(2,30)\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}(1,40)\\(2,40)\end{smallmatrix}\right\}$$

which means that:

$$f \in \left\{ \left\{\begin{smallmatrix}(1,30)\\(2,30)\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}(1,30)\\(2,40)\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}(1,40)\\(2,30)\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}(1,40)\\(2,40)\end{smallmatrix}\right\} \right\}$$

Let's use the notation "$A \to B$" for "the set of all functions from A to B".

Then $(A \to B) = \left\{ \left\{\begin{smallmatrix}(1,30)\\(2,30)\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}(1,30)\\(2,40)\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}(1,40)\\(2,30)\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}(1,40)\\(2,40)\end{smallmatrix}\right\} \right\}$
and $f : A \to B$
means $f \in (A \to B)$.

In Type Theory and λ-calculus "$a : A$" is pronounced "a is of type A", and the meaning of this is *roughly* "$a \in A$".
(We'll see the differences between '$\in$' and ':' (much) later).

Note that:
1. if $f : A \to B$ and $a : A$ then $f(a) : B$
2. if $a : A$ and $b : B$ then $(a, b) : A \times B$
3. if $p : A \times B$ then $\pi p : A$ and $\pi' p : B$, where
'π' means 'first projection' and
'π'' means 'second projection';
if $p = (2, 30)$ then $\pi p = 2$, $\pi' p = 30$.

If $p : A \times B$ and $g : B \to C$, then:

$$\underbrace{(\underbrace{\pi \underbrace{p}_{:A\times B}}_{:A}, \underbrace{\underbrace{g}_{:B\to C} (\underbrace{\pi' \underbrace{p}_{:A\times B}}_{:B}))}_{:C})}_{:A\times C}$$

Typed λ-calculus: trees

$A = \{1, 2\}$
$B = \{3, 4\}$
$C = \{30, 40\}$
$D = \{10, 20\}$
$A \times B = \left\{ \begin{matrix} (1,3), (1,4), \\ (2,3), (2,4) \end{matrix} \right\}$
$B \to C = \left\{ \left\{ \begin{smallmatrix} (3,30), \\ (4,30) \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} (3,30), \\ (4,40) \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} (3,40), \\ (4,30) \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} (3,40), \\ (4,40) \end{smallmatrix} \right\} \right\}$

If we know [the values of] a, b, f
then we know [the value of] $(a, f(b))$.
If $(a, b) = (2, 3)$ and $f = \left\{ \begin{smallmatrix} (3,30), \\ (4,40) \end{smallmatrix} \right\}$
then $(a, f(b)) = (2, 30)$.

$$\dfrac{\dfrac{(a,b)}{a}\,\pi \qquad \dfrac{\dfrac{(a,b)}{b}\,\pi' \qquad f}{f(b)}\,\mathsf{app}}{(a, f(b))}\,\mathsf{pair} \qquad\qquad \dfrac{\dfrac{(2,3)}{2}\,\pi \qquad \dfrac{\dfrac{(2,3)}{3}\,\pi' \qquad \{(3,30),(4,40)\}}{30}\,\mathsf{app}}{(2, 30)}\,\mathsf{pair}$$

If we know the *types* of a, b, f
we know the type of $(a, f(b))$.
If we know the types of p, f
we know the type of $(\pi p, f(\pi' p))$.
If we know the types of p, f
we know the type of $(\lambda p : A \times B.(\pi p, f(\pi' p)))$.

$$\dfrac{\dfrac{(a,b) : A \times B}{a : A}\,\pi \qquad \dfrac{\dfrac{(a,b) : A \times B}{b : B}\,\pi' \qquad f : B \to C}{f(b) : C}\,\mathsf{app}}{(a, f(b)) : A \times C}\,\mathsf{pair}$$

$$\dfrac{\dfrac{\dfrac{p : A \times B}{\pi p : A}\,\pi \qquad \dfrac{\dfrac{p : A \times B}{\pi' p : B}\,\pi' \qquad f : B \to C}{f(\pi' p) : C}\,\mathsf{app}}{(\pi p, f(\pi' p)) : A \times C}\,\mathsf{pair}}{(\lambda p : A \times B.(\pi p, f(\pi' p))) : A \times B \to A \times C}\,\lambda$$

Types: exercises

Let:
$A = \{1, 2\}$
$B = \{3, 4\}$
$C = \{30, 40\}$
$D = \{10, 20\}$
$f = \left\{ \begin{smallmatrix} (3,30), \\ (4,40) \end{smallmatrix} \right\}$
$g = \left\{ \begin{smallmatrix} (1,10), \\ (2,20) \end{smallmatrix} \right\}$
Note that $f : B \to C$ and $g : A \to D$.

a) Evaluate $A \times B$.
b) Evaluate $A \to D$.
c) Evaluate $(\pi p, f(\pi' p))$ for each of the four possible values of $p : A \times B$.
d) Evaluate $\lambda p{:}A{\times}B.(\pi p, f(\pi' p))$.
e) Is this true?

$$(\lambda p{:}A{\times}B.(\pi p, f(\pi' p))) = \left\{ \begin{smallmatrix} ((1,3),(1,30)), \\ ((1,4),(1,40)), \\ ((2,3),(2,30)), \\ ((2,4),(2,40)) \end{smallmatrix} \right\}$$

f) Let $p = (2, 3)$. Evaluate $(g(\pi p), f(\pi' p))$.
g) Let $p = (1, 2)$. Evaluate $(g(\pi p), f(\pi' p))$.
h) Check that if $p : A \times B$ then $(g(\pi p), f(\pi' p)) : D \times C$.
i) Check that

$$(\lambda p{:}A{\times}B.(g(\pi p), f(\pi' p))) : A \times B \to D \times C.$$

j) Evaluate $(\lambda p{:}A{\times}B.(g(\pi p), f(\pi' p)))$.

Type inference

Here is another notation for checking types:

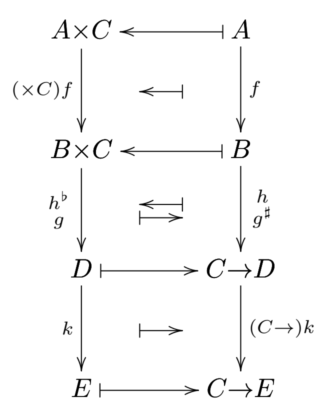
$$\underbrace{(\lambda \underbrace{p}_{:A\times B} : A\times B.\ \underbrace{(\underbrace{\pi \underbrace{p}_{:A\times B}}_{:A},\ \underbrace{\underbrace{f}_{:B\to C} (\underbrace{\pi' \underbrace{p}_{:A\times B}}_{:B})}_{:C})}_{:A\times C})}_{:A\times B\to A\times C}$$

Compare it with:

$$\dfrac{\dfrac{\dfrac{p:A\times B}{\pi p:A}\,\pi \qquad \dfrac{\dfrac{p:A\times B}{\pi' p:B}\,\pi' \quad f:B\to C}{f(\pi' p):C}\,\mathsf{app}}{(\pi p, f(\pi' p)):A\times C}\,\mathsf{pair}}{(\lambda p:A\times B.(\pi p, f(\pi' p))):A\times B\to A\times C}\,\lambda$$

Exercise:
Infer the type of each of the terms below (at the right of the ':=').
Use the two notations above.
The types of f, g, h, k are shown in the diagram below.

a) $(\times C)f \ := \ \lambda p{:}A{\times}C.(f(\pi p), \pi' p)$
b) $h^\flat \ := \ \lambda q{:}B{\times}C.(h(\pi q))(\pi' q)$
c) $g^\sharp \ := \ \lambda b{:}B.\lambda c{:}C.g(b,c)$
d) $(C{\to})k \ := \ \lambda \varphi{:}C{\to}D.\lambda c{:}C.k(\varphi c)$



Term inference

Exercises:

$$\dfrac{\dfrac{\dfrac{\dfrac{p : A \times C}{: A}\,\pi \qquad f : A \to B}{: B}\,\mathsf{app} \qquad \dfrac{p : A \times C}{: C}\,\pi'}{: B \times C}\,\mathsf{pair}}{: A \times C \to B \times C}\,\lambda$$

$$\dfrac{\dfrac{\dfrac{q : B \times C}{: C}\,\pi' \qquad \dfrac{\dfrac{q : B \times C}{: B}\,\pi \qquad h : B \to (C \to D)}{: C \to D}\,\mathsf{app}}{: D}\,\mathsf{app}}{: B \times C \to D}\,\lambda$$

$$\dfrac{\dfrac{\dfrac{\dfrac{b : B \quad c : C}{: B \times C}\,\mathsf{pair} \qquad g : B \times C \to D}{: D}\,\mathsf{app}}{: C \to D}\,\lambda}{: B \to (C \to D)}\,\lambda$$

$$\dfrac{\dfrac{\dfrac{\dfrac{c : C \quad \varphi : C \to D}{: D}\,\mathsf{app} \qquad k : D \to E}{: E}\,\mathsf{app}}{: (C \to E)}\,\lambda}{: (C \to D) \to (C \to E)}\,\lambda$$

Term inference: answers

$$\dfrac{\dfrac{\dfrac{\dfrac{p : A\times C}{\pi p : A}\,\pi \quad f : A\to B}{f(\pi p) : B}\,\mathsf{app} \quad \dfrac{p : A\times C}{\pi' p : C}\,\pi'}{(f(\pi p), \pi' p) : B\times C}\,\mathsf{pair}}{\lambda p{:}A{\times}C.(f(\pi p),\pi' p) : A\times C\to B\times C}\,\lambda$$

$$\dfrac{\dfrac{\dfrac{q : B\times C}{\pi' q : C}\,\pi' \quad \dfrac{\dfrac{q : B\times C}{\pi q : B}\,\pi \quad h : B\to(C\to D)}{h(\pi q) : C\to D}\,\mathsf{app}}{h(\pi q)(\pi' q) : D}\,\mathsf{app}}{\lambda q{:}B{\times}C.h(\pi q)(\pi' q) : B\times C\to D}\,\lambda$$

$$\dfrac{\dfrac{\dfrac{\dfrac{b : B \quad c : C}{(b,c) : B\times C}\,\mathsf{pair} \quad g : B\times C\to D}{g(b,c) : D}\,\mathsf{app}}{\lambda c{:}C.g(b,c) : C\to D}\,\lambda}{\lambda b{:}B.\lambda c{:}C.g(b,c) : B\to(C\to D)}\,\lambda$$

$$\dfrac{\dfrac{\dfrac{\dfrac{c : C \quad \varphi : C\to D}{\varphi c : D}\,\mathsf{app} \quad k : D\to E}{k(\varphi c) : E}\,\mathsf{app}}{\lambda c{:}C.k(\varphi c) : (C\to E)}\,\lambda}{\varphi{:}C{\to}D.\lambda c{:}C.k(\varphi c) : (C\to D)\to(C\to E)}\,\lambda$$

Contexts and '$\vdash$'

Suppose that A, B, C are known, and are sets.
(Jargon: "fix sets A, B, C".)
Then this

$$\underbrace{p:A\times B, f:B\to C}_{\substack{\text{“context”: a series of}\\ \text{declarations like}\\ \mathit{var}{:}\mathit{type}}} \vdash \underbrace{f(\pi'p):C}_{\mathit{expr}{:}\mathit{type}}$$

Means:
"In this context the expression *expr* makes sense, is not error, and its result is of type *type*."

Note that calculating $f(\pi'p)$ yields error if we do not know the values of f or p.

What happens if we add contexts to each $\mathit{term} : \mathit{type}$ in a tree?
The two bottom nodes in

$$\dfrac{\dfrac{\dfrac{p:A\times B}{\pi p:A}\,\pi \qquad \dfrac{\dfrac{p:A\times B}{\pi'p:B}\,\pi' \qquad f:B\to C}{f(\pi'p):C}\,\mathsf{app}}{(\pi p, f(\pi'p)):A\times C}\,\mathsf{pair}}{(\lambda p:A\times B.(\pi p,f(\pi'p))):A\times B\to A\times C}\,\lambda$$

would become:

$$f:B\to C, p:A\times B \vdash (\pi p, f(\pi'p)):A\times C$$
$$f:B\to C \vdash (\lambda p:A\times B.(\pi p, f(\pi'p))):A\times B\to A\times C$$

After the rule 'λ' the 'p' is no longer needed!

If we add the contexts and omit the types, the tree becomes:

$$\dfrac{\dfrac{\dfrac{p\vdash p}{p\vdash \pi p}\,\pi \qquad \dfrac{\dfrac{p\vdash p}{p\vdash \pi'p}\,\pi' \qquad f\vdash f}{f,p\vdash f(\pi'p)}\,\mathsf{app}}{f,p\vdash(\pi p, f(\pi'p))}\,\mathsf{pair}}{f\vdash(\lambda p{:}A{\times}B.(\pi p,f(\pi'p)))}\,\lambda \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{[p]^1}{\pi p}\,\pi \qquad \dfrac{\dfrac{[p]^1}{\pi'p}\,\pi' \qquad f}{f(\pi'p)}\,\mathsf{app}}{(\pi p, f(\pi'p))}\,\mathsf{pair}}{(\lambda p{:}A{\times}B.(\pi p,f(\pi'p)))}\,\lambda;1$$

Notational trick:
below the bar '$\lambda;1$' the value of p is no longer needed;
we say that the p is "discharged" (from the list of hypotheses)
and we mark the 'p' on the leaves of the tree with '$[\cdot]^1$';
a '$[\cdot]^1$' on a hypothesis means: "below the bar '$\lambda;1$' I am no longer a hypothesis".

Curry-Howard: introduction

We are learning a system called
“the simply-typed λ-calculus (with binary products)” —
system $\lambda 1$, for short.

In $\lambda 1$ in its fullest form,
its objects are trees of ‘$\ldots \vdash term : type$’s,
but we saw (evidence) that we can:
• reconstruct the full tree from just the ‘$term : type$’s,
• write just ‘$: type$’s (except on the leaves, to get the var names),
• reconstruct the full tree from just the bottom ‘$term : type$’...

For example, we can reconstruct the whole tree,
with contexts, from:

$$\dfrac{\dfrac{\dfrac{[p:A\times B]^1}{:A}\pi \qquad \dfrac{\dfrac{[p:A\times B]^1}{:B}\pi' \qquad f:B\to C}{:C}\mathsf{app}}{:A\times C}\mathsf{pair}}{:A\times B\to A\times C}\lambda$$

If we erase the terms and the ‘:’s and leave only the types,
we get something that is strikingly similar to a tree in
Natural Deduction,

$$\dfrac{\dfrac{\dfrac{[A\times B]^1}{A}\pi \qquad \dfrac{\dfrac{[A\times B]^1}{B}\pi' \qquad B\to C}{C}\mathsf{app}}{A\times C}\mathsf{pair}}{A\times B\to A\times C}\lambda$$

$$\rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{[P\land Q]^1}{P}\land E_1 \qquad \dfrac{\dfrac{[P\land Q]^1}{Q}\land E_2 \qquad Q\to R}{R}\to E}{P\land R}\land I}{P\land R\to P\land Q}\to I;1$$

which talks about *logic*.

Curry-Howard: Natural Deduction

The tree

$$\dfrac{\dfrac{[P\land Q]^1}{P}\land E_1 \qquad \dfrac{\dfrac{[P\land Q]^1}{Q}\land E_2 \qquad Q\to R}{R}\to E}{\dfrac{P\land R}{P\land R\to P\land Q}\to I;1}\land I$$

is in $ND_{\land\to}$ (or in $IPL_{\land\to}$), the fragment of Natural Deduction (or intuitionistic predicate logic) that only has the connectives $\land$ and $\to$.

Its rules are:

$$\dfrac{P \quad Q}{P\land Q}\land I \qquad \dfrac{P\land Q}{P}\land E_1 \qquad \dfrac{P\land Q}{Q}\land E_2$$

$$\dfrac{\begin{array}{c}P \quad [Q]^1\\ \vdots\\ R\end{array}}{Q\to R}\to I;1 \qquad \dfrac{P \quad P\to Q}{Q}\to E$$

New rules (for $\top$, $\bot$, $\lor$):
(not yet — see the whiteboard for 20170418)

Planar Heyting Algebras

Read sections 2–8 of:
http://angg.twu.net/LATEX/2017planar-has.pdf

Let $B = \begin{smallmatrix} & 32 & & & \\ & & 22 & & \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{smallmatrix}$ and $C = \begin{smallmatrix} & & & & 44 & & & & \\ & & & 43 & & 34 & & & \\ & & 42 & & 33 & & 24 & & \\ & 41 & & 32 & & 23 & & 14 & \\ 40 & & 31 & & 22 & & 13 & & 04 \\ & 30 & & 21 & & 12 & & 03 & \\ & & 20 & & 11 & & 02 & & \\ & & & 10 & & 01 & & & \\ & & & & 00 & & & & \end{smallmatrix}$.

Exercises:
Calculate, and represent in positional notation when possible:

a) $\lambda lr{:}B.l$
b) $\lambda lr{:}B.r$
c) $\lambda lr{:}B.(l \leq 1)$
d) $\lambda lr{:}B.(r \geq 1)$
e) $\lambda lr{:}B.lr \leq 11$
f) $\lambda lr{:}B.lr \wedge 12$
g) $\lambda lr{:}B.\ \mathsf{valid}\ (\langle l+1, r \rangle)$
h) $\lambda lr{:}B.lr\ \mathsf{leftof}\ 11$
i) $\lambda lr{:}B.lr\ \mathsf{leftof}\ 12$
j) $\lambda lr{:}B.lr\ \mathsf{above}\ 11$
k) $\lambda lr{:}B.\ \mathsf{ne}\ (lr)$
l) $\lambda lr{:}B.\ \mathsf{nw}\ (lr)$

m) $20 \to 11$
n) $02 \to 11$
o) $22 \to 11$
p) $00 \to 11$

q) $\lambda lr{:}B.\neg lr$
r) $\lambda lr{:}B.\neg\neg lr$
s) $\lambda lr{:}B.(lr = \neg\neg lr)$

t) $\lambda P{:}C.(P \to 22)$.
u) $\lambda Q{:}C.(22 \to Q)$.
v) find X such that $(\lambda P{:}C.P \leq X) = (\lambda P{:}C.(P \leq 22) \wedge (P \leq 13)))$.
w) find X such that $(\lambda R{:}C.X \leq R) = (\lambda R{:}C.(22 \leq R) \wedge (13 \leq R)))$.

Algebraic structures

A *ring* is a 6-uple

$$(R, 0_R, 1_R, +_R, -_R, \cdot_R)$$

where $R, 0_R, \ldots, \cdot_R$ have the following types,

R is a set,
$0_R \in R$,
$1_R \in R$,
$+_R : R \times R \to R$,
$-_R : R \to R$ (unary minus),
$\cdot_R : R \times R \to R$,

and where the components obey these equations ($\forall a, b, c \in R$):

$a+0_R = 0_R+a = a, \quad a+b = b+a, \quad a+(b+c) = (a+b)+c, \quad a+(-a) = 0,$
$a \cdot 1_R = 1_R \cdot a = a, \quad a \cdot b = b \cdot a, \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c,$
$a \cdot (b + c) = a \cdot b + a \cdot c.$

A *proto-ring* is a 6-uple $(R, 0_R, 1_R, +_R, -_R, \cdot_R)$
that obeys the typing conditions of a ring.
A *ring* is a proto-ring plus the assurance that it obeys the ring equations.

A *proto-Heyting Algebra* is a 7-uple

$$H = (\Omega, \leq_H, \top_H, \bot_H, \wedge_H, \vee_H, \to_H)$$

in which:

Ω is a set (the "set of truth values"),
$\leq_H \subset \Omega \times \Omega$ (partial order),
$\top_H \in \Omega$,
$\bot_H \in \Omega$,
$\wedge_H : \Omega \times \Omega \to \Omega$
$\vee_H : \Omega \times \Omega \to \Omega$
$\to_H : \Omega \times \Omega \to \Omega$

Sometimes we add operations '$\neg$' and '$\leftrightarrow$' to a (proto-)HA H,

$$H = (\Omega, \leq_H, \top_H, \bot_H, \wedge_H, \vee_H, \to_H, \neg_H, \leftrightarrow_H)$$

by *defining them* as $\neg P := P \to \bot$ and $P \leftrightarrow Q := (P \to Q) \wedge (Q \to P)$
(i.e., $\neg_H P := P \to_H \bot_H$
and $P \leftrightarrow_H Q := (P \to_H Q) \wedge_H (Q \to_H P)$).

This abuse of language is very common:
R "=" $(R, 0_R, 1_R, +_R, -_R, \cdot_R)$.

Protocategories

A *protocategory* is a 4-uple

$$\mathbf{C} = (\mathbf{C}_0, \mathrm{Hom}_{\mathbf{C}}, \mathrm{id}_{\mathbf{C}}, \circ_{\mathbf{C}})$$

where

$\mathbf{C}_0$ is a set (more precisely a "class"),
$\mathrm{Hom}_{\mathbf{C}} : \mathbf{C}_0 \times \mathbf{C}_0 \to \mathbf{Sets}$,
$\mathrm{id}_{\mathbf{C}}(A) \in \mathrm{Hom}_{\mathbf{C}}(A, A)$,
$(\circ_{\mathbf{C}})_{ABC} : \mathrm{Hom}_{\mathbf{C}}(B, C) \times \mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{C}}(A, C)$.

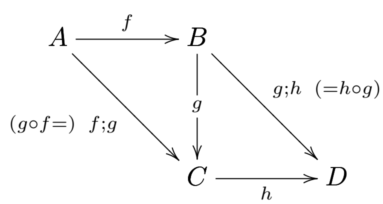
A *category* is a protocategory plus the assurance that identities behave as expected and composition is associative.

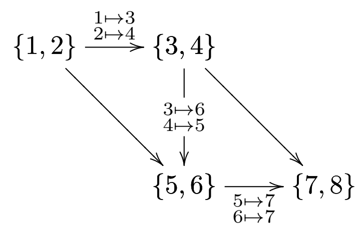
Sometimes we add an operation ';' to a category,

$$\mathbf{C} = (\mathbf{C}_0, \mathrm{Hom}_{\mathbf{C}}, \mathrm{id}_{\mathbf{C}}, \circ_{\mathbf{C}}, ;_{\mathbf{C}})$$

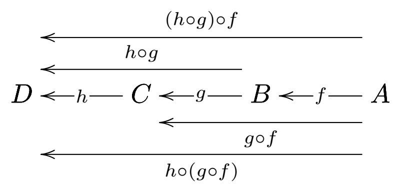
where ';' is the composition in other order: $f \circ g = g; f$.

Composition
(is associative)

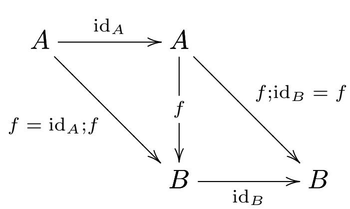


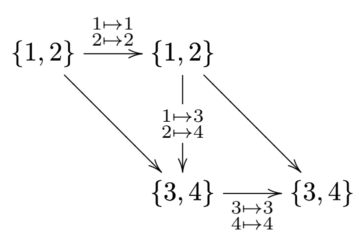


$$
\begin{aligned}
((h \circ g) \circ f)(a) &= (h \circ g)(f(a)) \\
&= h(g(f(a))) \\
&= h((g \circ f)(a)) \\
&= (h \circ (g \circ f))(a) \\
((h \circ g) \circ f)(a) &= (h \circ (g \circ f))(a) \\
(h \circ g) \circ f &= h \circ (g \circ f)
\end{aligned}
$$

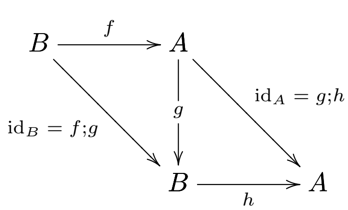


Identities:
If $f : A \to B$ then $\mathrm{id}_A; f = f = f; \mathrm{id}_B$





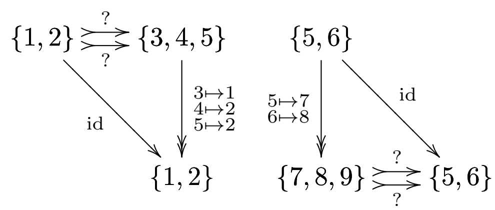
A theorem about lateral inverses:
If $f; g = \mathrm{id}$ and $g; h = \mathrm{id}$ then $f = h$



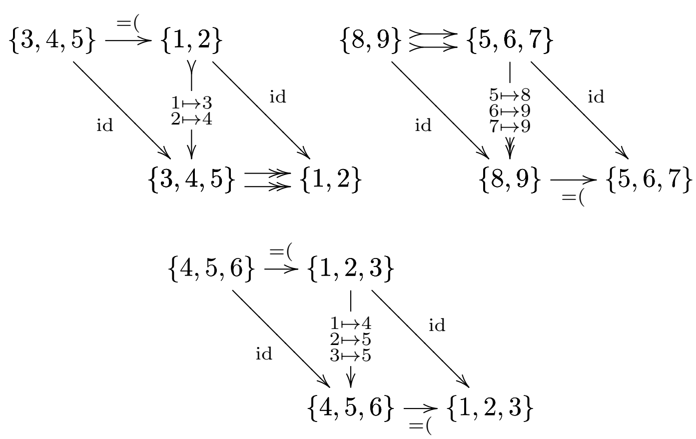
$(f; g); h = \mathrm{id}_B; h = h$
$f; (g; h) = f; \mathrm{id}_A = f$

$$
\begin{aligned}
f &= f; \mathrm{id}_A \\
&= f; (g; h) \\
&= (f; g); h \\
&= \mathrm{id}_B; h \\
&= h
\end{aligned}
$$

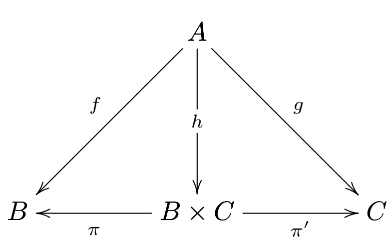
Multiple inverses

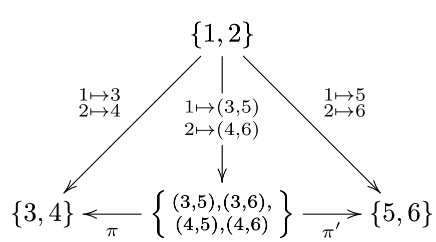


No inverses

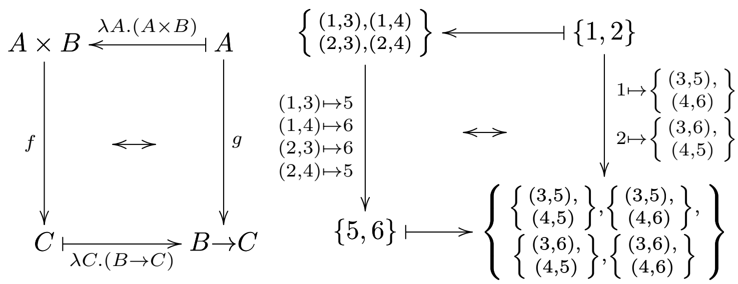


Products
Property: $\forall f, g.\exists! h.(f = h; \pi \ \&\ g = h; \pi')$
Solution: $h = \lambda a : A.(f(a), g(a))$
Bijection: $(f, g) \leftrightarrow h$
$(\rightarrow)$: $\lambda(f, g).(\lambda a : A.(f(a), g(a)))$
$(\leftarrow)$: $\lambda h.((h; \pi), (h; \pi'))$





Exponentials
Bijection: $f \leftrightarrow g$
$(\rightarrow)$ ("currying"): $g := \mathrm{cur}\, f := \lambda a : A.\lambda b : B.f(a, b)$
$(\leftarrow)$ ("uncurrying"): $f := \mathrm{uncur}\, g := \lambda(a, b) : A \times B.((g(a))(b))$



Properties: $\mathrm{cur}\,\mathrm{uncur}\, f = f, \quad \mathrm{uncur}\,\mathrm{cur}\, g = g$
where: $f \times f' := \langle \pi; f, \pi'; f' \rangle, \quad \mathrm{uncur}\, g := (g \times \mathrm{id}); \mathrm{ev}$
solving type equations:

$$\dfrac{\dfrac{\dfrac{\pi \quad f}{\pi; f} \qquad \dfrac{\pi' \quad f'}{\pi'; f'}}{\langle \pi; f, \pi'; f' \rangle}}{f \times f'}\ \mathrm{ren} \qquad \dfrac{\dfrac{\dfrac{\pi : A\times? \to A}{\pi; f : A\times? \to B} \qquad \dfrac{\pi' :? \times A' \to A' \quad f' : A' \to B'}{\pi'; f' :? \times A' \to B'}}{\langle \pi; f, \pi'; f' \rangle : A \times A' \to B \times B'}}{f \times f' : A \times A' \to B \times B'}\ \mathrm{ren}$$

$$\dfrac{\dfrac{\dfrac{g \quad \mathrm{id}}{g \times \mathrm{id}} \qquad \mathrm{ev}}{(g \times \mathrm{id}); \mathrm{ev}}}{\mathrm{uncur}\, g}\ \mathrm{ren} \qquad \dfrac{\dfrac{\dfrac{g : A \to (B{\to}C) \quad \mathrm{id} :? \to ?}{g \times \mathrm{id} : A\times? \to (B{\to}C)\times?} \qquad \mathrm{ev} : (B{\to}C) \times B \to C}{(g \times \mathrm{id}); \mathrm{ev} : A\times? \to C}}{\mathrm{uncur}\, g : A\times? \to C}$$