On some missing diagrams in the Elephant

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The abstract of the extended abstract that I submitted to the conference was just this:

> Imagine two category theorists, Aleks and Bob, who both think very visually and who have exactly the same background. One day Aleks discovers a theorem, T_1, and sends an e-mail, E_1, to Bob, stating and proving T_1 in a purely algebraic way; then Bob is able to reconstruct by himself Aleks's diagrams for T_1 exactly as Aleks has thought them. We say that Bob has reconstructed the *missing diagrams* in Aleks's e-mail.
>
> Now suppose that Carol has published a paper, P_2, with a theorem T_2. Aleks and Bob both read her paper independently, and both pretend that she thinks diagrammatically in the same way as them. They both "reconstruct the missing diagrams" in P_2 in the same way, even though Carol has never used those diagrams herself.
>
> Here we will reconstruct, in the sense above, some of the "missing diagrams" in two factorizations of geometric morphisms in section A4 of Johnstone's "Sketches of an Elephant", and also some "missing examples". Our criteria for determining what is "missing" and how to fill out the holes are essentially the ones presented in the "Logic for Children" workshop at the UniLog 2018; they are derived from a certain *definition* of "children" that turned out to be especially fruitful.

Its focus was the factorization in the last paragraph — let me call it simply "the factorization" from here on; it is explained in the section 6 of the poster.

In this poster I will try to do something different: I will try to give a broad view of the whole project, and show that the factorization is just a cherry on top of the sundae — a cherry that is there to please the mathematicians who believe that constructions and techniques only relevant when they *prove new theorems*.

The references in boldface in this poster are to papers, slides and notes that can accessed from the section titled "On some missing diagrams in the Elephant" in my webpage:

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http://angg.twu.net/math-b.html
http://angg.twu.net/math-b.html#missing-diagrams-elephant
```

They are:

[**IDARCT**]: **Internal Diagrams and Archetypal Reasoning in Category Theory**
(Logica Universalis, 2013)

[**PH1**]: **Planar Heyting Algebras for Children**
(Accepted @ South Americal Journal of Logic)

[**MDE**] **On some missing diagrams in the Elephant**
(My extended abstract to the ACT2019 — it's not in the conference website because poster session people are second-class people, duh)

[**NYo**]: **Notes on the Yoneda Lemma**
(In which each node and arrow can be interpreted precisely as a "term", and most of the interpretations are "obvious"; plus dictionaries!!!)
(slides, 2019)

1 "Is associated to"

The symbol '$\leftarrow\!\!\rightsquigarrow$' will be pronounced "is associated to", but the *formal* meaning of '$A \leftrightsquigarrow B$' will depend on the types of A and B. For example,

$$(P, A) \leftrightsquigarrow H$$

will mean that the 2-column graph (P, A) is associated to the Planar Heyting Algebra (a "ZHA") H through the standard bijection between 2CGs and ZHAs;

$$((P, A), Q) \leftrightsquigarrow (H, J)$$

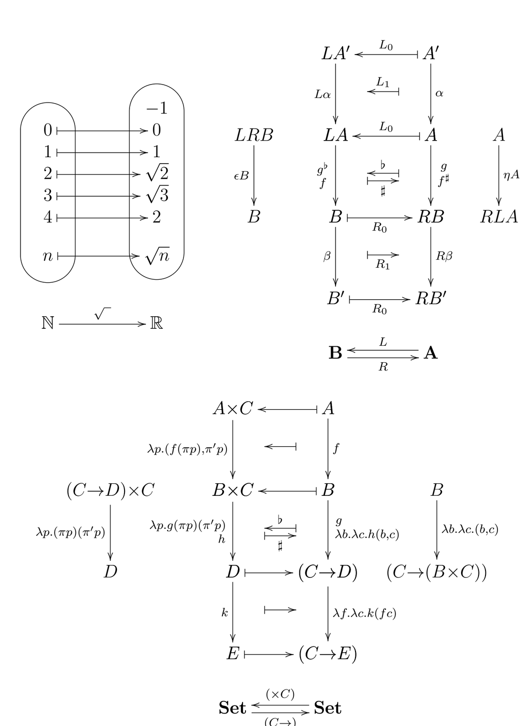
will mean that besides $(P, A) \leftrightsquigarrow J$ the equivalence relation induced by set of question marks Q is "the same" as the one induced by J-operator J; and

$$J \leftrightsquigarrow j$$

means that in the topos that we are talking about the J-operator J (J-operators are a concept that fell out of fashion) is associated to the local operator $j : \Omega \to \Omega$ — something is everyone (?!?) knows that induces a notion of "sheaf" on the topos we're in.

2 Parallel diagrams (of several kinds)

Look at the diagram below. Let's call its three subdiagrams ${}^{A\,B}_{\ C}$. Each of the subdiagrams A, B, C have an "external diagram", or an "external view", below, and an "internal diagram" or "internal view" above.



Our notions of "external" and "internal" are the same as in [LS97] or [LR03], but our use of arrows is much more precise. Note that in diagram A

- we distinguish '$\to$' and '$\mapsto$' in the usual way — $\mathbb{N} \overset{\sqrt{\ }}{\to} \mathbb{R}$ is a *typing*,
- one of the '$\mapsto$' arrows above $\mathbb{N} \overset{\sqrt{\ }}{\to} \mathbb{R}$ is "generic": $n \mapsto \sqrt{n}$ shows how to produce the image of a generic element $n \in \mathbb{N}$ — i.e., as an untyped λ-calculus term, $\sqrt{\ } = \lambda n.\sqrt{n}$,
- all the other '$\mapsto$' arrows above $\mathbb{N} \overset{\sqrt{\ }}{\to} \mathbb{R}$ are substituition instances of $n \mapsto \sqrt{n}$, maybe after some term reductions.

  "$3 \mapsto \sqrt{3}$" is just a substituion instance of "$n \mapsto \sqrt{n}$", but "$4 \mapsto 2$" is a substitution instance of "$n \mapsto \sqrt{n}$" followed by the reduction that transforms "$\sqrt{4}$" into "2".

The notion of "above" in these diagrams is very important. The blob above $\mathbb{N}$ shows some elements of $\mathbb{N}$; same for $\mathbb{R}$, and the '$\mapsto$' arrows above the '$\to$' are internal views of the '$\to$'.

The subfigure B shows the external view and the internal view of a (generic) adjunction. Things are much more complex now — we don't draw the blobs; above **B** we have six objects and four morphisms from the category **B**, and the same for **A**; the arrows marked 'L_0' above L are the action of L on objects and the arrow 'L_1' is the action of L on morphisms, and the same for R; we draw the transpositions $\flat$ and $\sharp$; we draw the unit and the counit.

The subfigure C is a particular case of the subfigure B — among other things it has exactly the same shape as B, and we can extract from it the definitions of L_0, L_1, etc in this particular case.

The "general case" B and the "particular case" C are evidently parallel diagrams, but in each of A, B, C we have a internal diagram parallel to an external diagram — but the internal diagrams are much bigger.

Note: these diagrams can be interpreted in proof assistants! See [**NYo**] for a reasonably big example worked out in all detail!

(A curiosity: many years ago I spent a lot of time studying [Jac99] and trying to create a bridge between his fibration-based categorical models for type theory and what I thought that were the "archetypal cases" motivating them. Some of my conventions for "above", "below", etc, comes from conventions for fibrations.)

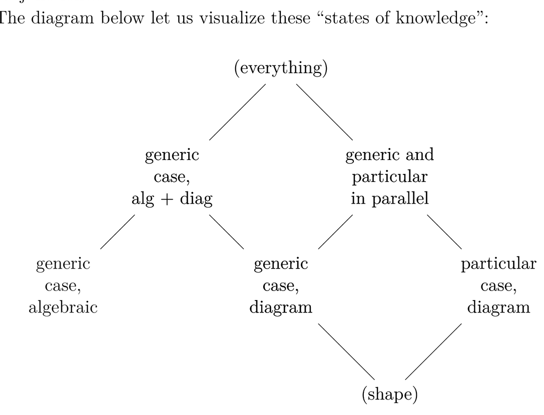
3 Projections and liftings

One of the first ideas presented in [**IDARCT**] is that we can treat operations that discard information, like *particularization*, as *projections*, and we can treat *generalization* as *lifting*... and the paper shows how to represent visually several operations that *erase* and that *reconstruct* information.

Suppose that we learned adjunctions from [Mac71]. His presentation is not very visual, so we worked a bit (or a lot) and found the diagrams for adjunctions in sec.2; we are now in a situation where we know the "algebraic" definition of an adjunction plus a diagrammatic way to represent it. We work a bit more and we get a diagram for a particular case – the adjunction $(\times B) \dashv (B\to)$.

There are some advantages in knowing the "algebraic version", plus having a diagram for the general case, that lets us play with some sequences operations as if they were shape and movement, plus having particular cases that let us test conjectures quickly... in recent texts about the Philosophy of Mathematics, like [Krö07], [Man08], [Cor04] we can even find *names* for each kind of reasoning, and each kind of intuition, that becomes accessible to us after we get another view of, say, adjunctions...

The diagram below let us visualize these "states of knowledge":



There are some processes that I discuss in [**IDARCT**] that are only mentioned very, very briefly in the three references above — namely, forgetting details and then reconstructing them, and "omitting diagrams" for publication followed by "reconstructing the diagrams that the author had in mind" as explained in the abstract in the beginning of the poster.

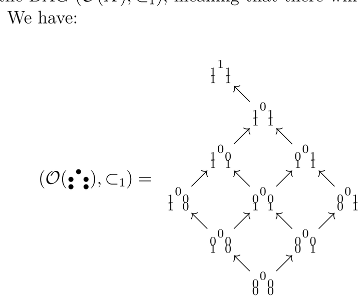
Also, there is an idea in [**IDARCT**] that I haven't seen anywhere else and that became a long-term project for me, that is that we can "project out" all the terms in a categorical construction or proof that mention equalities of morphisms — I call this "dropping the boring part and keeping only the fun part" of the categorical constructions and arguments —, then we do our constructions there, and *after we're finished with this part* we reconstruct the "boring" part in a way that matches the rest.

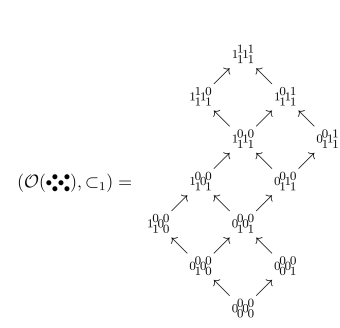
4 Drawing topologies

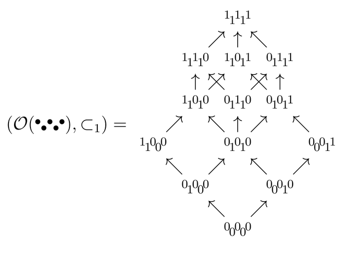
We can draw subsets of a set ⁘ by their characteristic functions: ${}^{0}{}^{1}{}_{1}{}^{1}{}_{0}$, ${}^{0}{}^{0}{}_{1}{}^{1}{}_{1}$, and so on. The *order topology* on a set ⁘ is defined by:

> a subset of ⁘ is *open*
> when its characteristic function
> *doesn't have* a '1' above a '0'.

We can draw topologies as directed graphs. Let "$A \subset_1 B$" stand for: "$A \subset B$, and the difference $B \backslash A$ has exactly one element". We will draw the topology $(X, \mathcal{O}(X))$ as the DAG $(\mathcal{O}(X), \subset_1)$, meaning that there will be an arrow $V \to U$ when $V \subset_1 U$. We have:





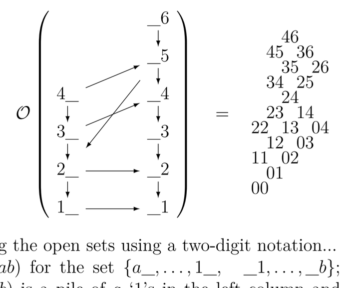


The DAGs $(\mathcal{O}(⁘), \subset_1)$ and $(\mathcal{O}(⁙), \subset_1)$ are planar (neat!), and $(\mathcal{O}(∴∵), \subset_1)$ is non-planar (clumsy!). The nonplanarity comes from ∴∵ having three independent points; the subDAG generated by the open sets of the form ${}^{?}{}_{1}{}^{?}{}_{1}{}^{?}$ is a cube.

4.1 2-Column Graphs ("2CGs")

One way to make sure that our topologies will be neat (i.e., planar) is to forbid having three independent points, and one way to do this is to declare that we are only interested in topologies in *2-columns graphs* (2CGs).

This is an example of a 2CG, and its topology:



but we are drawing the open sets using a two-digit notation...

We write $\mathsf{pile}(ab)$ for the set $\{a\_, \ldots, 1\_,\ \ \_1, \ldots, \_b\}$; the characteristic function of $\mathsf{pile}(a, b)$ is a pile of a '1's in the left column and b '1's in the right column.

We sometimes omit the 'pile's; for example, the 25 above means

$$25 \equiv \mathsf{pile}(2, 5) = \{2\_, 1\_,\ \ \_1, \_2, \_3, \_4, \_5\},$$

an open set. Note the $\mathsf{pile}(2, 1)$ is not an open set — its characteristic functions has a '1' (in the position 2_) pointing to a '0' (in the position _2).

A *specification* of a 2CG is a 4-uple (l, r, R, L); it generates a 2CG (P, A) with $P := \mathsf{pile}(lr)$, and with sets of intercolumn arrows R (going right) and L (going left). The set A of arrows of (P, A) is $R \cup L$ plus all the intracolumn arrows that point one step down. The 2CG above is generated by this specification:

$$(4, 6, \left\{\begin{smallmatrix} 4\_\to\_5, \\ 3\_\to\_4, \\ 2\_\to\_2, \\ 1\_\to\_1, \end{smallmatrix}\right\}, \{\, 2\_\leftarrow\_5 \,\})$$

In the library that I wrote to calculate with these objects that specification is represented as a string: `"46; 11 22 34 45, 25"`.

4.2 Planar Heyting Algebras ("ZHAs")

We can treat the topology above as a subset of $\mathbb{Z}^2$. Here's how.

> (x, y) means: start at $(0, 0)$, walk x steps east and y steps north.
> $\langle a, b\rangle$ means: start at $(0, 0)$, walk a steps northwest and b steps northeast — i.e., $\langle a, b\rangle = (0, 0) + a\overrightarrow{(-1, 1)} + b\overrightarrow{(1, 1)}$.
> $\mathbb{LR}$ is $\{\, \langle a, b\rangle \mid a, b \in \mathbb{N} \,\}$ — like $\mathbb{N}^2$ tilted 45° counterclockwise.
> ab is an abbreviation for $\langle a, b\rangle$.

Look at the topology above again. It is generated by a 2CG whose specification in `"46; 11 22 34 45, 25"`. The 46 is the top element of the topology, and the `"11 22 34 45"` and the `"25"` are the points where the left wall and the right wall have "bumps".

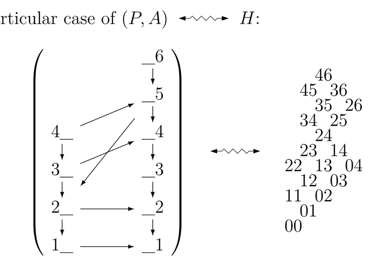
[**PH1**] defines ZHAs in sec.4 as a finite subset of $\mathbb{LR}$ "between a left wall and a right wall" (long story — see the paper), and it shows a correspondence that we write as:

$$(P, A) \leftrightsquigarrow H$$

we pronounce that as: "every 2CG (P, A) is associated to a ZHA H and vice-versa".

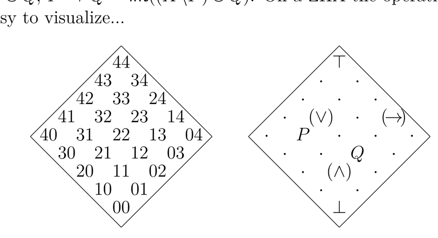
Topologies are Heyting Algebras, so the order topology $(P, \mathcal{O}_A(P))$ of a 2CG is a Heyting Algebra — and we can interpret intuitionistic propositional logic on it.

Here is a particular case of $(P, A) \leftrightsquigarrow H$:



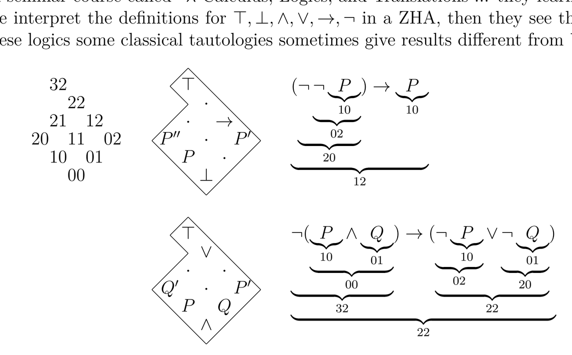
4.3 Logic in a ZHA

The standard way of interpreting logic on a topology $(X, \mathcal{O}(X))$ is: the truth values are the open sets $P, Q, R, \ldots \in \mathcal{O}(X)$; $\top = X$; $\bot = \emptyset$; $P \wedge Q = P \cap Q$; $P \vee Q = P \cup Q$; $P \to Q = \mathsf{int}((X\backslash P) \cup Q)$. On a ZHA the operations $\top, \bot, \wedge, \vee$ are very easy to visualize...



There is a way to calculate $P \to Q$ *visually*, but the formula has four subcases. For example, if P is "left of" Q then $P \to Q$ is $\mathsf{ne}(Q)$, i.e.: start from Q and walk northeast as many steps as possible. The diagram above shows that $31 \to 12 = 14$ — you can check that by calculating $\mathsf{int}((\mathsf{pile}(44)\backslash\mathsf{pile}(31)) \cup \mathsf{pile}(12))$.

By the way, this is one of the first things that I present to students in a hands-on seminar course called "λ-Calculus, Logics, and Translations"... they learn how the interpret the definitions for $\top, \bot, \wedge, \vee, \to, \neg$ in a ZHA, then they see that in these logics some classical tautologies sometimes give results different from $\top$,



and only then they learn what is a Heyting Algebra, and they check that ZHAs with these operations are Heyting Algebras, they learn topologies, etc, etc... the ordering of topics is very unusual — from particular to general — but it works!

4.4 J-operators on ZHAs

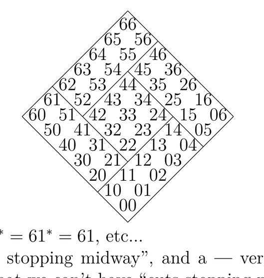
A *J-operator* on a Heyting Algebra H is a function $J : H \to H$ that obeys the axioms J1, J2, J3 below; we usually write J as $\cdot^* : \Omega \to \Omega$, and write the axioms as rules.

$$\frac{}{P \le P^*}\ \mathsf{J1} \qquad \frac{}{P^* = P^{**}}\ \mathsf{J2} \qquad \frac{}{(P \wedge Q)^* = P^* \wedge Q^*}\ \mathsf{J3}$$

> J1 says that the operation $\cdot^*$ is non-decreasing.
> J2 says that the operation $\cdot^*$ is idempotent.
> J3 is a bit mysterious but has amazing consequences.

A J-operator induces an equivalence relation: $P \sim Q$ iff $P^* = Q^*$. Let's write $[P]^J$ for the equivalence class of P.

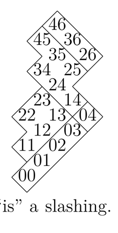
It is possible to prove that every equivalence class $[P]^J$ has a maximal element, a minimal element, and contains all the elements of H between the minimal and the maximal element, and nothing else. Also, for all elements $Q \in [P]^J$ we have that Q^* is the maximal element of $[P]^J$. So we can represent a J-operator graphically by the fences between its equivalence classes; if we interpret this as a J-operator,



then $32^* = 44^* = 44$, $30^* = 61^* = 61$, etc...

...but this has "cuts stopping midway", and a — very non-trivial — consequence of J1, J2, J3 is that we can't have "cuts stopping midway"!

Definition (informal): a *slashing* on a ZHA H is any drawing obtained by cutting H by diagonals cuts without "cuts stopping midway". This is an example of a ZHA with a slashing:



I said that every J-operator "is" a slashing. The converse is also true: every slashing "is" a J-operator.

4.5 Question marks

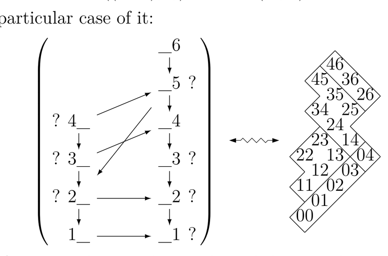
Take a 2CG (P, A), and choose a subset $Q \subseteq P$. This Q will be the *set of question marks*, and we will represent it by drawing '?'s at the side of its points. A structure $((P, A), Q)$ will be called a *2CG with question marks*.

Suppose that $(P, A) \leftrightsquigarrow H$. A set of question marks $Q \subseteq P$ induces an equivalence relation $\sim_Q$ on H: $ab \sim_Q cd$ if and only if $\mathsf{pile}(ab)$ and $\mathsf{pile}(cd)$ differ only on points of Q — or, more formally, $\mathsf{pile}(ab)\backslash Q = \mathsf{pile}(cd)\backslash Q$.

It is easy to convert a set of question marks to a slashing and vice-versa — and slashings correspond to J-operators and vice-versa. Our favourite way of writing these bijection is as:

$$((P, A), Q) \leftrightsquigarrow (H, J),$$

and here is a particular case of it:



Note that if we erase the question marks and the cuts above we get a particular case of $(P, A) \leftrightsquigarrow H$.

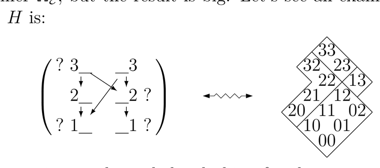
5 Toposes for children

My favourite very short *explanation* for what a topos is is:

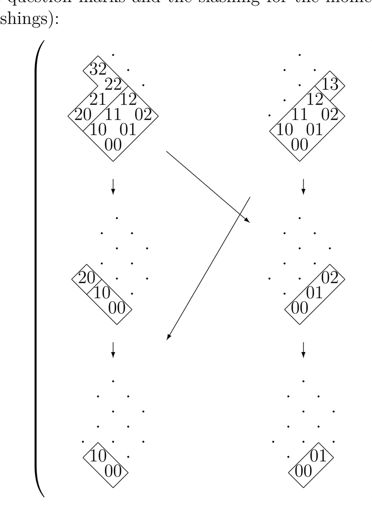
1. a Cartesian Closed Category (CCC) is a category with a terminal, binary products, and exponentials,
2. CCCs are exactly the categories in which we can interpret the simply-typed λ-calculus ("λ1"),
3. the archetypal CCC is **Set** (see [**IDARCT**]),
4. every Heyting Algebra is a CCC,
5. a topos is a CCC with pullbacks and a classifier object.
6. toposes are exactly the categories in which we can interpret a certain form of intuitionistic, typed Set Theory,
7. the archetypal topos is **Set**,
8. the subcategory $\mathrm{Sub}(1) \subset \mathcal{E}$ of a topos $\mathcal{E}$ is a Heyting Algebra — the "logic" of $\mathcal{E}$,
9. for every small category **A** the category $\mathbf{Set}^{\mathbf{A}}$ is a topos,
10. for every 2CG $(P, A) \leftrightsquigarrow H$ the category $\mathbf{Set}^{(P,A)}$ is a topos whose "logic" is H.

5.1 The classifier

Choose a 2CG (P, A). Let $\mathcal{E}$ be the topos $\mathbf{Set}^{(P,A)}$. There is an easy way to draw the classifier $\Omega_{\mathcal{E}}$, but the result is big. Let's see an example. Suppose that $(P, A) \leftrightsquigarrow H$ is:



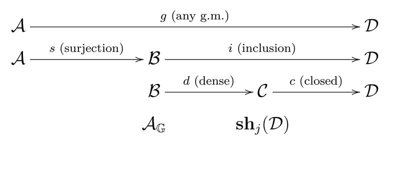
Ignore the question marks and the slashing for the moment. Then $\Omega_{\mathcal{E}}$ is (ignoring the slashings):

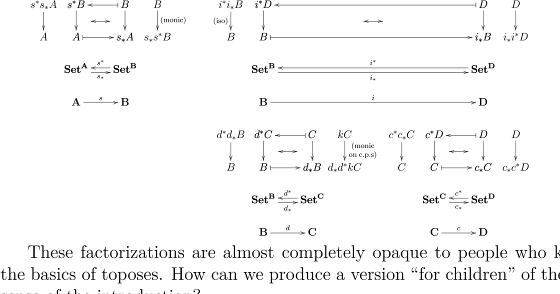


Now stop ignoring the question marks and the slashing above. We get an 2CG with question marks $((P, A), Q)$ and its associated ZHA with J-operator (H, J)...

6 The factorization

The Elephant presents in its sections A4.2 and A4.5 two factorizations of geometric morphisms that can be combined in a single diagram — see Figure **??**. An arbitrary geometry morphism $g : \mathcal{A} \to \mathcal{D}$ can be factored in an essentially unique way as a surjection followed by an inclusion ([EA4.2.10]), and an inclusion $i : \mathcal{B} \to \mathcal{D}$ can be factored in an essentially unique way as a dense g.m. followed by a closed g.m. ([EA4.5.20]). A canonical way to build these factorizations is by taking $\mathcal{B} := \mathcal{A}_{\mathbb{G}}$, where $\mathbb{G}$ is a certain comonad on $\mathcal{A}$ ([EA4.2.8]), and taking $\mathcal{C} := \mathbf{sh}_j(\mathcal{D})$, where j is a certain local operator on $\mathcal{D}$.





These factorizations are almost completely opaque to people who know just the basics of toposes. How can we produce a version "for children" of them in the sense of the introduction?

The trick is to start with geometric morphisms whose internal views can be drawn explicity — the ZGMs of section **??**. Actually we start with the lower level of Figure 2, and with the *belief* that all factorizations can be performed within ZGMs.

7 Two factorizations of ZGMs

The Elephant defines surjections by a list of equivalent conditions, and the same for inclusions and dense and closed geometric morphisms. Some of these conditions — the ones drawn in Figure **??** — are very easy to test on ZGMs.

1. $\mathbf{Set}^{\mathbf{A}} \overset{s}{\to} \mathbf{Set}^{\mathbf{B}}$ is a *surjection* when for every object $B \in \mathbf{Set}^{\mathbf{B}}$ the unit map ηB is monic ([EA4.2.6 (iv)]);
2. $\mathbf{Set}^{\mathbf{B}} \overset{i}{\to} \mathbf{Set}^{\mathbf{D}}$ is an *inclusion* when for every object $B \in \mathbf{Set}^{\mathbf{B}}$ the counit map ϵB is an iso ([EA4.2.8]);
3. $\mathbf{Set}^{\mathbf{B}} \overset{d}{\to} \mathbf{Set}^{\mathbf{C}}$ is an *dense* when for every constant presheaf kC the unit map ϵkC is a monic *[I can't find a reference for this now]*.

We also have two conditions for "dense" and "close" that are easy to state on "restrictions" in the sense of section **??** — but it's not trivial to derive them from the material in the Elephant. Let's state them anyway:

4. A restriction $\mathbf{Set}^{\mathbf{B}} \overset{d}{\to} \mathbf{Set}^{\mathbf{C}}$ is *dense* when all its question marks "have non-question marks ahead of them", i.e.: for every α in $\mathbf{C}$ such that $\alpha \in Q$ there is an arrow $\alpha \to \beta$ in $\mathbf{C}$ with $\beta \notin Q$;
5. A restriction $\mathbf{Set}^{\mathbf{C}} \overset{d}{\to} \mathbf{Set}^{\mathbf{D}}$ is *closed* when all its question marks "are at the end", i.e.: there are no arrows $\alpha \to \beta$ in $\mathbf{D}$ with $\alpha \in Q$ and $\beta \notin Q$.

We will say that a ZFunctor $\mathbf{A} \overset{s}{\to} \mathbf{B}$ *induces a surjection* when the ZGM $\mathbf{Set}^{\mathbf{A}} \overset{s}{\to} \mathbf{Set}^{\mathbf{B}}$ induced by it is a surjection; and the same for inclusion, dense, and closed.

How can we factor a ZFunctor $\mathbf{A} \overset{g}{\to} \mathbf{D}$ into ZFunctors $\mathbf{A} \overset{s}{\to} \mathbf{B} \overset{i}{\to} \mathbf{D}$ that induce a surjection and an inclusion, and how do we factor this $\mathbf{B} \overset{i}{\to} \mathbf{D}$ into $\mathbf{B} \overset{d}{\to} \mathbf{C} \overset{c}{\to} \mathbf{D}$? Here is a way to get a good part of a possible answer.

We can think that a ZFunctor $f : \mathbf{X} \to \mathbf{Y}$ does several *actions*. If we think that $\mathbf{X}$ is "before" and $\mathbf{Y}$ is "after", then f can, for example: create isolated objects, collapse isolated objects, collapse two ordered objects, create an arrow, create objects at the beginning of a connected set, create objects at the middle of a connected set, create objects at the end of a connected set... here are some examples that only do one of these actions each:

$$\begin{array}{rcl}
(1{\to}1') & \to & (1) \\
(1\quad 1') & \to & (1) \\
(1\quad 2) & \to & (1{\to}2) \\
(2) & \to & (1{\to}2) \\
(1{\to}3) & \to & (1{\to}2{\to}3) \\
(1) & \to & (1\quad 2) \\
(1) & \to & (1{\to}2)
\end{array}$$

When we take $\mathbf{A} \overset{g}{\to} \mathbf{D}$ as each one of the seven ZFunctors and we try to factor that g as $\mathbf{A} \overset{s}{\to} \mathbf{B} \overset{d}{\to} \mathbf{C} \overset{c}{\to} \mathbf{D}$ we see that the first three functors factor as $(s = g, d = \mathrm{id}, c = \mathrm{id})$, the next two as $(s = \mathrm{id}, d = g, c = \mathrm{id})$, and the last two as $(s = \mathrm{id}, d = \mathrm{id}, c = g)$. The leads to a:

Theorem. Take a ZFunctor $\mathbf{A} \overset{g}{\to} \mathbf{D}$. Factor it into $\mathbf{A} \overset{s}{\to} \mathbf{B} \overset{d}{\to} \mathbf{C} \overset{c}{\to} \mathbf{D}$ in the following way: s collapses objects and creates arrows; c creates objects at the middle and at the beginnings of connected sets; d creates objects at the ends of connected sets. Then this factorization of ZFunctors induces a surjective-dense-closed factorization of ZGMs.

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