How to almost teach Intuitionistic Logic to Discrete Mathematics Students

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http://angg.twu.net/math-b.html
http://www.logicauniversalis.org/wld
http://www.logicauniversalis.org/wld-rio-de-janeiro
"World Logic Day" — Rio de janeiro, 2019jan14

2019logicday January 14, 2019 14:35

Discrete Mathematics at PURO/UFF
I teach in a campus that UFF has in Rio das Ostras,
in the countryside of the state of Rio de Janeiro...
The entrance of the campus looks like this:

$\Leftarrow$

"PURO":
Pólo Universitário
de Rio das Ostras

Discrete Mathematics at PURO/UFF (2)
Discrete Mathematics is taught to first-semester
Computer Science students there.
Most of these students enter the university with
very little knowledge of how to use variables,
and when they attempt to spell out their ideas in Portuguese,
either speaking or writing, they are too imprecise...
They ask "can I do xxx yyy?" and if I say "yes" they
write weird atrocities in the test, and when I mark them
as wrong they blame me ("you said I could do that!");
if I say "no, you can't do that" then they do other
atrocites and blame me, too.
I don't have time to debug their Portuguese!

Discrete Mathematics at PURO/UFF (3)

...so I decided that I would stress mathetical notation —
Almost all ideas on the blackboard (whiteboard, actually)
would have examples in mathematical notation.

I structured the course of Discrete Mathematics
in three layers (presented more or less in parallel):

Layer 1: Calculating things
in a system with numbers, truth-values, sets and lists
where everything can be calculated in a finite number
of steps with almost no creativity required.

Layer 2: some infinite objects, like $\mathbb{N}$ and $\mathbb{Z}$, are allowed;
we learn how to "calculate", e.g., $\exists k \in \mathbb{Z}.10k = 23$

Layer 3: we add formal proofs.

Layer 1: Calculating things

...in a system with numbers, truth-values, sets and lists
where everything can be calculated in a finite number
of steps with almost no creativity required.

Example:

$$
\begin{aligned}
(\forall a \in \{2,3,5\}.a^2 < 10) &= (a^2 < 10)[a := 2] \land \\
&\quad (a^2 < 10)[a := 3] \land \\
&\quad (a^2 < 10)[a := 5] \\
&= (2^2 < 10) \land (3^2 < 10) \land (4^2 < 10) \\
&= (4 < 10) \land (9 < 10) \land (16 < 10) \\
&= \mathbf{T} \land \mathbf{T} \land \mathbf{F} \\
&= \mathbf{F}
\end{aligned}
$$

Note the substituion operator:
$(a^2 < 10)[a := 3] = (3^2 < 10)$.
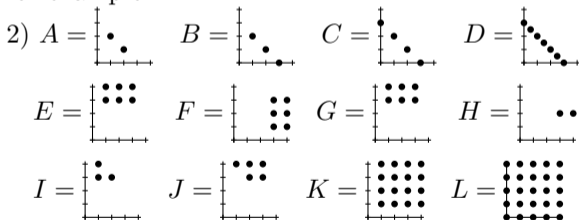
Layer 1: Set Comprehensions

I wrote a lengthy explanation of set comprehensions, using "generators", "filters" and a "result expression". The students started by learning how to calculate things like these (note the ';' instead of a '|'; these '$\{\ldots;\ldots\}$'s are calculated from left to right!):

$$\underbrace{a \in \{1,2\}}_{\text{gen}}, \underbrace{b \in \{2,3\}}_{\text{gen}}, \underbrace{a \neq b}_{\text{filt}}; \underbrace{(a,b)}_{\text{expr}}\}$$

$$= \{(1,2),(1,3),(2,3)\}$$

$$= \vdots \begin{smallmatrix} \bullet & \bullet \\ \bullet & \end{smallmatrix}$$

...then $\{\, a \in \{2,3,4\} \mid a^2 < 10 \,\}$
and $\{\, 10a + b \mid a \in \{1,2\}, b \in \{3,4\} \,\}$.

Layer 1: Set Comprehensions (2)

The "lengthy explanation of set comprehensions"
using "generators", "filters" and a "result expression", has:
2 pages of explanations,
2 pages of exercises (5+19+16+9+16+7 exercises),
1 page of answers, all using a graphical notation —
for example:

2) $A =$ $B =$ $C =$ $D =$

$E =$ $F =$ $G =$ $H =$

$I =$ $J =$ $K =$ $L =$

Positional notations

...then we adapt the graphical notation for subsets of $\mathbb{Z}^2$...
we define a convention for omitting the axes,

$$K = \left\{ \begin{array}{c} (1,3), \\ (0,2), \quad\quad (2,2), \\ (1,1), \\ (1,0) \end{array} \right\} = \quad = \quad$$

we define a positional notation for functions,

$$f = \left\{ \begin{array}{c} ((1,3),1), \\ ((0,2),0), \quad\quad ((2,2),2), \\ ((1,1),1), \\ ((1,0),1) \end{array} \right\} = \begin{array}{c} 1 \\ 0 \quad 2 \\ 1 \\ 1 \end{array}$$

and for subsets, partial functions, and characteristic functons:

$$\subset \qquad \begin{array}{c} 1 \\ \cdot \quad 2 \\ 1 \\ \cdot \end{array} \qquad \begin{array}{c} 1 \\ 0 \quad 1 \\ 1 \\ 0 \end{array}$$

Propositions

Let $W = \{0,1,2,3\} \times \{0,1,2\} = \begin{smallmatrix}\bullet\;\bullet\;\bullet\;\bullet\\\bullet\;\bullet\;\bullet\;\bullet\\\bullet\;\bullet\;\bullet\;\bullet\end{smallmatrix}$.

(Our "set of worlds").

A *proposition* on A is a function $P : A \to \{\mathbf{F}, \mathbf{T}\}$.

Let P, Q, R be these propositions on W:

$P = \{\,((x,y),z) \mid (x,y) \in W, z = (x \leq 1 \wedge y \geq 1)\,\}$

$Q = \{\,((x,y),z) \mid (x,y) \in W, z = (1 \leq x \leq 2 \wedge y \geq 1)\,\}$

$R = \{\,((x,y),z) \mid (x,y) \in W, z = (0 \leq x \leq 2 \wedge y \leq 1)\,\}$

or:

$P = P(x,y) = (x \leq 1 \wedge y \geq 1) \qquad = \begin{smallmatrix}\mathbf{T}\;\mathbf{T}\;\mathbf{F}\;\mathbf{F}\\\mathbf{T}\;\mathbf{T}\;\mathbf{F}\;\mathbf{F}\\\mathbf{F}\;\mathbf{F}\;\mathbf{F}\;\mathbf{F}\end{smallmatrix}$

$Q = Q(x,y) = (1 \leq x \leq 2 \wedge y \geq 1) \quad = \begin{smallmatrix}\mathbf{F}\;\mathbf{T}\;\mathbf{T}\;\mathbf{F}\\\mathbf{F}\;\mathbf{T}\;\mathbf{T}\;\mathbf{F}\\\mathbf{F}\;\mathbf{F}\;\mathbf{F}\;\mathbf{F}\end{smallmatrix}$

$R = R(x,y) = (0 \leq x \leq 2 \wedge y \leq 1) \quad = \begin{smallmatrix}\mathbf{F}\;\mathbf{F}\;\mathbf{F}\;\mathbf{F}\\\mathbf{T}\;\mathbf{T}\;\mathbf{T}\;\mathbf{F}\\\mathbf{T}\;\mathbf{T}\;\mathbf{T}\;\mathbf{F}\end{smallmatrix}$

Propositions (2)
In a (long) series of exercises the students learned to visualize
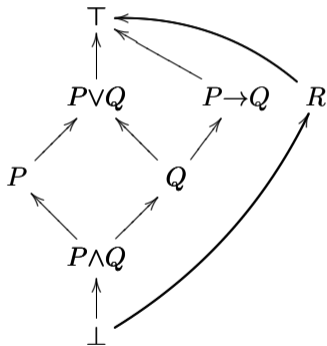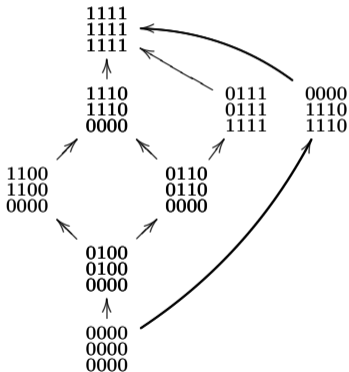these and lots of other propositions on W — actually this set
of propositions,
$\mathcal{S} = \{\top,\ \bot,\ P,\ Q,\ R,\ P \wedge Q,\ P \vee Q,\ P \rightarrow Q\}$
and I asked them to draw the Hasse diagram of the partial
order on $\mathcal{S}$. They got this:
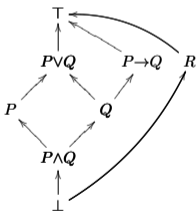
Propositions (3)
Using '0's and '1's instead of '**F**'s and '**T**'s, what they got was:

Comprehensions → contexts → sequents
The part at the left of the ';' in a '$\{\ldots;\ldots\}$'
is called the "context". For example:

$$\underbrace{\{\underbrace{a \in \{1,2\}}_{\text{gen}}, \underbrace{b \in \{2,3\}}_{\text{gen}}, \underbrace{a \neq b}_{\text{filt}}, \underbrace{c \in \{1,\ldots,a+b\}}_{\text{gen}}}_{\text{context}}; \underbrace{10a+c\}}_{\text{expr}}$$

The *set of possible values* for this context is:

$$\underbrace{\{\underbrace{a \in \{1,2\}}_{\text{gen}}, \underbrace{b \in \{2,3\}}_{\text{gen}}, \underbrace{a \neq b}_{\text{filt}}, \underbrace{c \in \{1,\ldots,a+b\}}_{\text{gen}}}_{\text{context}}; \underbrace{(a,b,c)\}}_{\text{expr}}$$

Comprehensions → contexts → sequents
This is surprisingly similar to contexts in sequents!

$$\underbrace{\underbrace{a \in \mathbb{Z}}_{\text{gen}}, \underbrace{b \in \{1, 2, 3\}}_{\text{gen}}, \underbrace{\mathsf{prime}(4a + b)}_{\text{filt}} \vdash b = 3}_{\text{context}}$$

The sequent above can be seen as a (false!) proposition.
We used this in the course to debug proofs.
Our formal proofs were written as series of numbered lines,
each line starting by either "Suppose" of "Then".
Each "Then" line had an associated sequent —
its context was made of all the open "Suppose"s.
For each "Then" line in a valid proof its associated sequent
had to be a true proposition.

Part 2: a course
about λ-Calculus,
Logic and Categories,
with almost no prerequisites

See: http://angg.twu.net/math-b.html#lclt

λ-Calculus, Logics and Translations
In the semesters 2016.1, 2016.2, 2017.1, 2017.2, and 2018.1
I gave a hands-on seminar course (an "optativa" in Portuguese)
called "λ-Calculus, Logics and Translations" —
"LCLT" from here on.
It was all based on exercises.
I organized it to be as beginner-friendly as possible,
and I expected it to have some Psychology students
at least in the first classes, but they never came
(even though they said they would)...

Evaluating expressions
It started as this. Suppose that we forget all the algebra
we know; we only know how to calculate expressions step-by-
step by adding, subtracting, or multiplying two *numbers*.
Then there are several paths from any initial expression
to its final result, and we can draw that as a directed graph
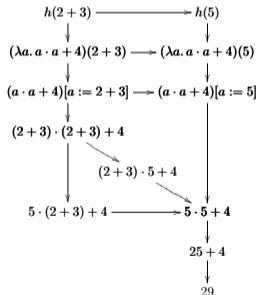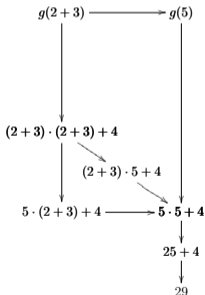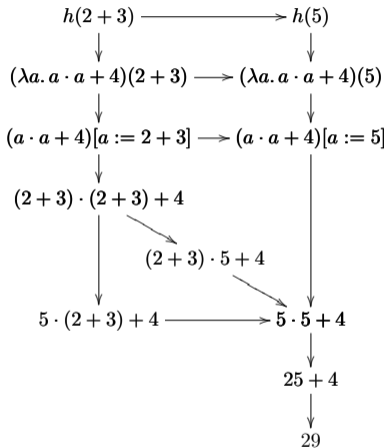where each arrow $\alpha \to \beta$ means a "one-step reduction"...
Example:

$$2 \cdot 3 + 4 \cdot 5 \longrightarrow 2 \cdot 3 + 20$$

$$6 + 4 \cdot 5 \longrightarrow 6 + 20 \longrightarrow 26$$

Evaluating expressions (2)
Then I added rules for evaluating
a named function $g(a) = a \cdot a + 4$,
an unnamed function $\lambda a.\, a \cdot a + 4$, and
I defined $h = \lambda a.\, a \cdot a + 4$...

$$g(2+3) \longrightarrow g(5)$$

$$(2+3) \cdot (2+3) + 4$$

$$(2+3) \cdot 5 + 4$$

$$5 \cdot (2+3) + 4 \longrightarrow \mathbf{5 \cdot 5 + 4}$$

$$25 + 4$$

$$29$$

$$h(2+3) \longrightarrow h(5)$$

$$(\lambda a.\, a \cdot a + 4)(2+3) \longrightarrow (\lambda a.\, a \cdot a + 4)(5)$$

$$(a \cdot a + 4)[a := 2+3] \longrightarrow (a \cdot a + 4)[a := 5]$$

$$(2+3) \cdot (2+3) + 4$$

$$(2+3) \cdot 5 + 4$$

$$5 \cdot (2+3) + 4 \longrightarrow \mathbf{5 \cdot 5 + 4}$$

$$25 + 4$$

$$29$$

$g(2+3) \longrightarrow g(5)$

$h(2+3) \longrightarrow h(5)$

$(\lambda a.\, a \cdot a + 4)(2+3) \longrightarrow (\lambda a.\, a \cdot a + 4)(5)$

$(a \cdot a + 4)[a := 2+3] \longrightarrow (a \cdot a + 4)[a := 5]$

$(2+3) \cdot (2+3) + 4$

$(2+3) \cdot 5 + 4$

$(2+3) \cdot (2+3) + 4$

$(2+3) \cdot 5 + 4$

$5 \cdot (2+3) + 4 \longrightarrow 5 \cdot 5 + 4$

$5 \cdot (2+3) + 4 \longrightarrow 5 \cdot 5 + 4$
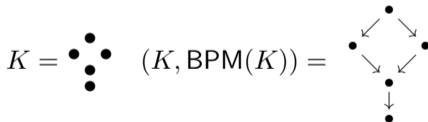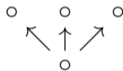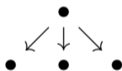
$25 + 4$

$25 + 4$

29

29
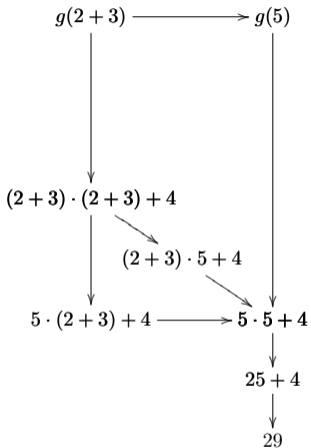
Directed graphs
Only then I introduced the concept of directed graph,
and how DGs are represented formally in Mathematics
as pairs of the form (points, arrows)...
And I introduced two ways to convert subsets of $\mathbb{Z}^2$ to
graphs: adding black (or white) pawns moves...

$$K = \quad (K, \mathsf{BPM}(K)) =$$

Wet paint

A *ZSet* is a finite subset of $\mathbb{Z}^2$. (Modulo a detail)

A *ZDAG* is a ZSet plus its black (or white) pawns moves.

Let A be a ZSet. Let B be a subset of A.

We will represent B as the characteristic function of $B \subseteq A$;

for example, $B = {}_{0}^{1}{}_{1}^{}{}^{1}_{0}$.

Imagine that the '1's are painted with wet black paint,

that can flow down through the arrows (black pawns moves).

If there is an arrow $1 \to 0$ then black paint will flow

from the 1 to the 0 and will paint the 0 black.

This is unstable: ${}_{0}^{1}{}_{1}^{}{}^{1}_{0}$ — it changes when the paint flows.

This is stable: ${}_{0}^{0}{}_{1}^{}{}^{1}_{1}$ — it doesn't change.

Stable subsets

I gave the students a ZSet (a finite subset of $\mathbb{Z}^2$), for example, $H = \vcenter{\hbox{$\begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\end{smallmatrix}$}}$, and I asked them to find all stable subsets of it... Notation:

$\mathcal{P}(H)$ is the set of all subsets of H,

$\mathcal{O}(H)$ is the set of all stable subsets of H.

We have $\mathcal{O}(H) =$

$\left\{ {}_0^0\,{}_0^0,\; {}_0^0\,{}_1^0,\; {}_0^0\,{}_1^0,\; {}_1^0\,{}_0^0,\; {}_0^0\,{}_1^0,\; {}_1^0\,{}_1^0,\; {}_1^0\,{}_0^0,\; {}_1^0\,{}_1^0,\; {}_1^0\,{}_1^0,\; {}_1^1\,{}_1^1, \right\}$

Stable subsets: partial order
Then I asked them to draw the partial order on $\mathcal{O}(H)$,
with $\begin{smallmatrix}1&1&1\\1& &1\end{smallmatrix}$ at the top, and an arrow $\alpha \to \beta$ when
β is α with one '0' changed to a '1'...

$$(H, \mathsf{BPM}(H)) = \qquad (\mathcal{O}(H), \subset_1) =$$

$$(\mathcal{O}(H), \subset_1) =$$

The Hasse diagram, where each node is written as a value c over a 2×2 block $\begin{smallmatrix}a&b\\d&e\end{smallmatrix}$:

- Top: $\overset{1}{\begin{smallmatrix}1&1\\1&1\end{smallmatrix}}$

- $\overset{0}{\begin{smallmatrix}1&1\\1&1\end{smallmatrix}}$ (covered by the top node)

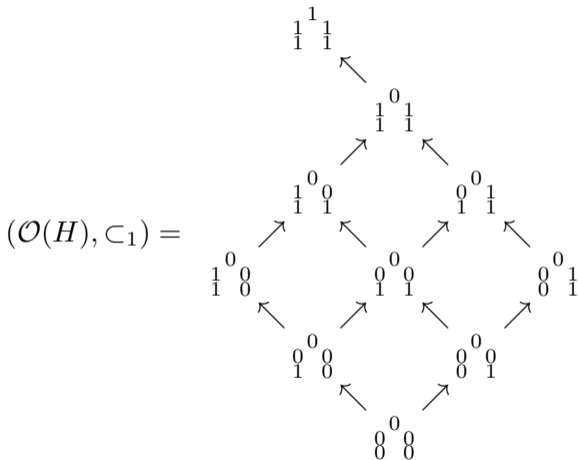- $\overset{0}{\begin{smallmatrix}1&0\\1&1\end{smallmatrix}}$ and $\overset{0}{\begin{smallmatrix}0&1\\1&1\end{smallmatrix}}$ (both covered by $\overset{0}{\begin{smallmatrix}1&1\\1&1\end{smallmatrix}}$)

- $\overset{0}{\begin{smallmatrix}1&0\\1&0\end{smallmatrix}}$, $\overset{0}{\begin{smallmatrix}0&0\\1&1\end{smallmatrix}}$, $\overset{0}{\begin{smallmatrix}0&1\\0&1\end{smallmatrix}}$

- $\overset{0}{\begin{smallmatrix}0&0\\1&0\end{smallmatrix}}$ and $\overset{0}{\begin{smallmatrix}0&0\\0&1\end{smallmatrix}}$

- Bottom: $\overset{0}{\begin{smallmatrix}0&0\\0&0\end{smallmatrix}}$

with covering relations indicated by the arrows:

$$\overset{0}{\begin{smallmatrix}0&0\\0&0\end{smallmatrix}} \longrightarrow \overset{0}{\begin{smallmatrix}0&0\\1&0\end{smallmatrix}},\ \overset{0}{\begin{smallmatrix}0&0\\0&1\end{smallmatrix}} \longrightarrow \overset{0}{\begin{smallmatrix}1&0\\1&0\end{smallmatrix}},\ \overset{0}{\begin{smallmatrix}0&0\\1&1\end{smallmatrix}},\ \overset{0}{\begin{smallmatrix}0&1\\0&1\end{smallmatrix}} \longrightarrow \overset{0}{\begin{smallmatrix}1&0\\1&1\end{smallmatrix}},\ \overset{0}{\begin{smallmatrix}0&1\\1&1\end{smallmatrix}} \longrightarrow \overset{0}{\begin{smallmatrix}1&1\\1&1\end{smallmatrix}} \longrightarrow \overset{1}{\begin{smallmatrix}1&1\\1&1\end{smallmatrix}}$$

Order topologies

In standard terms, $\mathcal{O}(W)$ is the *order topology* on W.
More gerally, let (P, A) be a directed graph; $A \subseteq P \times P$.
Each arrow $p \to q$ in A, i.e., $(p, q) \in A$, can be seen as a
condition on open sets: if an open set contains p then it
has to contain q.

$(P, A) = (\{1, 2, 3\}, \left\{ \begin{smallmatrix} (1,3), \\ (2,3) \end{smallmatrix} \right\})$

$\Rightarrow \ \mathcal{O}_A(P) = \{\, U \subseteq P \mid \left(\begin{smallmatrix} 1 \in U \to 3 \in U \\ 2 \in U \to 3 \in U \end{smallmatrix} \ \wedge \ \right) \}$

Notation:
If W is a ZSet,
$\mathcal{O}(W) = \mathcal{O}_{\mathsf{BPM}(W)}(W)$
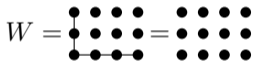The "set of stable subsets of W" is a topology.

Topologies are Heyting Algebras

We can interpret Intuituionistic Propositional Logic ("IPL")
on topologies: $\top, \bot, \wedge, \vee, \rightarrow, \neg, \leftrightarrow$.
Remember — from the big exercise for MD students —
that when our set of worlds W is

$$W = \begin{smallmatrix} \bullet\ \bullet\ \bullet\ \bullet \\ \bullet\ \bullet\ \bullet\ \bullet \\ \bullet\ \bullet\ \bullet\ \bullet \end{smallmatrix} = \begin{smallmatrix} \bullet\ \bullet\ \bullet\ \bullet \\ \bullet\ \bullet\ \bullet\ \bullet \\ \bullet\ \bullet\ \bullet\ \bullet \end{smallmatrix}$$

then we can represent propositions by the
set of worlds in which they're true,
and we use characteristic functions to draw them...

$$P = \begin{smallmatrix} 1100 \\ 1100 \\ 0000 \end{smallmatrix} \quad Q = \begin{smallmatrix} 0110 \\ 0110 \\ 0000 \end{smallmatrix} \quad P \wedge Q = \begin{smallmatrix} 0100 \\ 0100 \\ 0000 \end{smallmatrix}$$

Propositional Logic on a topology

How do we interpret Intuitionistic Propositional Logic
on a topology? Conjunction an disjunction are easy...

If $P = \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & 1 & 1 \end{smallmatrix}$ and $Q = \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 0 & 1 \end{smallmatrix}$ then

$P \wedge Q = \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & 0 & 1 \end{smallmatrix}$ and

$P \vee Q = \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & 1 \end{smallmatrix}$...

But $P \to Q$ is problematic...

IPL on a topology (2)

$$P \to Q \;=\; \neg \underbrace{P}_{\substack{0\ \ 0\\1\ \ 1}} \vee \underbrace{Q}_{\substack{0\ \ 0\\0\ \ 1}} \;=\; {\color{red}\substack{\ 1\\0\ \ 1}}, \text{ not stable/open!}$$

$$\underbrace{}_{\color{red}\substack{\ 1\\0\ \ 0}}$$

$$\underbrace{}_{\color{red}\substack{\ 1\\0\ \ 1}}$$

The fix is totally artificial.
We distinguish modal operations (easy)
from intuitionistic operations (hard — use interior).

IPL on a topology (3)

$$\neg_{\mathsf{M}} P \;:=\; \top \backslash P$$
$$P \to_{\mathsf{M}} Q \;:=\; \neg_M P \vee Q$$
$$
\begin{aligned}
P \to Q \;&:=\; P \to_{\mathsf{I}} Q \\
&:=\; \mathsf{Int}(P \to_{\mathsf{M}} Q) \\
&=\; \mathsf{Int}(\neg_{\mathsf{M}} P \vee Q) \\
&=\; \mathsf{Int}((\top \backslash P) \vee Q)
\end{aligned}
$$
$$
\begin{aligned}
\neg P \;&:=\; P \to \bot \\
&=\; P \to_I \bot \\
&=\; \mathsf{Int}(P \to_{\mathsf{M}} \bot) \\
&=\; \mathsf{Int}(\neg_M P \vee \bot) \\
&=\; \mathsf{Int}(\neg_M P) \\
&=\; \mathsf{Int}(\top \backslash P)
\end{aligned}
$$

$$\begin{smallmatrix}0&0\\1&1\end{smallmatrix} \to \begin{smallmatrix}0&0\\0&1\end{smallmatrix} = \mathsf{Int}\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix} \vee \begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right) = \mathsf{Int}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) = \begin{smallmatrix}0&0\\0&1\end{smallmatrix}$$

How do we visualize the intuitionistic '→'?
How do we visualize how the
intuitionistic implication works?
⊤ and ⊥ are easy: the top and the bottom elements,
∧ and ∨ are easy: "meet" and "join" in the lattice.
We need a more convenient way to draw our topologies.
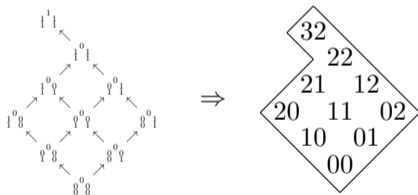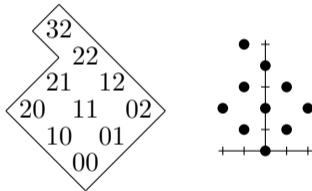

Each open set will be written as two digits.

LR-coordinates

(x, y) means: start at $(0,0)$, walk x steps E, y steps N.

$\langle l, r \rangle$ means: start at $(0,0)$, walk l steps NW, r steps NE.

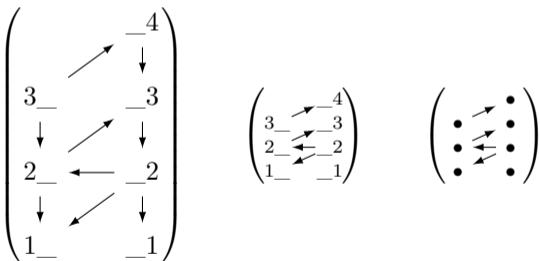We abbreviate $\langle l, r \rangle$ as lr.

Check:

$20 = \langle 2, 0 \rangle = (0,0) + 2\overrightarrow{(-1,1)} + 0\overrightarrow{(1,1)} = (-2, 2)$

$32 = \langle 2, 0 \rangle = (0,0) + 3\overrightarrow{(-1,1)} + 2\overrightarrow{(1,1)} = (-1, 5)$

2-column graphs
A 2-column graph ("2CG") has two columns of points,
all vertical arrows pointing one step down, and
some intercolumn arrows.
An example:

$$
\begin{pmatrix}
 & \nearrow & \_4 \\
 & & \downarrow \\
3\_ & & \_3 \\
\downarrow & \nearrow & \downarrow \\
2\_ & \longleftarrow & \_2 \\
\downarrow & \swarrow & \downarrow \\
1\_ & & \_1
\end{pmatrix}
\qquad
\begin{pmatrix}
 & \nearrow \_4 \\
3\_ \nearrow \_3 \\
2\_ \twoheadleftarrow \_2 \\
1\_ \nearrow \_1
\end{pmatrix}
\qquad
\begin{pmatrix}
\bullet & \nearrow & \bullet \\
\bullet & \nearrow & \bullet \\
\bullet & \twoheadleftarrow & \bullet \\
\bullet & \nearrow & \bullet
\end{pmatrix}
$$

2-column graphs (2)

The operation 2CG builds a 2-column graph formally from
the height of its left column,
the height of its right column,
the intercolumn arrows going right,
the intercolumn arrows going left.

$$2CG(3, 4, \left\{ \begin{smallmatrix} 3\_\rightarrow\_4, \\ 2\_\rightarrow\_3 \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} 2\_\leftarrow\_2, \\ 1\_\leftarrow\_2 \end{smallmatrix} \right\})$$

$$= \left(\left\{ \begin{smallmatrix} 3\_, 2\_, 1\_, \\ \_4, \_3, \_2, \_1 \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} 3\_\rightarrow2\_, 2\_\rightarrow1\_, \\ \_4\rightarrow\_3, \_3\rightarrow\_2, \_2\rightarrow\_1, \\ 3\_\rightarrow\_4, 2\_\rightarrow\_3, \\ 2\_\leftarrow\_2, 1\_\leftarrow\_2 \end{smallmatrix} \right\} \right)$$

$$\begin{pmatrix} & & \_4 \\ 3\_ & \nearrow & \_3 \\ 2\_ & \nwarrow & \_2 \\ 1\_ & \swarrow & \_1 \end{pmatrix} \qquad \begin{pmatrix} & & \bullet \\ \bullet & \nearrow & \bullet \\ \bullet & \nwarrow & \bullet \\ \bullet & & \bullet \end{pmatrix}$$

2CGs and piles

$2\mathsf{CG}(3, 4, \{ \begin{smallmatrix} 3\_\to\_4, \\ 2\_\to\_3 \end{smallmatrix} \}, \{ \begin{smallmatrix} 2\_\leftarrow\_2, \\ 1\_\leftarrow\_2 \end{smallmatrix} \}) = (\{ \begin{smallmatrix} 3\_, 2\_, 1\_, \\ \_4, \_3, \_2, \_1 \end{smallmatrix} \}, \{ \cdots \})$

$\mathsf{pile}(3, 4) = \{ \begin{smallmatrix} 3\_, 2\_, 1\_, \\ \_4, \_3, \_2, \_1 \end{smallmatrix} \} = \begin{pmatrix} & & 1 \\ 1 & \nearrow & 1 \\ 1 & \rightleftarrows & 1 \\ 1 & \swarrow & 1 \end{pmatrix}$

$\mathsf{pile}(2, 1) = \{ \qquad \begin{smallmatrix} 2\_, 1\_, \\ \_1 \end{smallmatrix} \} = \begin{pmatrix} & & 0 \\ 0 & \nearrow & 0 \\ 1 & \rightleftarrows & 0 \\ 1 & \swarrow & 1 \end{pmatrix}$

All open sets of a 2CG are piles,

but some piles are not open sets...

for example, $\mathsf{pile}(2, 1)$ is not open — it has 2_ but not _3.

2CGs and piles

Let $(P, A) = \mathsf{2CG}(3, 4, \left\{ \begin{smallmatrix} 3\_\to\_4, \\ 2\_\to\_3 \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} 2\_\leftarrow\_2, \\ 1\_\leftarrow\_2 \end{smallmatrix} \right\})$

$$= \begin{pmatrix} 3\_\nearrow^{\;\;\_4} \\ {}^{\;\;\_3} \\ 2\_\nwarrow_{\searrow\_2} \\ 1\_\nearrow\_1 \end{pmatrix} .$$

Then $\mathcal{O}_A(P) =$


```
        34
          24
        23
      22  13
    21  12
      11
    10  01
      00
```

Can you visualize $\mathcal{O}(\mathbb{R})$? I find that very hard!

2CGs and implication

We are using

$$(P, A) = \begin{pmatrix} {}_{3}\!\!-\!\!\nearrow\!\!-\!\!^{4} \\ {}_{3}\!\!-\!\!\nearrow\!\!-\!\!_{3} \\ {}_{2}\!\!-\!\!\longleftarrow\!\!-\!\!_{2} \\ {}_{1}\!\!-\!\!\swarrow\!\!-\!\!_{1} \end{pmatrix}, \quad \mathcal{O}_P(A) = \begin{array}{c} 34 \\ 24 \\ 23 \\ 22 \quad 13 \\ 21 \quad 12 \\ 11 \\ 10 \quad 01 \\ 00 \end{array}$$

Then $21 \to 12 = \begin{pmatrix} 0\!\!\nearrow\!\!0 \\ 0\!\!\nearrow\!\!0 \\ 1\!\!\swarrow\!\!0 \\ 1\!\!\swarrow\!\!1 \end{pmatrix} \to \begin{pmatrix} 0\!\!\nearrow\!\!0 \\ 0\!\!\nearrow\!\!1 \\ 0\!\!\swarrow\!\!1 \\ 1\!\!\swarrow\!\!1 \end{pmatrix} = \mathsf{Int}(\dots)\dots$

2CGs and implication (2)

Then $21 \to 12 = \begin{pmatrix} & \nearrow & 0 \\ 0 & \nearrow & 0 \\ 1 & \rightleftarrows & 0 \\ 1 & \nwarrow & 1 \end{pmatrix} \to \begin{pmatrix} & \nearrow & 0 \\ 0 & \nearrow & 0 \\ 0 & \rightleftarrows & 1 \\ 1 & \nwarrow & 1 \end{pmatrix}$

$= \mathsf{Int}(\begin{pmatrix} & \nearrow & 1 \\ 1 & \nearrow & 1 \\ 0 & \rightleftarrows & 1 \\ 0 & \nwarrow & 0 \end{pmatrix} \vee \begin{pmatrix} & \nearrow & 0 \\ 0 & \nearrow & 0 \\ 0 & \rightleftarrows & 1 \\ 1 & \nwarrow & 1 \end{pmatrix})$

$= \mathsf{Int}(\begin{pmatrix} & \nearrow & 1 \\ 1 & \nearrow & 1 \\ 0 & \rightleftarrows & 1 \\ 1 & \nwarrow & 1 \end{pmatrix})$

$= \begin{pmatrix} & \nearrow & 0 \\ 0 & \nearrow & 1 \\ 0 & \rightleftarrows & 1 \\ 1 & \nwarrow & 1 \end{pmatrix} = 13$

2CGs and implication (3)

In $(P, A) = \begin{pmatrix} 3 & \nearrow & 4 \\ 3 & \nearrow & 3 \\ 2 & \rightleftarrows & 2 \\ 1 & \swarrow & 1 \end{pmatrix}$, $\mathcal{O}_P(A) = \begin{array}{c} 34 \\ 24 \\ 23 \\ 22 \quad 13 \\ 21 \quad 12 \\ 11 \\ 10 \quad 01 \\ 00 \end{array}$,

$21 \to 12 = 13$.

Back to LCLT
In LCLT I followed a different path.
The students learned first a lot of λ-calculus
(it's useful for them, they're from CompSci!)
then a bit of Curry-Howard,
then topologies, then Heyting Algebras,
and we used 2CGs and finite,
planar Heyting Algebras ("ZHAs")
to develop a lot of intuition about IPL —
we saw several classical tautologies that are
falsifiable in ZHAs.
Then we saw Categories and Cartesian Closed
Categories ("CCCs"), and how using Int to
calculate '$\rightarrow$' corresponds to an adjunction.

Where do we go from here?
This is the abstract that I submitted to EBL 2019:

Five applications of the "Logic for Children" project to Category Theory
Category Theory is usually presented in a way that is too abstract, with concrete examples of each given structure being mentioned briefly, if at all. One of the themes of the "Logic for Children" workshop, held in the UNILOG 2018, was a set of tools and techniques for drawing diagrams of categorical concepts in a canonical shape, and for drawing diagrams of particular cases of those concepts in essentialy the same shape as the general case; these diagrams for a general and a particular case can be draw side by side "in parallel" in a way that lets us transfer knowledge from the particular case to the

general, and back.

In this talk we will present briefly five applications of these techniques: 1) a way to visualize planar, finite Heyting Algebras — we call them "ZHAs" — and to develop a feeling for how the logic connectives in a ZHA work; 2) a way to build a topos with a given logic (when that "logic" is a ZHA); 3) a way to represent a closure operator on a ZHA by a "slashing on that ZHA by diagonal cuts with no cuts stopping midway"; 4) a way to extend a slashing on a ZHA H to a "notion of sheafness" on the associated topos; 5) a way to start from a certain very abstract factorization of geometric morphisms between toposes, described in Peter Johnstone's "Sketches of an Elephant", and derive some intuitive meaning for what that factorization "means": basically, we draw the diagrams, plug

in it some very simple geometric morphisms, and check which ones the factorization classifies as "surjections", "inclusions", "closed", and "dense".

A topology that is not planar

$$
\begin{array}{c}
1_1 1_1 1^1 \\
\nearrow \quad \uparrow \quad \nwarrow \\
1_1 1_1 0 \quad 1_1 0_1 1 \quad 0_1 1_1 1 \\
\uparrow \quad \rtimes\!\!\!\ltimes \quad\quad \rtimes\!\!\!\ltimes \quad \uparrow \\
1_1 0_1 0 \quad 0_1 1_1 0 \quad 0_1 0_1 1 \\
(\mathcal{O}(W), \subset_1) = \qquad \nearrow \quad \nwarrow \ \uparrow \ \nearrow \quad\quad \nwarrow \\
1_1 0_0 0 \qquad 0_1 0_1 0 \qquad 0_0 0_1 1 \\
\nwarrow \quad \nearrow \quad \nwarrow \quad \nearrow \\
0_1 0_0 0 \quad\quad 0_0 0_1 0 \\
\nwarrow \quad \nearrow \\
0_0 0_0 0
\end{array}
$$