A diagram for the Yoneda Lemma

(In which each node and arrow can be interpreted precisely as a "term", and most of the interpretations are "obvious"; plus dictionaries!!!)

Eduardo Ochs (UFF, Rio das Ostras, Brazil)
http://angg.twu.net/#intro-tys-lfc

$$\begin{tikzcd}
 & A \arrow[d, "\gamma", ""{name=G}] \\
C \arrow[r, mapsto] & RC \\
(C\to\_) \arrow[r, "T"', ""{name=T}] & (A\to R\_)
\arrow[from=T, to=G, bend left, <->]
\end{tikzcd}$$

$T_0 := \lambda D.\lambda g.(\gamma; Rg)$
$\gamma := TC(\mathrm{id}_C)$

Three Yoneda Lemmas

These are our diagrams for three “Yoneda Lemmas”.

$$\begin{tikzcd}
& A \arrow[d, "\gamma"] \\
C \arrow[r, mapsto] & RC \\
(C{\to}\_) \arrow[r, "T"] & (A{\to}R\_)
\end{tikzcd}$$

$$\begin{tikzcd}
& 1 \arrow[d, "\gamma"] \\
C \arrow[r, mapsto] & RC \\
(C{\to}\_) \arrow[r, "T"] \arrow[dr, "T'"'] & (1{\to}R\_) \arrow[d, leftrightarrow] \\
& R
\end{tikzcd}$$

$$\begin{tikzcd}
& 1 \arrow[d, "\gamma"] \\
C \arrow[r, mapsto] & (B{\to}\,C) \\
(C{\to}\_) \arrow[r, "T"] \arrow[dr, "T'"'] & (1{\to}(B{\to}\_)) \arrow[d, leftrightarrow] \\
& (B{\to}\_)
\end{tikzcd}$$

In all cases we have bijections between ‘γ’s and ‘T’s.
The ‘γ’s are morphisms, the ‘T’s are natural transformations.
Right: the most concrete and familiar YL, a bijection
$\mathrm{Hom}(B, C) \leftrightarrow \mathrm{Nat}((C{\to}\_), (B{\to}\_))$.
Left: the most abstract YL.

Three Yoneda Lemmas (and their names)

$$\begin{tikzcd}
 & A \arrow[d, "\gamma"] \\
C \arrow[r, mapsto] & RC
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 1 \arrow[d, "\gamma"] \\
C \arrow[r, mapsto] & RC
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 1 \arrow[d, "\gamma"] \\
C \arrow[r, mapsto] & (B \to C)
\end{tikzcd}$$

$$\begin{tikzcd}
(C \to \_) \arrow[r, "T"] & (A \to R\_)
\end{tikzcd}
\qquad
\begin{tikzcd}
(C \to \_) \arrow[r, "T"] \arrow[dr, "T'"'] & (1 \to R\_) \arrow[d, leftrightarrow] \\
 & R
\end{tikzcd}
\qquad
\begin{tikzcd}
(C \to \_) \arrow[r, "T"] \arrow[dr, "T'"'] & (1 \to (B \to \_)) \arrow[d, leftrightarrow] \\
 & (B \to \_)
\end{tikzcd}$$

Left: an (obscure?) lemma from adjunctions.
Middle: the Yoneda Lemma.
Right: the Yoneda Embedding.

Left: $\mathrm{Hom}(A, RC) \cong \mathrm{Nat}(\mathrm{Hom}(C, -), \mathrm{Hom}(A, R-))$
Middle: $RC \cong \mathrm{Nat}(\mathrm{Hom}(C, -), R)$
Right: $\mathrm{Hom}(B, C) \cong \mathrm{Nat}(\mathrm{Hom}(C, -), \mathrm{Hom}(B, -))$

Three Yoneda Lemmas, co- and contravariant

We can modify each of our three Yoneda lemmas to get a covariant version of it... for example, the Yoneda embedding that we just saw says that the contravariant functor $X \mapsto (X {\to} \_)$ at the left below is full and faithful...

$$\begin{tikzcd}
C \arrow[r, mapsto] \arrow[dr, phantom, "\leftrightarrow"] & (C {\to} \_) \arrow[d] \\
B \arrow[u] \arrow[r, mapsto] & (B {\to} \_)
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, mapsto] \arrow[d] \arrow[dr, phantom, "\leftrightarrow"] & (\_ {\to} C) \arrow[d] \\
D \arrow[r, mapsto] & (\_ {\to} D)
\end{tikzcd}$$

$$\mathbf{C}^{\mathrm{op}} \xrightarrow{X \mapsto (X {\to} \_)} \mathbf{Set}^{\mathbf{C}} \qquad \mathbf{C} \xrightarrow{Y \mapsto (\_ {\to} Y)} \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$$

It is also possible to show that the covariant functor $Y \mapsto (\_ {\to} Y)$ at the right is full and faithful.

Next slides: the six lemmas, in my notation and in Riehl's.

$$\begin{tikzcd}
& A \arrow[d, "\gamma"] \\
C \arrow[r, mapsto] & RC \\
\mathbf{C} \arrow[r, "R"] & \mathbf{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
& 1 \arrow[d, "\gamma"] \\
C \arrow[r, mapsto] & RC \\
\mathbf{C} \arrow[r, "R"] & \mathbf{Set}
\end{tikzcd}
\qquad
\begin{tikzcd}
& 1 \arrow[d, "\gamma"] \\
C \arrow[r, mapsto] & (B\to C) \\
\mathbf{C} \arrow[r, "(B\to\_)"] & \mathbf{Set}
\end{tikzcd}$$

$$\begin{tikzcd}
(C\to\_) \arrow[r, "T"] & (A\to R\_)
\end{tikzcd}
\qquad
\begin{tikzcd}
(C\to\_) \arrow[r, "T"] \arrow[dr, "T'"'] & (1\to R\_) \arrow[d, leftrightarrow] \\
& R
\end{tikzcd}
\qquad
\begin{tikzcd}
(C\to\_) \arrow[r, "T"] \arrow[dr, "T'"'] & (1\to(B\to\_)) \arrow[d, leftrightarrow] \\
& (B\to\_)
\end{tikzcd}$$

$$\begin{tikzcd}
& A \arrow[d, "\gamma"] \\
C \arrow[r, mapsto] & FC \\
\mathbf{C}^{\mathrm{op}} \arrow[r, "F"] & \mathbf{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
& 1 \arrow[d, "\gamma"] \\
C \arrow[r, mapsto] & FC \\
\mathbf{C}^{\mathrm{op}} \arrow[r, "F"] & \mathbf{Set}
\end{tikzcd}
\qquad
\begin{tikzcd}
& 1 \arrow[d, "\gamma"] \\
C \arrow[r, mapsto] & (C\to D) \\
\mathbf{C}^{\mathrm{op}} \arrow[r, "(\_\to D)"] & \mathbf{Set}
\end{tikzcd}$$

$$\begin{tikzcd}
(\_\to C) \arrow[r, "T"] & (A\to F\_)
\end{tikzcd}
\qquad
\begin{tikzcd}
(\_\to C) \arrow[r, "T"] \arrow[dr, "T'"'] & (1\to F\_) \arrow[d, leftrightarrow] \\
& F
\end{tikzcd}
\qquad
\begin{tikzcd}
(\_\to C) \arrow[r, "T"] \arrow[dr, "T'"'] & (1\to(\_\to D)) \arrow[d, leftrightarrow] \\
& (\_\to D)
\end{tikzcd}$$

TODO:
Translate the diagrams from the previous slide to the notation in Section 2.2 of Emily Riehl's "Categories in Context"...

http://www.math.jhu.edu/~eriehl/context.pdf

The first (most abstract) Yoneda Lemma
Choose (locally small) categories $\mathbf{A}$ and $\mathbf{C}$, objects $A \in \mathbf{A}$ and $C \in \mathbf{C}$, a functor $R : \mathbf{C} \to \mathbf{A}$, and a morphism $\gamma : A \to RC$.

$$\begin{tikzcd}
 & A \arrow[d, "\gamma"] \\
C \arrow[r, mapsto] & RC
\end{tikzcd}$$

$$(C{\to}\_) \xrightarrow{T} (A{\to}R\_)$$

We need to understand the functors $(C \to \_) : \mathbf{C} \to \mathbf{Set}$ and $(A \to R\_) : \mathbf{C} \to \mathbf{Set}$ and see how the morphism $\gamma : A \to RC$ induces a natural transformation T...

The two functors: internal views

To understand the functors $(C{\to}\_)$ and $(A{\to}R\_)$ we
1) draw an auxiliar diagram (left),
2) draw their internal views (middle, right).

$$\begin{tikzcd}
 & A \arrow[d, "\gamma"] \arrow[dd, bend left, "\delta"] \\
C \arrow[r, mapsto] \arrow[d, "g"'] & RC \arrow[d, "Rg"] \\
D \arrow[r, mapsto] \arrow[d, "h"'] & RD \arrow[d, "Rh"] \\
E \arrow[r, mapsto] & RE \\
\mathbf{C} \arrow[r, "R"] & \mathbf{A}
\end{tikzcd}$$

$$\begin{tikzcd}
D \arrow[r, mapsto] \arrow[d, "h"'] & (C{\to}D) \arrow[d, "\lambda g.(g;h)"] & g \arrow[d, mapsto] \\
E \arrow[r, mapsto] & (C{\to}E) & g;h \\
\mathbf{C} \arrow[r, "(C\to\_)"] & \mathbf{Set} &
\end{tikzcd}$$

$$\begin{tikzcd}
D \arrow[r, mapsto] \arrow[d, "h"'] & (A{\to}RD) \arrow[d, "\lambda\delta.(\delta;Rh)"] & \delta \arrow[d, mapsto] \\
E \arrow[r, mapsto] & (A{\to}RE) & \delta;Rh \\
\mathbf{C} \arrow[r, "(A\to R\_)"] & \mathbf{Set} &
\end{tikzcd}$$

$(C{\to}\_)_0(D) = \mathrm{Hom}_{\mathbf{C}}(C, D)$, $(C{\to}\_)_1(h) = \lambda g.(g; h)$,
$(A{\to}R\_)_0(D) = \mathrm{Hom}_{\mathbf{A}}(A, RD)$, $(A{\to}R\_)_1(h) = \lambda\delta.(\delta; Rh)$.

The natural transformation: internal view

To understand the NT $T : (C\to\_) \to (A\to R\_)$
we start by seeing how it produces, for objects $D, E \in \mathbf{C}$,
morphisms TD and TE...

$$\begin{tikzcd}
 & A \arrow[d, "\gamma"] \arrow[dd, bend left=60, "\delta"] \\
C \arrow[r, mapsto] \arrow[d, "g"'] & RC \arrow[d, "Rg"] \\
D \arrow[r, mapsto] \arrow[d, "h"'] & RD \arrow[d, "Rh"] \\
E \arrow[r, mapsto] & RE
\end{tikzcd}$$

$$\begin{tikzcd}
D & (C\to D) \arrow[r, "TD"] & (A\to RD) & g \arrow[r, mapsto] & \gamma; Rg \\
E & (C\to E) \arrow[r, "TE"] & (A\to RE) & g;h \arrow[r, mapsto] & \gamma; R(g;h) \\
 & (C\to\_) \arrow[r, "T"] & (A\to R\_) & &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "R"] & \mathbf{A}
\end{tikzcd}$$

So $TD = \lambda g.(R; g)$,
$T_0 = \lambda D.\lambda g.(R; g)$.

The natural transformation: internal view (2)

Now we want to check that this T obeys sqcond_T, i.e. that for every morphism $h : D \to E$ the "obvious" induced square commutes.

$$\begin{tikzcd}
 & A \arrow[d, "\gamma"] \arrow[dd, "\delta", shift left=4ex] \\
C \arrow[r, mapsto] \arrow[d, "g"'] & RC \arrow[d, "Rg"] \\
D \arrow[r, mapsto] \arrow[d, "h"'] & RD \arrow[d, "Rh"] \\
E \arrow[r, mapsto] & RE \\
\mathbf{C} \arrow[r, "R"] & \mathbf{A}
\end{tikzcd}$$

$$\begin{tikzcd}
D \arrow[d, "h"'] \\
E
\end{tikzcd}$$

$$\begin{tikzcd}
(C{\to}D) \arrow[r, "TD"] \arrow[d, "(C{\to}\_)h"'] & (A{\to}RD) \arrow[d, "(A{\to}R\_)h"'] \\
(C{\to}E) \arrow[r, "TE"] & (A{\to}RE) \\
(C{\to}\_) \arrow[r, "T"] & (A{\to}R\_)
\end{tikzcd}$$

$$\begin{tikzcd}
g \arrow[r, mapsto] \arrow[dd, mapsto] & \gamma; Rg \arrow[d, mapsto] & \delta \arrow[d, mapsto] \\
 & (\gamma; Rg); Rh & \delta; Rh \\
g; h \arrow[r, mapsto] & \gamma; R(g; h) &
\end{tikzcd}$$

It commutes because we have $\forall g.((\gamma; Rg); Rh = \gamma; R(g; h))$.

The diagram of the first Yoneda Lemma

We now understand all nodes and arrows in this diagram...
Remember that γ induced T.

$$\begin{tikzcd} & A \arrow[d, "\gamma"] \\ C \arrow[r, mapsto] & RC \end{tikzcd}$$

$$\begin{tikzcd} (C{\to}\_) \arrow[r, "T"] & (A{\to}R\_) \end{tikzcd}$$

$A \in \mathbf{A}$
$C \in \mathbf{C}$
$R : \mathbf{A} \to \mathbf{C}$
$\gamma : A \to RC$

$(C{\to}\_) : \mathbf{C} \to \mathbf{Set}$
$(C{\to}\_)_0(D) = \mathrm{Hom}_{\mathbf{C}}(C, D)$
$(C{\to}\_)_1(h) = \lambda g.(g; h)$

$(A{\to}R\_) : \mathbf{C} \to \mathbf{Set}$
$(A{\to}R\_)_0(D) = \mathrm{Hom}_{\mathbf{A}}(A, RD)$
$(A{\to}R\_)_1(h) = \lambda \delta.(\delta; Rh)$

$T : (C{\to}\_) \to (A{\to}R\_)$
$T_0(D) := \lambda g.(\gamma; Rg)$

The diagram of the first Yoneda Lemma (2)

We started with a morphism γ and defined T from it.
We can also do the inverse!

$$\begin{tikzcd}
 & A \arrow[d, "\gamma"] \\
C \arrow[r, mapsto] & RC
\end{tikzcd}$$

$$(C{\to}\_) \xrightarrow{T} (A{\to}R\_)$$

$A \in \mathbf{A}$
$C \in \mathbf{C}$
$R : \mathbf{A} \to \mathbf{C}$
$\gamma : A \to RC$
$\gamma := TC(\mathrm{id}_C)$

$(C{\to}\_) : \mathbf{C} \to \mathbf{Set}$
$(C{\to}\_)_0(D) = \mathrm{Hom}_{\mathbf{C}}(C, D)$
$(C{\to}\_)_1(h) = \lambda g.(g; h)$

$(A{\to}R\_) : \mathbf{C} \to \mathbf{Set}$
$(A{\to}R\_)_0(D) = \mathrm{Hom}_{\mathbf{A}}(A, RD)$
$(A{\to}R\_)_1(h) = \lambda \delta.(\delta; Rh)$

$T : (C{\to}\_) \to (A{\to}R\_)$

The bijection

Fact: the operations
$T := \lambda D.\lambda g.(\gamma; Rg)$ and
$\gamma := TC(\mathrm{id}_C)$
are inverses to one another.
Let's rewrite them as "T_γ" and "γ_T"...

$$T_\gamma = \lambda D.\lambda g.(\gamma; Rg)$$
$$\gamma_T = TC(\mathrm{id}_C)$$
$$T_{(\gamma_T)} = \lambda D.\lambda g.(\gamma_T; Rg) = \lambda D.\lambda g.((TC(\mathrm{id}_C)); Rg)$$
$$\gamma_{(T_\gamma)} = T_\gamma C(\mathrm{id}_C) = (\lambda D.\lambda g.(\gamma; Rg))C(\mathrm{id}_C)$$

We want to check that $\gamma_{(T_\gamma)} = \gamma$ (easy)
and that $T_{(\gamma_T)} = T$ (harder).

The bijection (2)

It's easy to check that $\gamma_{(T_\gamma)} = \gamma$:

$$\begin{aligned}
\gamma_{(T_\gamma)} &= T_\gamma C(\mathrm{id}_C) \\
&= (\lambda D.\lambda g.(\gamma; Rg))C(\mathrm{id}_C) \\
&= (\lambda g.(\gamma; Rg))(\mathrm{id}_C) \\
&= \gamma; R(\mathrm{id}_C) \\
&= \gamma; \mathrm{id}_{RC} \\
&= \gamma
\end{aligned}$$

The bijection (3)

Remember that
$T_{(\gamma_T)} = \lambda D.\lambda g.((TC(\mathrm{id}_C)); Rg)$, and so
$T_{(\gamma_T)}D(g) = TC(\mathrm{id}_C); Rg$.

We want to check this:
$\forall D.\forall g.(T_{(\gamma_T)}D(g) = TD(g))$, i.e.,
$\forall D.\forall g.(TC(\mathrm{id}_C); Rg = TD(g))$...

This is a consequence of sqcond_T!

The bijection (4)

We want to check this:
$\forall D. \forall g. (TC(\mathrm{id}_C); Rg = TD(g))$...

This is a consequence of sqcond_T!
Look:

$$\begin{tikzcd}
C \arrow[d, "g"'] & (C\to C) \arrow[r, "TC"] \arrow[d, "(C\to\_)g"'] & (A\to RC) \arrow[d, "(A\to R\_)g"'] & \mathrm{id}_C \arrow[r, mapsto] \arrow[dd, mapsto] & TC(\mathrm{id}_C) \arrow[d, mapsto] \\
D & (C\to D) \arrow[r, "TD"] & (A\to RD) & & TC(\mathrm{id}_C); Rg \\
 & & & g \arrow[r, mapsto] & TD(g)
\end{tikzcd}$$

$$\begin{tikzcd}
(C\to\_) \arrow[r, "T"] & (A\to R\_)
\end{tikzcd}$$

Drawing the bijection

A honest way to draw the bijection between 'γ's and 'T's would be diagram with the curved arrow in the middle... But we will commit an abuse of (diagrammatical) language and use a vertical arrow, as in the diagram at the right.

$$\begin{tikzcd}
& A \arrow[d, "\gamma"] \\
C \arrow[r, mapsto] & RC \\
(C\to\_) \arrow[r, "T"] & (A\to R\_)
\end{tikzcd}$$

$$\begin{tikzcd}
& A \arrow[d, "\gamma", ""{name=G, left}] \\
C \arrow[r, mapsto] & RC \\
(C\to\_) \arrow[r, "T"', ""{name=T}] & (A\to R\_)
\arrow[from=T, to=G, leftrightarrow, bend left]
\end{tikzcd}$$

$$\begin{tikzcd}
& A \arrow[d, "\gamma"] \\
C \arrow[r, mapsto, ""{name=M, below}] & RC \\
(C\to\_) \arrow[r, "T"', ""{name=T}] & (A\to R\_)
\arrow[from=M, to=T, leftrightarrow, shorten=2mm]
\end{tikzcd}$$

Now we have a shape for the (first) Yoneda Lemma and we can use it to compare several notations... But it's better to do that with the tree YLs at once, so let's prove the other two.

Changing the category $\mathbf{A}$ to $\mathbf{Set}$
Remember that $A, RC \in \mathbf{A}$ and $C \in \mathbf{C}$...
This is not shown in the diagram, but it appears
in the terms and types in lots of places.
Let's take a particular case: $\mathbf{A}$ becomes $\mathbf{Set}$...
In the notation of (simultaneous) substitution: $[\mathbf{A} := \mathbf{Set}]$.
The diagram does not change, but we can now
take a particular case of A too: $[A := 1]$. We get:

$$\left(\begin{tikzcd} & A \arrow[d, "\gamma", ""{name=G}] \\ C \arrow[r, mapsto] & RC \\ (C{\to}\_) \arrow[r, "T"', ""{name=T}] & (A{\to}R\_) \arrow[from=G, to=T, leftrightarrow, bend right] \end{tikzcd}\right) \begin{bmatrix}\mathbf{A} := \mathbf{Set} \\ A := 1\end{bmatrix} = \left(\begin{tikzcd} & 1 \arrow[d, "\gamma", ""{name=G}] \\ C \arrow[r, mapsto] & RC \\ (C{\to}\_) \arrow[r, "T"', ""{name=T}] & (1{\to}R\_) \arrow[from=G, to=T, leftrightarrow, bend right] \end{tikzcd}\right)$$

Getting rid of the '1's

Convention: 1 is the singleton set,
with single element $*$: $* \in 1 \in \mathbf{Set}$, $1 = \{*\}$.
If $B \in \mathbf{Set}$ then an arrow $\beta : 1 \to B$
"selects" an element $b \in B$...
We have a bijection between elements of $b \in B$
and arrows $\beta : 1 \to B$,
that we write as $B \leftrightarrow (1{\to}B)$,
or as two operations as $b := \beta(*)$, $\beta := \lambda *.b$ ...

My favourite way to represent a bijection $\begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \end{tikzcd}$

in a type system is as a 6-uple $(A, B, f, g, \mathsf{wdl}, \mathsf{wdr})$,
where $f : A \to B$, $g : B \to A$, and wdl and wdr assure
that $\forall a \in A.(g \circ f)(a) = a$ and $\forall b \in A.(f \circ g)(b) = b$
respectively.

Bijections and isos in type systems

One of my reasons for writing these notes was to show how these diagrams can be interpreted in a formal way in type systems and in proof assistants, so let me be type-ish for a moment...

Thorsten Altenkirch — in his book chapter "Naïve Type Theory" (from 2018(?), available from this home page) — uses the notation $[\![P]\!]$ for the "set of evidence" for the proposition P.

I prefer to call $[\![P]\!]$ the "set of proofs" of P (which *suggests* that we are in the BHK interpretation), or the "set of witnesses" of P (which *suggests* a model with proof-irrelevance and every $[\![P]\!]$ being either empty or a singleton)...

So...

Bijections and isos in type systems (2)

So:

$$\begin{aligned}
(A \underset{g}{\overset{f}{\rightleftarrows}} B) \quad = \quad & (A, B, f, g, \mathsf{wdl}, \mathsf{wdr}) \\
& \text{where:} \\
& A \text{ is a set,} \\
& B \text{ is a set,} \\
& f : A \to B, \\
& g : B \to A, \\
& \mathsf{wdl} : [\![\forall a \in A. (g \circ f)(a) = a]\!], \\
& \mathsf{wdr} : [\![\forall b \in B. (f \circ g)(b) = b]\!].
\end{aligned}$$

This is easy to adapt to define isos in a category.

$(A \overset{f}{\longleftrightarrow} B)$ is interpreted as $(A \underset{f^{-1}}{\overset{f}{\rightleftarrows}} B)$.

Getting rid of the ‘1’s (2)

The (nameless) bijection $(1{\to}B) \leftrightarrow B$ can be interpreted as:

$$\begin{tikzcd}
(1{\to}B) \arrow[d, leftrightarrow] & \beta \arrow[d, mapsto, shift right] \\
B & b \arrow[u, mapsto, shift right]
\end{tikzcd}$$

$b \in B$
$\beta \in (1{\to}B)$
$b := \beta(*)$
$\beta := \lambda *.b$

and written as:

$$\begin{tikzcd}
B \arrow[r, shift left, "\lambda b.\lambda *.b"] & (1{\to}B) \arrow[l, shift left, "\lambda\beta.\beta(*)"]
\end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd}
(1{\to}B) \arrow[r, shift left, "\lambda\beta.\beta(*)"] & B \arrow[l, shift left, "\lambda b.\lambda *.b"]
\end{tikzcd}$$

The components `wdl` and `wdr` of the 6-uples are treated as “obvious”, and are omitted.

Getting rid of the ‘1’s (3)

If $R : \mathbf{C} \to \mathbf{Set}$ then we have a (nameless) natural transformation $(1{\to}R\_) \leftrightarrow R$ between these functors:

$$\begin{tikzcd}
D \arrow[rr, mapsto] \arrow[dd, "h"'] & & RD \arrow[dd, "Rh"] \\
& \longmapsto & \\
E1 \arrow[rr, mapsto] & & RE
\end{tikzcd}
\qquad
\begin{tikzcd}
D \arrow[rr, mapsto] \arrow[dd, "h"'] & & (1{\to}RD) \arrow[dd, "{\begin{array}{l}(1{\to}R\_)(h)\\:= \lambda\delta.(\delta;Rh)\end{array}}"] \\
& \longmapsto & \\
E \arrow[rr, mapsto] & & (1{\to}RE)
\end{tikzcd}$$

$$\mathbf{C} \xrightarrow{R} \mathbf{Set} \qquad\qquad \mathbf{C} \xrightarrow{(1{\to}R\_)} \mathbf{Set}$$

Note that in type theory $R = (R_0, R_1, \dots)$, $(1{\to}R\_) = ((1{\to}R\_)_0, (1{\to}R\_)_1, \dots)$, and the diagrams above give us enough information to let us build $(1{\to}R\_)$ as a term.

Getting rid of the '1's (4)
If $R : \mathbf{C} \to \mathbf{Set}$ then we have a (nameless)
natural isomorphism $(1{\to}R\_) \leftrightarrow R$
between the functors defined in the previous page...
If $F, G : \mathbf{A} \to \mathbf{B}$ then a natural transformation $T : F \to G$
is formalized in TT as a pair (T_0, sqcond_T),
where T_0 is its "action on objects"
and sqcond_T is its "square condition".
The nameless natual iso $(1{\to}R\_) \leftrightarrow R$ can be interpreted as
a nameless NT $(1{\to}R\_) \to R$,
a nameless NT $R \to (1{\to}R\_)$,
and guarantees that their composites are identity functors...

Getting rid of the '1's (5)

The nameless natual iso $(1{\to}R\_) \leftrightarrow R$ can be interpreted as
a nameless NT $(1{\to}R\_) \to R$,
a nameless NT $R \to (1{\to}R\_)$,
and guarantees that their composites are identity functors...

Their actions on objects can be defined from this:
$((1{\to}R\_) \to R)_0(D) : (1{\to}RD) \to RD)$
$((1{\to}R\_) \to R)_0(D) = \lambda\delta.\delta(*)$
$(R \to (1{\to}R\_))_0(D) : D \to (1{\to}RD)$
$(R \to (1{\to}R\_))_0(D) = \lambda d.\lambda *.d$

(I will omit the details)

Changing the category A to Set (2)

With the nameless natural iso $(1{\to}R\_) \leftrightarrow R$
we can add an extra level to the basement our diagram,
and this yields an “obvious” bijection between ‘γ’s and ‘T'’s.
This new diagram “is” our Second Yoneda Lemma.

$$\begin{tikzcd}
 & 1 \arrow[d, "\gamma", ""{name=G}] \\
C \arrow[r, mapsto] & RC \\
(C{\to}\_) \arrow[r, "T"', ""{name=T}] & (1{\to}R\_)
\arrow[from=T, to=G, <->, bend left]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 1 \arrow[d, "\gamma", ""{name=G}] \\
C \arrow[r, mapsto] & RC \\
(C{\to}\_) \arrow[r, "T"', ""{name=T}] \arrow[dr, "T'"'] & (1{\to}R\_) \arrow[d, <->] \\
 & R
\arrow[from=T, to=G, <->, bend left]
\end{tikzcd}$$

Changing R to $(B{\to})$

Choose an object $B \in \mathbf{C}$.
It induces a functor $(B{\to}) : \mathbf{C} \to \mathbf{Set}$.
Several slides ago we did this substitution
on the diagram of the first Yoneda Lemma:

$$\begin{bmatrix} \mathbf{A} & := & \mathbf{Set} \\ A & := & 1 \end{bmatrix}$$

Now we will do this substitution
on the diagram of the second Yoneda Lemma:

$$\begin{bmatrix} R & := & (B{\to}) \end{bmatrix}$$

very little will change in the diagram,
but a lot will change in the terms and types.

Changing R to $(B{\to})$ (2)

After the substitution $\left[R\ :=\ (B{\to})\right]$
the diagram for the Second Yoneda Lemma (left) becomes
the diagram for the Third Yoneda Lemma (right):

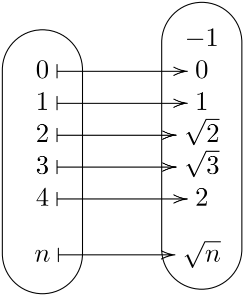
$$\begin{tikzcd}
 & 1 \arrow[d, "\gamma", ""{name=G, near start}] \\
C \arrow[r, mapsto] & RC \\
(C{\to}\_) \arrow[r, "T"', ""{name=T, near end}] \arrow[dr, "T'"'] & (1{\to}R\_) \arrow[d, leftrightarrow] \\
 & R
\arrow[from=G, to=T, bend right, leftrightarrow]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 1 \arrow[d, "\gamma", ""{name=G2, near start}] \\
C \arrow[r, mapsto] & (B{\to}C) \\
(C{\to}\_) \arrow[r, "T"', ""{name=T2, near end}] \arrow[dr, "T'"'] & (1{\to}(B{\to}\_)) \arrow[d, leftrightarrow] \\
 & (B{\to}\_)
\arrow[from=G2, to=T2, bend right, leftrightarrow]
\end{tikzcd}$$

Our Third Yoneda Lemma is usually stated as this bijection:
$(B{\to}C) \leftrightarrow ((C{\to}\_) \to (B{\to}\_))$, where the right side is
the space of natural transformations from $(C{\to}\_)$ to $(B{\to}\_))$.

READING INTERNAL DIAGRAMS

Motivation: blob-sets

$$\begin{array}{rccc} \sqrt{}\,: & \mathbb{N} & \to & \mathbb{R} \\ & n & \mapsto & \sqrt{n} \end{array}$$



$$\begin{tikzcd} \mathbb{N} \arrow[r, "\sqrt{}"] & \mathbb{R} \end{tikzcd}$$

Motivation: blob-sets (2)

“Internal diagrams are a tool that lets us lower the level of abstraction. (...) Look at the figure at the left in the previous slide and compare its ‘$\mathbb{N} \to \mathbb{R}$’ in the upper line (the external view), with the ‘$n \mapsto \sqrt{n}$’ in the lower line (the internal view); the $n \mapsto \sqrt{n}$ shows a (generic) element and its image.

The figure at the right shows the external view at the bottom and an internal view at the top; note that all elements in the blobs for $\mathbb{N}$ and $\mathbb{R}$ are named, but only a few of the elements are shown (...) the arrows like $3 \mapsto \sqrt{3}$ and $4 \mapsto 2$, that show elements and their images, are substitution instances of the generic $n \mapsto \sqrt{n}$, maybe after some calculations (or “reductions” in λ-calculus terminology).”

The internal view of a functor

We usually draw internal views above external views,
general cases at the left, particular cases at the right.
Remember that a functor $F : \mathbf{C} \to \mathbf{D}$
is a 4-uple: $(F0, F1, \mathsf{respids}_F, \mathsf{respcomp}_F)$.
The diagram at the right below defines $(X\times)$. How?

$$\begin{tikzcd}
A \arrow[r, mapsto, "F_0"] \arrow[d, "g"'] \arrow[dr, phantom, "\overset{F_1}{\longmapsto}"] & FA \arrow[d, "Fg"] \\
B \arrow[r, mapsto, "F_0"'] & FB \\
\mathbf{C} \arrow[r, "F"] & \mathbf{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, mapsto] \arrow[d, "f"'] \arrow[dr, phantom, "\longmapsto"] & X\times Y \arrow[d, "\langle \pi, f \circ \pi' \rangle"] \\
Z \arrow[r, mapsto] & X\times Z \\
\mathbf{Set} \arrow[r, "(X\times)"] & \mathbf{Set}
\end{tikzcd}$$

The internal view of a functor (2)

Compare:

$$\begin{tikzcd}
A \arrow[r, mapsto, "F_0"] \arrow[d, "g"'] \arrow[dr, phantom, "\overset{F_1}{\longmapsto}" description] & FA \arrow[d, "Fg"] \\
B \arrow[r, mapsto, "F_0"'] & FB \\
\mathbf{C} \arrow[r, "F"] & \mathbf{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, mapsto] \arrow[d, "f"'] \arrow[dr, phantom, "\longmapsto" description] & X{\times}Y \arrow[d, "\langle\pi, f\circ\pi'\rangle"] \\
Z \arrow[r, mapsto] & X{\times}Z \\
\mathbf{Set} \arrow[r, "(X\times)"] & \mathbf{Set}
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, mapsto, "F_0"] \arrow[d, "g"'] \arrow[dr, phantom, "\overset{F_1}{\longmapsto}" description] & FA \arrow[d, "Fg"] \\
B \arrow[r, mapsto, "F_0"'] & FB \\
\mathbf{C} \arrow[r, "F"] & \mathbf{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, mapsto, "(X\times)_0"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{(X\times)_1}{\longmapsto}" description] & X{\times}Y \arrow[d, "\langle\pi, f\circ\pi'\rangle"] \\
Z \arrow[r, mapsto, "(X\times)_0"] & X{\times}Z \\
\mathbf{Set} \arrow[r, "(X\times)"] & \mathbf{Set}
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, mapsto, "(X\times)_0"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{(X\times)_1}{\longmapsto}" description] & (X\times)_0(Y) \arrow[d, "(X\times)_1(f)"] \\
Z \arrow[r, mapsto, "(X\times)_0"] & (X\times)_0(Z) \\
\mathbf{Set} \arrow[r, "(X\times)"] & \mathbf{Set}
\end{tikzcd}$$

So $(X\times)_0 = \lambda Y.(X \times Y)$,
and $(X\times)_1 = \lambda f.\langle\pi, f \circ \pi'\rangle$.

TO DO: internal diagrams of NTs and adjunctions
(I have lots of those diagrams — plus monads, etc —
but I never explained the conventions in them very clearly...)

This is an adjunction:

$$\begin{tikzcd}
LRB \arrow[dd, "\epsilon B"'] & LA \arrow[dd, "f"'] & & & A \arrow[lll, mapsto, "L_0"'] \arrow[dd, "g"] & A \arrow[dd, "\eta A"] \\
& & {} \arrow[r, mapsto, shift right, "\sharp_{AB}"'] & {} \arrow[l, mapsto, shift right, "\flat_{AB}"'] & & \\
B & B \arrow[rrr, mapsto, "R_0"'] & & & RB & RLA
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{B} \arrow[r, shift right, "R"'] & \mathbf{A} \arrow[l, shift right, "L"']
\end{tikzcd}$$

HELP
NEEDED

Help needed: proof assistants

I was never able to learn enough Coq or Agda...

I guess that it would be easy to formalize the figure with the three Yoneda Lemmas in Coq or Agda. We can number its objects as

	o11		o21		o31
o12	o13	o22	o23	o32	o33
o14	o15	o24	o25	o34	o35
			o26		o36

and choose some convention for the ascii names for arrows, and for the ascii names for arrows between arrows.

Help needed: proof assistants (2)

Smart proof assistants should be able to find by themselves the proofs that we said that were "obvious". Besides the obvious proofs I've said that some constructions are "obvious". Finding obvious "constructions" needs term inference instead of proof inference, and implementation of term inference are rare.

Help needed: Agda

There are several implementation of CT in Agda.

There is even a "Big list of formalizations of Category Theory in proof assistants" in MathOverflow!

I'm trying to read — with Juan Meleiro — a big blog post by Jannis Limpberg called "Yoneda's Lemma in Excruciating Detail", that has an implementation in Agda.

We will try to:

1. draw its diagrams in our favourite shapes,
2. assign coordinates to some of his terms,
3. sketch a way to read our 2D diagrams directly in Agda...

Jannis Limpberg
“Yoneda’s Lemma in Excruciating Detail” (blog post):
https://limperg.de/posts/2018-07-27-yoneda.html
The functor Ap:

$$\begin{tikzcd}
(F,C) \arrow[r, mapsto] \arrow[d, "{(\theta,f)}"'] \arrow[dr, phantom, "\longmapsto"] & FC \arrow[d, "Gf\circ\theta_C"] \\
(G,D) \arrow[r, mapsto] & GD
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[d, "f"'] & FC \arrow[r, "\theta_C"] \arrow[d, "Ff"'] & GC \arrow[d, "Gf"] \\
D & FD \arrow[r, "\theta_D"] & GD
\end{tikzcd}$$

$$(\mathbf{C}\to\mathbf{D})\times\mathbf{C} \xrightarrow{\mathrm{Ap}} \mathbf{D}$$

$$F \xrightarrow{\theta} G$$

Jannis Lindberg (2)

The functor Hom:

$$\begin{tikzcd}
C & D \arrow[dd, "g"] & (C,D) \arrow[r, mapsto] \arrow[dd, "{(f,g)}"'] & \mathrm{Hom}(C,D) \arrow[dd, "\lambda h.(g\circ h\circ f)"] & h \arrow[dd, mapsto] \\
 & & {} \arrow[r, phantom, "\longmapsto"] & {} & \\
C' \arrow[uu, "f"] & D' & (C',D') \arrow[r, mapsto] & \mathrm{Hom}(C',D') & g\circ h\circ f
\end{tikzcd}$$

$$\mathbf{C}^{\mathrm{op}}\times\mathbf{C} \xrightarrow{\mathrm{Ap}} \mathbf{Sets}$$

Jannis Limpberg (3)

The functor y:

$$\begin{tikzcd}
D \arrow[rr, mapsto] \arrow[dd, "f"'] & & \mathrm{Hom}(\bullet, D) \arrow[dd, "yf"] \\
& \longmapsto & \\
D' \arrow[rr, mapsto] & & \mathrm{Hom}(\bullet, D')
\end{tikzcd}$$

$$\begin{tikzcd}
C & \mathrm{Hom}(C, D) \arrow[r, "(yf)_C"] \arrow[d, "(yD)(g)"'] & \mathrm{Hom}(C, D') \arrow[d, "(yD')(g)"] & h \arrow[r, mapsto] \arrow[d, mapsto] & f \circ h \arrow[d, mapsto] \\
C' \arrow[u, "g"] & \mathrm{Hom}(C', D) \arrow[r, "(yf)_{C'}"] & \mathrm{Hom}(C', D') & h \circ g \arrow[r, mapsto] & f \circ h \circ g
\end{tikzcd}$$

$$\mathbf{C} \xrightarrow{y} \mathbf{C}^{\mathrm{op}} \Rightarrow \mathbf{Sets}$$

$$\mathrm{Hom}(\bullet, D) \xrightarrow{yf} \mathrm{Hom}(\bullet, D')$$

DICTIONARIES

Same shape, several notations

Now that we have a shape for the three Yoneda Lemmas we can change the notation — i.e., what is written in each of the nodes that we named o11, o12, ..., o36 a few slides ago, and also change what is written in the arrows...

For typographical reasons — I don't have good ways to put labels along curved arrows — I will have to commit the abuse of diagrammatical language explained in the slide "Drawing the bijection" (p.13), and draw the curved bijections as just their vertical-ish lower halves.

Categories for the Working Mathematician
Here is how MacLane states our YLs in his CWM.
Our first YL is implicit in his Proposition 1 in p.59:

> **Proposition 1.** *For a functor* $S : D \to C$ *a pair* $\langle r, u : c \to Sr \rangle$ *is universal from* c *to* S *if and only if the function sending each* $f' : r \to d$ *into* $Sf'u : c \to Sd$ *is a bijection of hom-sets*
>
> $$D(r, d) \cong C(c, Sd). \qquad (1)$$
>
> *This bijection is natural in* d*. Conversely, given* r *and* c*, any natural isomorphism (1) is determined in this way by a unique arrow* $u : c \to Sr$ *such that* $\langle r, u \rangle$ *is universal from* c *to* S*.*

Categories for the Working Mathematician (2)

Our second YL appears in p.61 of CWM, as this:

> **Lemma** *(Yoneda). If* $K : D \to$ **Set** *is a functor from* D *and* r *an object in* D *(for* D *a category with small hom-sets), there is a bijection*
>
> $$y : \mathrm{Nat}(D(r,-), K) \cong Kr$$
>
> *which sends each natural transformation* $\alpha : D(r,-) \overset{\bullet}{\to} K$ *to* $\alpha_r 1_r$*, the image of the identity* $r \to r$*.*

Categories for the Working Mathematician (3)

Our third YL also appears in p.61 of CWM, as a corollary:

> **Corollary.** *For objects* $r, s \in D$*, each natural transformation* $D(r, -) \to D(s, -)$ *has the form* $D(h, -)$ *for a unique arrow* $h : s \to r$*.*

Categories for the Working Mathematician (4)

$$\begin{tikzcd}
 & c \arrow[d, "u"] \\
r \arrow[r, mapsto] \arrow[d, mapsto, shift right] & Sr \\
D(r,-) \arrow[r, "T"'] \arrow[u, mapsto, shift right] & C(c,S-)
\end{tikzcd}$$

$$\begin{tikzcd}
 & * \arrow[d, "u"] \\
r \arrow[r, mapsto] \arrow[d, mapsto, shift right] & Kr \\
D(r,-) \arrow[r] \arrow[u, mapsto, shift right, "y"'] \arrow[dr, "T'"'] & \mathbf{Set}(*,K-) \arrow[d, leftrightarrow] \\
 & K
\end{tikzcd}$$

$$\begin{tikzcd}
 & * \arrow[d, "f"] \\
r \arrow[r, mapsto] \arrow[d, mapsto, shift right, "Y"'] & D(s,r) \\
D(r,-) \arrow[r] \arrow[u, mapsto, shift right] \arrow[dr, "{D(f,-)}"'] & \mathbf{Set}(*,D(s,-)) \arrow[d, leftrightarrow] \\
 & D(s,-)
\end{tikzcd}$$