1 J-operators

A *J-operator* on a Heyting Algebra $H = (\Omega, \leq, \top, \bot, \wedge, \vee, \to, \leftrightarrow, \neg)$ is a function $J : \Omega \to \Omega$ that obeys the axioms J1, J2, J3 below; we usually write J as $\cdot^* : \Omega \to \Omega$, and write the axioms as rules.

$$\frac{}{P \leq P^*}\,\mathsf{J1} \qquad \frac{}{P^* = P^{**}}\,\mathsf{J2} \qquad \frac{}{(P \wedge Q)^* = P^* \wedge Q^*}\,\mathsf{J3}$$

J1 says that the operation $\cdot^*$ is non-decreasing.
J2 says that the operation $\cdot^*$ is idempotent.
J3 is a bit mysterious but will have interesting consequences.

Note that when H is a ZHA then any slash-operator on H is a J-operator on it; see secs.**??** and **??**.

A J-operator induces an equivalence relation and equivalence classes on Ω, like slashings do:

$$\begin{array}{rcl} P \sim_J Q & \text{iff} & P^* = Q^* \\ [P]^J & := & \{\, Q \in \Omega \mid P^* = Q^* \,\} \end{array}$$

The equivalence classes of a J-operator J are called *J-regions*.

The axioms J1, J2, J3 have many consequences. The first ones are listed in Figure 1 as derived rules, whose names mean:

Mop (monotonicity for products): a lemma used to prove Mo,
Mo (monotonicity): $P \leq Q$ implies $P^* \leq Q^*$,
Sand (sandwiching): all truth values between P and P^* are equivalent,
EC$\wedge$: equivalence classes are closed by '&',
EC$\vee$: equivalence classes are closed by '$\vee$',
ECS: equivalence classes are closed by sandwiching,

Take a J-equivalence class, $[P]^J$, and list its elements: $[P]^J = \{P_1, \ldots, P_n\}$. Let $P_\wedge := ((P_1 \wedge P_2) \wedge \ldots) \wedge P_n$ and $P_\vee := ((P_1 \vee P_2) \vee \ldots) \vee P_n$. Clearly $P_\wedge \leq P_i \leq P_\vee$ for each i, so $[P]^J \subseteq [P_\wedge, P_\vee]$.

Using EC$\wedge$ and EC$\vee$ several times we see that:

$$\begin{array}{rr} P_1 \wedge P_2 \sim_J P & P_1 \vee P_2 \sim_J P \\ (P_1 \wedge P_2) \wedge P_3 \sim_J P & (P_1 \vee P_2) \vee P_3 \sim_J P \\ \vdots & \vdots \\ ((P_1 \wedge P_2) \wedge \ldots) \wedge P_n \sim_J P & ((P_1 \vee P_2) \vee \ldots) \vee P_n \sim_J P \\ P_\wedge \sim_J P & P_\vee \sim_J P \\ P_\wedge \in [P]^J & P_\vee \in [P]^J \end{array}$$

$$\dfrac{}{(P \wedge Q)^* \leq Q^*}\,\mathsf{Mop} \quad := \quad \dfrac{\dfrac{}{(P \wedge Q)^* = P^* \wedge Q^*}\,\mathsf{J3} \qquad \dfrac{}{P^* \wedge Q^* \leq Q^*}}{(P \wedge Q)^* \leq Q^*}$$

$$\dfrac{P \leq Q}{P^* \leq Q^*}\,\mathsf{Mo} \quad := \quad \dfrac{\dfrac{\dfrac{P \leq Q}{P = P \wedge Q}}{P^* = (P \wedge Q)^*} \qquad \dfrac{}{(P \wedge Q)^* \leq Q^*}\,\mathsf{Mop}}{P^* \leq Q^*}$$

$$\dfrac{P \leq Q \leq P^*}{P^* = Q^*}\,\mathsf{Sand} \quad := \quad \dfrac{\dfrac{P \leq Q}{P^* \leq Q^*}\,\mathsf{Mo} \qquad \dfrac{\dfrac{Q \leq P^*}{Q^* \leq P^{**}}\,\mathsf{Mo} \qquad \dfrac{}{P^{**} = P^*}\,\mathsf{J2}}{Q^* \leq P^*}}{P^* = Q^*}$$

$$\dfrac{P^* = Q^*}{P^* = Q^* = (P \wedge Q)^*}\,\mathsf{EC}\wedge \quad := \quad \dfrac{\dfrac{P^* = Q^*}{P^* = Q^* = P^* \wedge Q^*} \qquad \dfrac{}{P^* \wedge Q^* = (P \wedge Q)^*}\,\mathsf{J3}}{P^* = Q^* = (P \wedge Q)^*}$$

$$\dfrac{P^* = Q^*}{P^* = Q^* = (P \vee Q)^*}\,\mathsf{EC}\vee \quad := \quad \dfrac{P^* = Q^* \qquad \dfrac{\dfrac{\dfrac{}{P \leq P \vee Q} \qquad \dfrac{\dfrac{}{P \leq P^*}\,\mathsf{J1} \qquad \dfrac{\dfrac{}{Q \leq Q^*}\,\mathsf{J1} \qquad \dfrac{P^* = Q^*}{Q^* = P^*}}{Q \leq P^*}}{P \vee Q \leq P^*}}{P \leq P \vee Q \leq P^*}}{P^* = (P \vee Q)^*}\,\mathsf{Sand}}{P^* = Q^* = (P \vee Q)^*}$$

$$\dfrac{P \leq Q \leq R \qquad P^* = R^*}{P^* = Q^* = R^*}\,\mathsf{ECS} \quad := \quad \dfrac{\dfrac{\dfrac{P \leq Q \leq R \qquad \dfrac{}{R \leq R^*}\,\mathsf{J1} \qquad \dfrac{P^* = R^*}{R^* = P^*}}{P \leq Q \leq P^*}}{P^* = Q^*}\,\mathsf{Sand} \qquad P^* = R^*}{P^* = Q^* = R^*}$$

Figure 1: J-operators: basic derived rules

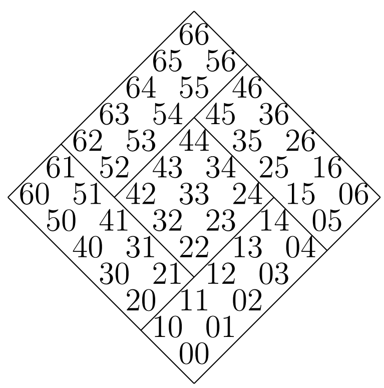
and using ECS we can see that all elements between $P_\wedge$ and $P_\vee$ are J-equivalent to P:

$$\dfrac{\dfrac{P_\wedge \leq Q \leq P_\vee \qquad \dfrac{\dfrac{P_\wedge \sim_J P}{P_\wedge{}^* = P^*} \qquad \dfrac{P_\vee \sim_J P}{P_\vee{}^* = P^*}}{P_\wedge{}^* = P_\vee{}^*}}{P_\wedge{}^* = Q^* = P_\vee{}^*}\,\mathsf{ECS} \qquad P_\vee{}^* = P^*}{\dfrac{Q^* = P^*}{Q \sim_J P}}$$

so $[P_\wedge, P_\vee] \subseteq [P]^J$. This means that *J-regions are intervals.*

1.1 Cuts stopping midway

Look at the figure below, that shows a partition of a ZHA $A = [00, 66]$ into five regions, each region being an interval; this partition does not come from a slashing, as it has cuts that stop midway. Define an operation '$\cdot^*$' on A, that works by taking each truth-value P in it to the top element of its region; for example, $30^* = 61$.



It is easy to see that '$\cdot^*$' obeys J1 and J2; however, it does *not* obey J3 — we will prove that in sec.1.2. As we will see, *the partitons of a ZHA into intervals that obey* J1, J2, J3 *ae exactly the slashings;* or, in other words, *every J-operator comes from a slashing.*

1.2 The are no Y-cuts and no λ-cuts

We want to see that if a partition of a ZHA H into intervals has "Y-cuts" or "λ-cuts", like these parts of the last diagram in sec.1.1,

$$\begin{array}{ccc} & 22 & \\ 21 & & 12 \\ & 11 & \end{array} \quad \Leftarrow \quad \text{this is a Y-cut}$$

$$\begin{array}{ccc} & 25 & \\ 24 & & 15 \\ & 14 & \end{array} \quad \Leftarrow \quad \text{this is a } \lambda\text{-cut}$$

then the operation J that takes each element to the top of its equivalence class cannot obey J1, J2 and J3 at the same time. We will prove that by deriving rules that say that if $11 \sim_J 12$ then $21 \sim_J 22$, and that if $15 \sim_J 25$ then $14 \sim_J 24$; actually, our rules will say that if $11^* = 12^*$ then $(11 \vee 21)^* = (12 \vee 21)^*$, and that if $15^* = 25^*$ then $(15 \wedge 24)^* = (25 \wedge 24)^*$. The rules are:

$$\dfrac{P^* = Q^*}{(P \vee R)^* = (Q \vee R)^*}\ \textsf{NoYcuts} \quad := \quad \begin{array}{c} P^* = Q^* \\ \hline P \vee R^* = Q \vee R^* \\ \hline (P \vee R^*)^* = (Q \vee R^*)^* \\ \hline\hline (P \vee R)^* = (Q \vee R)^* \end{array}\ \ovee_6 = \ovee_4$$

$$\dfrac{P^* = Q^*}{(P \wedge R)^* = (Q \wedge R)^*}\ \textsf{No}\lambda\textsf{cuts} \quad := \quad \begin{array}{c} P^* = Q^* \\ \hline P^* \wedge R^* = Q^* \wedge R^* \\ \hline (P \wedge R)^* = (Q \wedge R)^* \end{array}\ \textsf{J3}$$

The expansion of double bar labeled '$\ovee_6 = \ovee_4$' in the top derivation uses twice the derived rule $\ovee_6 = \ovee_4$, that is easy to prove using the cubes of sec.1.3.

1.3 How J-operators interact with connectives

The axiom J3 says that $(P \wedge Q)^* = P^* \wedge Q^*$ — it says something about how '$\cdot^*$' interacts with '$\wedge$'. Let's introduce a shorter notation. There are eight ways to replace each of the '?'s in $(P^? \wedge Q^?)^?$ by either nothing or a star. We establish that the three '?'s in $(P^? \wedge Q^?)^?$ are "worth" 1, 2 and 4 respectively, and we use $P \mathbin{⦸}_n Q$ to denote $(P^? \wedge Q^?)^?$ with the bits "that belong to n" replaced by stars. So:

$$
\begin{array}{rclrcl}
⦸_0 & = & P \wedge Q, & ⦸_4 & = & (P \wedge Q)^*, \\
⦸_1 & = & P^* \wedge Q, & ⦸_5 & = & (P^* \wedge Q)^*, \\
⦸_2 & = & P \wedge Q^*, & ⦸_6 & = & (P \wedge Q^*)^*, \\
⦸_3 & = & P^* \wedge Q^*, & ⦸_7 & = & (P^* \wedge Q^*)^*.
\end{array}
$$

We omit the arguments of $⦸_n$ when they are P and Q — so we can rewrite $(P \wedge Q)^* = P^* \wedge Q^*$ as $⦸_4 = ⦸_3$. These conventions also hold for $⦹$ and $⊖\!\!\to$.

It is easy to prove each one of the arrows in the cubes below ($A \longrightarrow B$ means $A \leq B$):

$$
\begin{tikzcd}
 & (P^*{\wedge}Q^*)^* & \\
(P^*{\wedge}Q)^* \arrow[ur] & P^*{\wedge}Q^* \arrow[u] & (P{\wedge}Q^*)^* \arrow[ul] \\
P^*{\wedge}Q \arrow[u] \arrow[ur] & (P{\wedge}Q)^* \arrow[ul] \arrow[ur] & P{\wedge}Q^* \arrow[ul] \arrow[u] \\
 & P{\wedge}Q \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (P^*{\vee}Q^*)^* & \\
(P^*{\vee}Q)^* \arrow[ur] & P^*{\vee}Q^* \arrow[u] & (P{\vee}Q^*)^* \arrow[ul] \\
P^*{\vee}Q \arrow[u] \arrow[ur] & (P{\vee}Q)^* \arrow[ul] \arrow[ur] & P{\vee}Q^* \arrow[ul] \arrow[u] \\
 & P{\vee}Q \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (P^*{\to}Q^*)^* \arrow[dr] & \\
(P^*{\to}Q)^* \arrow[ur] \arrow[dr] & P^*{\to}Q^* \arrow[u] \arrow[dr] & (P{\to}Q^*)^* \\
P^*{\to}Q \arrow[u] \arrow[ur] \arrow[dr] & (P{\to}Q)^* \arrow[ur] & P{\to}Q^* \arrow[u] \\
 & P{\to}Q \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

Let's write their sets of elements as $⦸_{0...7} := \{⦸_0, \ldots, ⦸_7\}$, $⦹_{0...7} := \{⦹_0, \ldots, ⦹_7\}$, and $⊖\!\!\to_{0...7} := \{⊖\!\!\to_0, \ldots, ⊖\!\!\to_7\}$. The cubes above — we will call them the "obvious and-cube", the "obvious or-cube", and the "obvious implication-cube" — can be interpreted as directed graphs $(⦸_{0...7}, \mathsf{OCube}_\wedge)$, $(⦹_{0...7}, \mathsf{OCube}_\vee)$, $(⊖\!\!\to_{0...7}, \mathsf{OCube}_\to)$.

The "extended cubes" will be the directed graphs with the arrows above plus the ones coming from these derived rules:

$$\dfrac{}{(P^*\wedge Q^*)^* = P^*\wedge Q^* = (P\wedge Q)^*}\;\owedge_7=\owedge_3=\owedge_4 \qquad :=$$

$$\dfrac{\dfrac{\dfrac{}{P^{**}=P^*}\;\mathsf{J2}\qquad \dfrac{}{Q^{**}=Q^*}\;\mathsf{J2}}{(P^*\wedge Q^*)^* = P^{**}\wedge Q^{**} = P^*\wedge Q^* = (P\wedge Q)^*}\;\mathsf{J3}}{(P^*\wedge Q^*)^* = P^*\wedge Q^* = (P\wedge Q)^*}$$

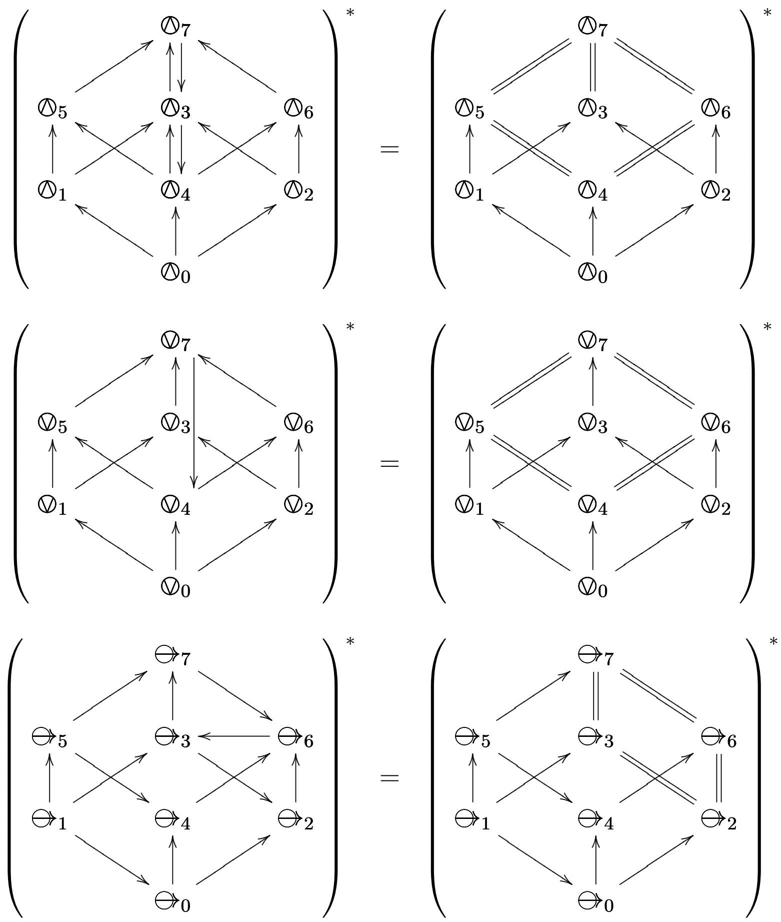
$$\dfrac{}{(P^*\vee Q^*)^*\le(P\vee Q)^*}\;\ovee_7\le\ovee_3 \qquad := \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P\le P\vee Q}}{P^*\le(P\vee Q)^*}\;\mathsf{Mo}\qquad \dfrac{\dfrac{}{Q\le P\vee Q}}{Q^*\le (P\vee Q)^*}\;\mathsf{Mo}}{P^*\vee Q^*\le (P\vee Q)^*}}{(P^*\vee Q^*)^*\le(P\vee Q)^{**}}\;\mathsf{Mo}}{(P^*\vee Q^*)^*\le(P\vee Q)^*}\;\mathsf{J2}$$

$$\dfrac{}{(P\to Q^*)^*\le P^*\to Q^*}\;\text{\textcircled{$\to$}}_6\le\text{\textcircled{$\to$}}_3 \qquad := \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P\to Q^*\le P\to Q^*}}{(P\to Q^*)\wedge P\le Q^*}}{((P\to Q^*)\wedge P)^*\le Q^{**}}\;\mathsf{Mo}}{((P\to Q^*)\wedge P)^*\le Q^*}\;\mathsf{J2}}{(P\to Q^*)^*\wedge P^*\le Q^*}\;\mathsf{J3}}{(P\to Q^*)^*\le P^*\to Q^*}$$

where $\owedge_7=\owedge_3=\owedge_4$ will be interpreted as these arrows:

$$(P^*\wedge Q^*)^* \leftrightarrows P^*\wedge Q^* \leftrightarrows (P\wedge Q)^*$$

The directed graphs of these "extended cubes" will be called $(\owedge_{0\ldots7},\mathsf{ECube}_\wedge)$, $(\ovee_{0\ldots7},\mathsf{ECube}_\vee)$, $(\text{\textcircled{$\to$}}_{0\ldots7},\mathsf{ECube}_\to)$. We are interested in the (non-strict) partial orders that they generate, and we want an easy way to remember these partial orders. The figure below shows these extended cubes at the left, and at the right the "simplified cubes", $\mathsf{SCube}_\wedge$, $\mathsf{SCube}_\vee$, and $\mathsf{SCube}_\to$, that generate the same partial orders that the extended cubes.



From these cubes it is easy to see, for example, that we can prove $\ovee_5 = \ovee_6$ (as a derived rule).

1.4 Valuations

Let $H_\odot$ and $J_\odot$ be a ZHA and a J-operator on it, and let $v_\odot$ be a function from the set $\{P, Q\}$ to H. By an abuse of language $v_\odot$ will also denote the triple $(H_\odot, J_\odot, v_\odot)$ — and by a second abuse of language $v_\odot$ will also denote the obvious extension of $v_\odot : \{P, Q\} \to H$ to the set of all valid expressions formed from P, Q, $\cdot^*$, $\top$, $\bot$, and the connectives.

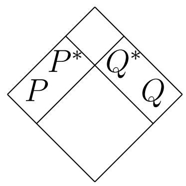
Let $i, j \in \{0, \ldots, 7\}$. Then $(\owedge_i, \owedge_j) \in \mathsf{SCube}^*_\wedge$ means that $\owedge_i \leq \owedge_j$ is a theorem, and so $v_\odot(\owedge_i) \leq v_\odot(\owedge_j)$ holds; i.e.,

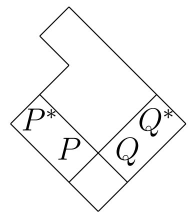
$$\mathsf{SCube}^*_\wedge \subseteq \{\, (\owedge_i, \owedge_j) \mid i,j \in \{0,\dots,7\},\ v_\odot(\owedge_i) \leq v_\odot(\owedge_j) \,\}$$

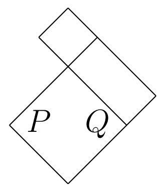
and the same for:

$$\begin{array}{r}\mathsf{SCube}^*_\vee \subseteq \{\, (\ovee_i, \ovee_j) \mid i,j \in \{0,\dots,7\},\ v_\odot(\ovee_i) \leq v_\odot(\ovee_j) \,\} \\ \mathsf{SCube}^*_\to \subseteq \{\, (\ominus\!\!\!\!\rightarrow_i, \ominus\!\!\!\!\rightarrow_j) \mid i,j \in \{0,\dots,7\},\ v_\odot(\ominus\!\!\!\!\rightarrow_i) \leq v_\odot(\ominus\!\!\!\!\rightarrow_j) \,\}\end{array}$$

Some valuations that turn these '$\subseteq$'s into '$=$'. Let

$(H_\wedge, J_\wedge, v_\wedge) =$ 

$(H_\vee, J_\vee, v_\vee) =$ 

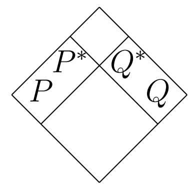
$(H_\to, J_\to, v_\to) =$ 

then

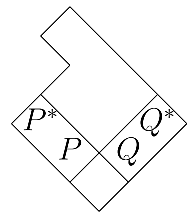
$$\begin{array}{r}\mathsf{SCube}^*_\wedge = \{\, (\owedge_i, \owedge_j) \mid i,j \in \{0,\dots,7\},\ v_\wedge(\owedge_i) \leq v_\wedge(\owedge_j) \,\} \\ \mathsf{SCube}^*_\vee = \{\, (\ovee_i, \ovee_j) \mid i,j \in \{0,\dots,7\},\ v_\vee(\ovee_i) \leq v_\vee(\ovee_j) \,\} \\ \mathsf{SCube}^*_\to = \{\, (\ominus\!\!\!\!\rightarrow_i, \ominus\!\!\!\!\rightarrow_j) \mid i,j \in \{0,\dots,7\},\ v_\to(\ominus\!\!\!\!\rightarrow_i) \leq v_\to(\ominus\!\!\!\!\rightarrow_j) \,\}\end{array}$$

or, in more elementary terms:

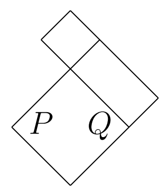
A very important fact. For any i and j,

$\owedge_i \leq \owedge_j$ is a theorem iff it is true in 

,

$\ovee_i \leq \ovee_j$ is a theorem iff it is true in 

,

$\ominus\!\!\!\to_i \leq \ominus\!\!\!\to_j$ is a theorem iff it is true in 

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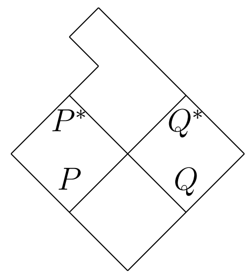
The very important fact, and the valuations $v_\wedge$, $v_\vee$, $v_\to$, give us:

- a way to *remember* which sentences of the forms $\owedge_i \leq \owedge_j$, $\ovee_i \leq \ovee_j$, $\ominus\!\!\!\to_i \leq \ominus\!\!\!\to_j$ are theorems;
- countermodels for all the sentences of these forms not in $\mathsf{SCube}_\wedge$, $\mathsf{SCube}_\vee$, $\mathsf{SCube}_\to$. For example, $\ovee_7 \leq \ovee_4$ is not in $\mathsf{SCube}_\vee$; and $v_\vee(\ovee_7) \leq v_\vee(\ovee_4)$, which shows that $\ovee_7 \leq \ovee_4$ can't be a theorem.

An observation. I arrived at the cubes $\mathsf{ECube}^*_\wedge$, $\mathsf{ECube}^*_\vee$, $\mathsf{ECube}^*_\to$ by taking the material in the corollary 5.3 of chapter 5 in [?] and trying to make it fit into less mental space (as discussed in [?]); after that I wanted to be sure that each arrow that is not in the extended cubes has a countermodel, and I found the countermodels one by one; then I wondered if I could find a single countermodel for all non-theorems in $\mathsf{ECube}^*_\wedge$ (and the same for $\mathsf{ECube}^*_\vee$ and $\mathsf{ECube}^*_\to$), and I tried to start with a valuation that distinguished *some* equivalence classes in $\mathsf{ECube}^*_\wedge$, and change it bit by bit, getting valuations that distinguished more equivalence classes at every step. Eventually I arrived at $v_\wedge$, $v_\vee$ and at $v_\to$, and at the — surprisingly nice — "very important fact" above.

Note that this valuation

$$(H_{\wedge\vee}, J_{\wedge\vee}, v_{\wedge\vee}) \quad =$$



distinguishes all equivalence classes in $\mathsf{ECube}^*_\wedge$ and in $\mathsf{ECube}^*_\vee$, but not in $\mathsf{ECube}^*_\to$... it “thinks” that $P \to Q$ and $P^* \to Q$ are equal.