Using Planar Heyting Algebras to develop visual intuition about Intuitionistic Propositional Logic

Eduardo Ochs (UFF, Brazil)
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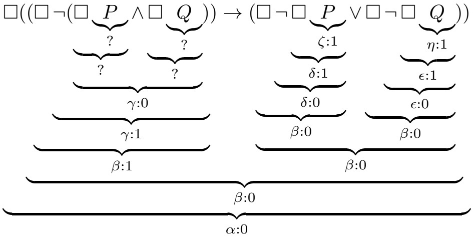
The Gödel translation $T : \mathsf{IPL} \to \mathsf{S4}$

Let $\mathsf{DeM} = \neg(P \wedge Q) \to (\neg P \vee \neg Q)$,
the part of the DeMorgan laws that is not
a theorem in IPL. Its Gödel translation to S4 is:

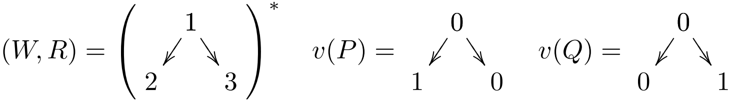
$$\underbrace{T(\underbrace{\neg(\underbrace{\underbrace{P}_{\Box P} \wedge \underbrace{Q}_{\Box Q}}_{\Box P \wedge \Box Q})}_{\Box\neg(\Box P \wedge \Box Q)} \to (\underbrace{\underbrace{\neg \underbrace{P}_{\Box P}}_{\Box\neg\Box P} \vee \underbrace{\neg \underbrace{Q}_{\Box Q}}_{\Box\neg\Box Q}}_{\Box\neg\Box P \vee \Box\neg\Box Q}))}_{\Box((\Box\neg(\Box P \wedge \Box Q)) \to (\Box\neg\Box P \vee \Box\neg\Box Q))}$$

An S4-tableau for DeM



$\forall\ \beta$ ahead of α,
$\forall\ \gamma$ ahead of β,
$\exists\ \delta$ ahead of β,
$\exists\ \epsilon$ ahead of β,
$\forall\ \zeta$ ahead of δ,
$\forall\ \eta$ ahead of ϵ,
$(\Box P \wedge \Box Q)_\gamma = 0$,
$P_\zeta = 1$,
$Q_\eta = 1$.

A solution, i.e., a countermodel for $T(\mathsf{DeM})$:



The value of the subexpressions

Once we fix (W, R), $v(P)$, $v(Q)$ we can visualize the value of each subexpression... I will drop the 'v's from here on.

$$\underbrace{\Box(\underbrace{(\underbrace{\Box\underbrace{\neg(\underbrace{\underbrace{\Box\underbrace{P}_{{}_1{}^0{}_0}}_{{}_1{}^0{}_0}\wedge\underbrace{\Box\underbrace{Q}_{{}_0{}^0{}_1}}_{{}_0{}^0{}_1}}_{{}_0{}^0{}_0})}_{{}_1{}^1{}_1}}_{{}_1{}^1{}_1}) \to (\underbrace{\underbrace{\Box\underbrace{\neg\underbrace{\Box\underbrace{P}_{{}_1{}^0{}_0}}_{{}_1{}^0{}_0}}_{{}_0{}^1{}_1}}_{{}_0{}^0{}_1}\vee\underbrace{\Box\underbrace{\neg\underbrace{\Box\underbrace{Q}_{{}_0{}^0{}_1}}_{{}_0{}^0{}_1}}_{{}_1{}^1{}_0}}_{{}_1{}^0{}_0}}_{{}_1{}^0{}_1})}_{{}_1{}^0{}_1})}_{{}_1{}^0{}_1}$$

The value of the subexpressions (2)

Start with any IPL expression A —
we're using $A = \neg(P \wedge Q) \to (\neg P \vee \neg Q)$ —
and an $\mathsf{S4}$ valuation $((W, R), v)$.
What happens if we calculate the values of the subexpressions of A? Take any subexpression B of A; calculate $v(T(B))$ —

$$T(\underbrace{\underbrace{\neg(\underbrace{\underbrace{P}_{\Box P} \wedge \underbrace{Q}_{\Box Q}}_{\Box P \wedge \Box Q})}_{\Box\neg(\Box P \wedge \Box Q)} \to (\underbrace{\underbrace{\neg \underbrace{P}_{\Box P}}_{\Box\neg\Box P} \vee \underbrace{\neg \underbrace{Q}_{\Box Q}}_{\Box\neg\Box Q}}_{\Box\neg\Box P \vee \Box\neg\Box Q})}_{\Box((\Box\neg(\Box P \wedge \Box Q)) \to (\Box\neg\Box P \vee \Box\neg\Box Q))})$$

The value of the subexpressions (3)

It turns out that the value of (the translation of) each subexpression B of A is $\Box$-stable, in the following sense: $\Box\, v(T(B)) = v(T(B))$. Also, the intuitionistic operations $\top, \bot, \wedge, \vee, \to_I, \neg_I, \leftrightarrow_I$ can coexist with the $\mathsf{S4}$ operations in the same system...

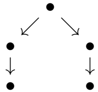
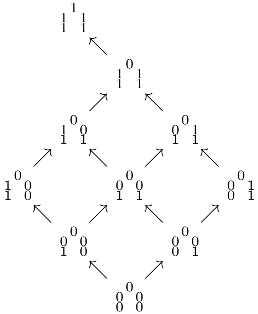
$$T(\underbrace{\underbrace{\neg_I(\underbrace{\underbrace{P}_{{}_1{}^0{}_0} \wedge \underbrace{Q}_{{}_0{}^0{}_1}}_{{}_0{}^0{}_0})}_{{}_1{}^1{}_1} \to_I \underbrace{(\underbrace{\neg_I \underbrace{P}_{{}_1{}^0{}_0}}_{{}_0{}^0{}_1} \vee \underbrace{\neg_I \underbrace{Q}_{{}_0{}^0{}_1}}_{{}_1{}^0{}_0})}_{{}_1{}^0{}_1}}_{{}_1{}^0{}_1}})$$

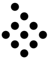
The lattice of $\Box$-stable truth-values

When the intuitionistic operations $\top, \bot, \wedge, \vee, \to_I, \neg_I, \leftrightarrow_I$ receive $\Box$-stable inputs they return $\Box$-stable outputs.
How do we visualize:
a) the set of $\Box$-stable truth values in an $\mathsf{S4}$-frame (W, R^*)?
b) the operations $\top, \bot, \wedge, \vee, \to_I, \neg_I, \leftrightarrow_I$?

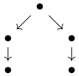
Tricks:
The stable truth-values are the open sets in a certain topology, and $(W, \mathcal{O}_R(W))$ is an order topology (see [PH1], secs.11–17).
Topologies are Heyting Algebras, that are lattices with some extra properties...
And for some '(W, R)'s these lattices are planar.

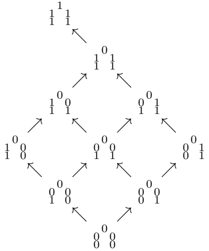
An example (planar)

$(H, \mathsf{BPM}(H)) =$  $(\mathcal{O}(H), \subset_1) =$ 

Compact notation: $(\mathcal{O}(H), \subseteq) =$ 

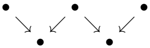
Some technicalities

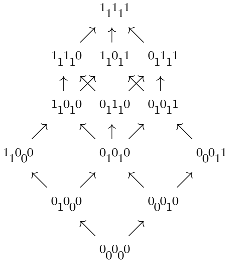
$(H, \mathsf{BPM}(H)) =$ 

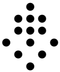
$(\mathcal{O}(H), \subset_1) =$ 

If $A \subseteq \mathbb{Z}^2$ then $\mathsf{BPM}(A)$ is the set of "black pawns moves" on A ([PH1], sec.2); $U \subset_1 V$ means "$U \subseteq V$ and they differ by exactly one element" ([PH1], sec.13). The transitive-reflexive closure of $(\mathcal{O}_R(W), \subset_1)$ is the lattice $(\mathcal{O}_R(W), \subseteq)$.

An example (non-planar)

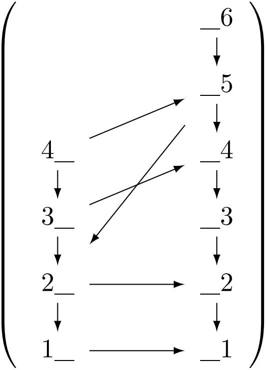
$(W, \mathsf{BPM}(W)) =$ 

$(\mathcal{O}(W), \subset_1) =$ 

The Heyting Algebra $(\mathcal{O}(W), \subseteq)$
can't be recovered from this diagram: 

2-column graphs and LR-coordinates

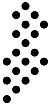
$(\mathcal{O}$



$, \subseteq) \quad =$

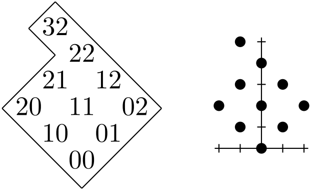
46
45 36
35 26
34 25
24
23 14
22 13 04
12 03
11 02
01
00

$=$



LR-coordinates

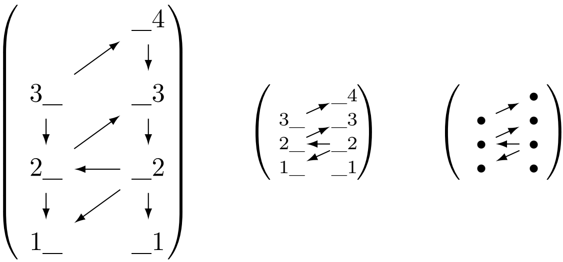
(x, y) means: start at $(0, 0)$, walk x steps E, y steps N.
$\langle l, r\rangle$ means: start at $(0, 0)$, walk l steps NW, r steps NE.
We abbreviate $\langle l, r\rangle$ as lr.
Check:



$$20 = \langle 2, 0\rangle = (0, 0) + 2\overrightarrow{(-1, 1)} + 0\overrightarrow{(1, 1)} = (-2, 2)$$
$$32 = \langle 2, 0\rangle = (0, 0) + 3\overrightarrow{(-1, 1)} + 2\overrightarrow{(1, 1)} = (-1, 5)$$

2-column graphs

A 2-column graph ("2CG") has two columns of points, all vertical arrows pointing one step down, and *some* intercolumn arrows.
An example:

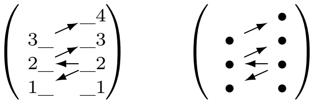


2-column graphs (2)

The operation 2CG builds a 2-column graph formally from
the height of its left column,
the height of its right column,
the intercolumn arrows going right,
the intercolumn arrows going left.

$$\mathsf{2CG}(3, 4, \left\{ \begin{smallmatrix} 3\_ \to \_4, \\ 2\_ \to \_3 \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} 2\_ \leftarrow \_2, \\ 1\_ \leftarrow \_2 \end{smallmatrix} \right\})$$

$$= \left(\left\{ \begin{smallmatrix} & 3\_, & 2\_, & 1\_, \\ \_4, & \_3, & \_2, & \_1 \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} 3\_ \to 2\_, \; 2\_ \to 1\_, \\ \_4 \to \_3, \; \_3 \to \_2, \; \_2 \to \_1, \\ 3\_ \to \_4, \; 2\_ \to \_3, \\ 2\_ \leftarrow \_2, \; 1\_ \leftarrow \_2 \end{smallmatrix} \right\} \right)$$



2CGs and piles

$\mathsf{2CG}(3,4,\left\{ {3\_ \to \_4, \atop 2\_ \to \_3} \right\}, \left\{ {2\_ \leftarrow \_2, \atop 1\_ \leftarrow \_2} \right\}) = \left(\left\{ {\_4,}\, {3\_, \atop \_3,}\, {2\_, \atop \_2,}\, {1\_, \atop \_1} \right\}, \{ \cdots \} \right)$

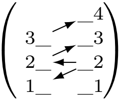
$$\mathsf{pile}(3,4) = \left\{ {\_4,}\, {3\_, \atop \_3,}\, {2\_, \atop \_2,}\, {1\_, \atop \_1} \right\} = \begin{pmatrix} & & 1 \\ 1 & \nearrow & 1 \\ 1 & \nearrow\leftarrow & 1 \\ 1 & \swarrow & 1 \end{pmatrix}$$

$$\mathsf{pile}(2,1) = \left\{ {2\_,}\, {1\_, \atop \_1} \right\} = \begin{pmatrix} & & 0 \\ 0 & \nearrow & 0 \\ 1 & \nearrow\leftarrow & 0 \\ 1 & \swarrow & 1 \end{pmatrix}$$

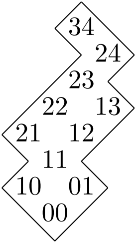
All open sets of a 2CG are piles,
but some piles are not open sets...
for example, $\mathsf{pile}(2,1)$ is not open — it has 2_ but not _3.

2CGs and piles (2)

Let $(P, A) = \mathsf{2CG}(3, 4, \left\{ \begin{smallmatrix} 3\_\!\!\to\!\!\_4, \\ 2\_\!\!\to\!\!\_3 \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} 2\_\!\!\leftarrow\!\!\_2, \\ 1\_\!\!\leftarrow\!\!\_2 \end{smallmatrix} \right\})$

$=$ 

.

Then $\mathcal{O}_A(P) =$ 

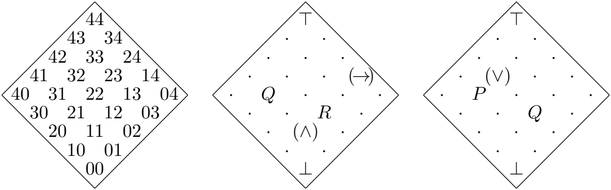
Can you visualize $\mathcal{O}(\mathbb{R})$? I find that very hard!

Heyting Algebras in elementary terms
Here's what we did up to this point:
We saw briefly the Gödel translation and believed it,
We saw briefly that topologies are HAs and believed that,
We saw briefly how to build planar HAs from 2CGs,
We saw some very compact notations for planar HAs ("ZHAs")...

But we don't know yet how $\top, \bot, \land, \lor, \to, \neg, \leftrightarrow$ behave in the compact notation, and how to visualize them!

This can be done almost from scratch, using the way to define $\top, \bot, \land, \lor, \to, \neg, \leftrightarrow$ from the partial order of a lattice.

Heyting Algebras in elementary terms (2)
Left: H, the ZHA that we will use in the next slides.
Middle: $(\wedge) = (Q \wedge_H R)$, $(\to) = (Q \to_H R)$.
Right: $(\vee) = (P \vee_H Q)$.



We will see that:
a) if $Q = 31$ and $R = 12$ then $Q \wedge_H R = 11$,
b) if $Q = 31$ and $R = 12$ then $Q \to_H R = 14$.
c) if $P = 31$ and $Q = 12$ then $P \vee_H Q = 32$,

Heyting Algebras in elementary terms (3)

Motivation:

$$\bot \to P$$
$$P \to \top$$
$$(P \to (Q \wedge R)) \leftrightarrow ((P \to Q) \wedge (P \to R))$$
$$((P \vee Q) \to R) \leftrightarrow ((P \to R) \wedge (Q \to R))$$
$$(P \to (Q \to R)) \leftrightarrow ((P \wedge Q) \to R)$$

$$\bot \leq P$$
$$P \leq \top$$

$$(P \leq (Q \wedge R)) \leftrightarrow ((P \leq Q) \wedge (P \leq R))$$
$$(P \leq (Q \to R)) \leftrightarrow ((P \wedge Q) \leq R)$$

$$((P \vee Q) \leq R) \leftrightarrow ((P \leq R) \wedge (Q \leq R))$$

Heyting Algebras in elementary terms (4)

$\top$ and $\bot$ are the elements of the ZHA H that obey this for all $P \in H$:

$\bot \leq P$
$P \leq \top$

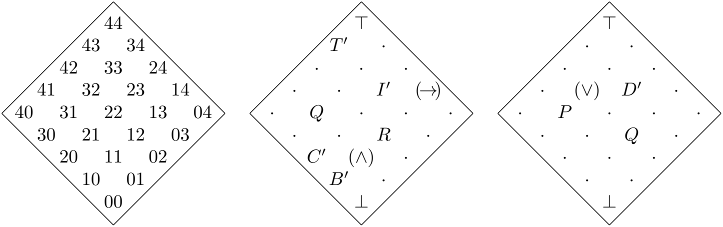
Fix Q and R. $(Q \land R)$ and $(Q \to R)$ are the elements of H that obey this for all P:

$(P \leq (Q \land R)) \leftrightarrow ((P \leq Q) \land (P \leq R))$
$(P \leq (Q \to R)) \leftrightarrow ((P \land Q) \leq R)$

Fix P and Q. $(P \lor Q)$ is the element of H that obeys this for all R:

$((P \lor Q) \leq R) \leftrightarrow ((P \leq R) \land (Q \leq R))$

Guessing (wrongly) and testing



B' is a bad guess for $\bot$: it fails when $P = 01$.
T' is a bad guess for $\top$: it fails when $P = 44$.
C' is a bad guess for $(Q \wedge R)$: it fails when $P = 11$.
I' is a bad guess for $(Q \to R)$: it fails when $P = 14$.
D' is a bad guess for $(P \vee Q)$: it fails when $R = 32$.