Planar Heyting Algebras for Children 2: Local Operators, J-Operators, and Slashings

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Abstract

Choose a topos $\mathcal{E}$. There are several different "notions of sheafness" on $\mathcal{E}$. How do we visualize them?

Let's refer to the classifier object of $\mathcal{E}$ as Ω, and to its Heyting Algebra of truth-values, $\mathrm{Sub}(1_{\mathcal{E}})$, as H; we will sometimes call H the "logic" of the topos. There is a well-known way of representing notions of sheafness as morphisms $j : \Omega \to \Omega$, but these 'j's yield big diagrams when we draw them explicitly; here we will see a way to represent these 'j's as maps $J : H \to H$ in a way that is much more manageable.

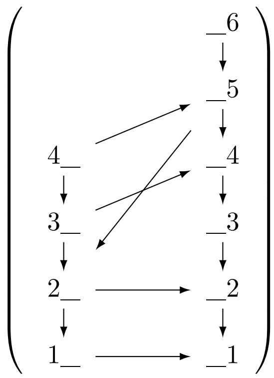
In the previous paper of this series — called [PH1] from here on — we showed how certain toy models of Heyting Algebras, called "ZHAs", can be used to develop visual intuition for how Heyting Algebras and Intuitionistic Propositional Logic work; here we will extend that to sheaves. The full idea is this: *notions of sheafness* correspond to *local operators* and vice-versa; *local operators* correspond to *J-operators* and vice-versa; if our Heyting Algebra H is a ZHA then *J-operators* correspond to *slashings* on H, and vice-versa; *slashings* on H correspond to *"sets of question marks"* and vice-versa, and each set of question marks induces a notion of *erasing and reconstructing*, which induces a sheaf. Also, every ZHA H corresponds to an (acyclic) 2-column graph, and vice-versa, and for any two-column graph (P, A) the logic of the topos $\mathbf{Set}^{(P,A)}$ is exactly the ZHA H associated to (P, A).

The introduction of [PH1] discusses two different senses in which a mathematical text can be "for children". The first sense involves some precise metamathetical tools for transfering knowledge back and forth between a general case "for adults" and a toy model "for children"; the second sense is simply that the text's presentation has few prerequisites and never becomes too abstract. Here we will use the second sense: everything here, except for the last section, should be accessible to students who have taken a course on Discrete Mathematics and read [PH1]. This means that categories, toposes, sheaves and the maps $j : \Omega \to \Omega$ only appear in the last section, and before that we deal only with the J-operators $J : H \to H$, how they correspond to slashings and sets of question marks, and how they form an algebra.

Contents

1 Basic definitions

One of the main constructions of [PH1] is a correspondence between 2-column graphs ("2CGs") and Planar Heyting Algebras ("ZHAs"), as in this example:



$$\leftrightsquigarrow \qquad \begin{array}{ccccc}
 & & 46 & & \\
 & 45 & & 36 & \\
 & & 35 & & 26 \\
 & 34 & & 25 & \\
 & & 24 & & \\
 & 23 & & 14 & \\
22 & & 13 & & 04 \\
 & 12 & & 03 & \\
11 & & 02 & & \\
 & 01 & & & \\
00 & & & & \\
\end{array}$$

The arrows in the 2CG (P, A) (mnemonic: "points" and "arrows") are interpreted as conditions that subsets of P must obey to the open: for example, the arrow $(4\_, \_5) \in A$ means that if an open set $U \subseteq A$ contains the point $4\_$ then it also has to contains $\_5$. This generates an *order topology* on P, that we denote by $\mathcal{O}_A(P)$, and the ZHA H at the right of the squiggly arrow in the figure is this $\mathcal{O}_A(P)$ drawn in a very compacty way — by using the operation "pile", and abbreviating it.

We write $\mathsf{pile}(ab)$ for the subset of P formed by pile of a elements at the left and a pile of b elements at the right, as in:

$$25 \equiv \mathsf{pile}(25) = \{2\_, 1\_, \;\; \_1, \_2, \_3, \_4, \_5\},$$

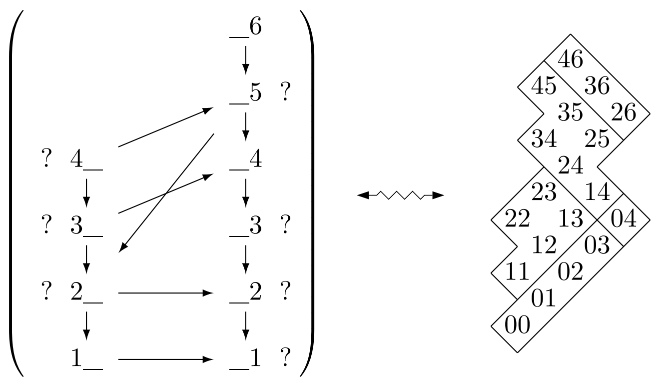
The '$\equiv$' in "$25 \equiv \mathsf{pile}(2, 5)$" means a change of notation — it means that sometimes 'ab' will be an abbreviation for "$\mathsf{pile}(ab)$". With this abreviation it is easy to check that the H above is exactly the topology $\mathcal{O}_A(P)$. Note that, for example, $21 \not\in H$; this is because $\mathsf{pile}(21) = \{2\_, 1\_, \;\; \_1\}$, and this set does not obey all the conditions associated to the arrows in A: we have $(2\_, \_2) \in A$ but $2\_ \in \mathsf{pile}(21)$ and $\_2 \not\in \mathsf{pile}(21)$.

Let's now introduce some new ideas.

1.1 Question marks and slashings

A *set of question marks* on a 2CG (P, A) is a subset $Q \subseteq P$. We write a 2CG with question marks as $((P, A), Q)$, and we represent this Q graphically by writing a '?' close to each element of P that belongs to Q, as in the figure below. The intended meaning of these question marks is that we want to forget the information on them and then see which elements of $\mathcal{O}_A(P)$ become indistinguishable after this forgetting: two elements $ab, cd \in H$ are *Q-equivalent*, written as $ab \sim_Q cd$, iff $\mathsf{pile}(ab)\backslash Q = \mathsf{pile}(cd)\backslash Q$. In the $((P, A), Q)$ of the figure below we have $23 \sim_Q 13 \not\sim_Q 14$.

A *slashing* S on a ZHA H is a set of diagonal cuts on H "that do not stop midway". These cuts are interpreted as fences that divide H in separate regions, and two elements $ab, cd \in H$ are *S-equivalent*, written as $ab \sim_S cd$, if they belong to the same region. In the slashing at the right in the figure below we have $11 \sim_S 23 \not\sim_S 14$.



In [PH1] we used the notation $(P, A) \leftrightsquigarrow H$ to say that H is the ZHA associated to the 2CG (P, A); this "is associated to" was interpreted formally as $\mathcal{O}_A(P) = H$. We are now extending this to $((P, A), Q) \leftrightsquigarrow (H, S)$ — a 2CG with question marks $((P, A), Q)$ is associated to the ZHA with slashing (H, S) when we have $\mathcal{O}_A(P) = H$ and the equivalence relations $\sim_Q, \sim_S \subseteq H \times H$ coincide. Note that the two '$\leftrightsquigarrow$'s are both pronounced as "is associated to", but they have different formal meanings.

1.2 Piccs and slashings

A picc ("partition into contiguous classes") of a "discrete interval" $I = \{0, \ldots, n\}$ is a partition P of I that obeys this condition ("picc-ness"):

$$\forall a, b, c \in \{0, \ldots, n\}. \; (a < b < c \wedge a \sim_P c) \to (a \sim_P b \wedge b \sim_P c).$$

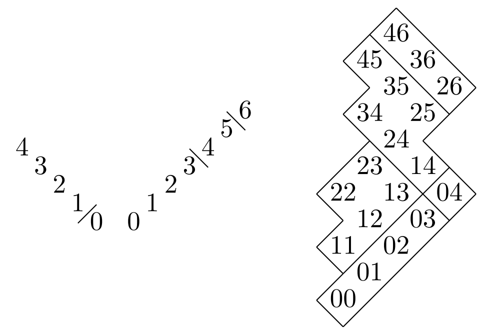
So $P = \{\{0\}, \{1, 2, 3\}, \{4, 5\}\}$ is a picc of $\{0, \ldots, 5\}$, and $P' = \{\{0\}, \{1, 2, 4, 5\}, \{3\}\}$ is a partition of $\{0, \ldots, 5\}$ that is not a picc.

A short notation for piccs is this:

$$0|123|45 \equiv \{\{0\}, \{1, 2, 3\}, \{4, 5\}\}$$

we list all digits in the (discrete) interval in order, and we put bars to indicate where we change from one equivalence class to another.

We will represent a slashing S formally as pairs of piccs, one for the left digit and one for the right digit. Our notation for slashings as pairs will be based on this figure:



The slashing S that we are using in our examples will be represented as:

$$\begin{array}{rcl} S & = & (L,R) \\ & = & (\{\{0\},\{1,2,3,4\}\},\ \{\{0,1,2,3\},\{4,5\},\{6\}\}) \\ & = & (0|1234, 0123|45|6) \\ & = & (4321/0,\ 0123\backslash 45\backslash 6) \end{array}$$

We use '/'s and '\'s instead of '|'s to remind us of the direction of the cuts: the '/'s correspond to cuts that go northeast and the '\'s to cuts that go northwest.

We can now define the equivalence relation $\sim_S$ formally: if $S=(L,R)$ then $ab \sim_S cd$ iff $a \sim_L c$ and $c \sim_R d$.

The expression "$S=(L,R)$ is a slashing on H" will mean: H is a ZHA, L is a picc on $\{0,\dots,l\}$, and R a picc on $\{0,\dots,r\}$, where lr is the top element of H. The domain of the equivalence relation $\sim_S$ will be considered to be H, not $\{0,\dots,l\}\times\{0,\dots,r\}$.

1.3 Slash-operators

When $S=(L,R)$ is a slashing on H we will use the notations $[\cdot]^L$, $[\cdot]^R$, $[\cdot]^S$ for the equivalence classes of L, R, S and the notations $\cdot^L$, $\cdot^R$, $\cdot^S$ for the highest element in those equivalence class. In our example we have $[2]^L=\{1,2,3,4\}$, $[2]^R=\{0,1,2,3\}$, $[22]^S=\{11,12,13,22,23\}$, $2^L=4$, $2^R=4$, $2^S=23$. Note that $[\cdot]^S$ and $\cdot^S$ depend on the ZHA.

A *slash-operator* on a ZHA H is a function $f: H \to H$ that is equal to some $\cdot^S$.

Take any function $f: H \to H$ on a ZHA. Let:

$$\begin{array}{rcl} S_0 & = & \{\,(ab, f(ab)) \mid ab \in H\,\} \\ L_0 & = & \{\,(a,c) \mid (ab,cd) \in S_0\,\} \\ R_0 & = & \{\,(b,d) \mid (ab,cd) \in S_0\,\} \\ L & = & {L_0}^* \\ R & = & {R_0}^* \\ S & = & (L,R) \end{array}$$

The function f is a slash-operator if and only if these L and R are piccs and $f = \cdot^S$.

1.4 From slashings to question marks and vice-versa

Choose any path from the bottom element of the ZHA to its top element that is made of one unit steps northwest or northeast — for example, this one:

$$(a_0b_0, a_1b_1, \ldots a_{10}b_{10}) = (00, 01, 02, 03, 04, 14, 24, 34, 35, 36, 46)$$

If we apply 'pile' to each element of that path we get a sequence of sets,

$$(\mathsf{pile}(a_0b_0), \mathsf{pile}(a_1b_1), \ldots, \mathsf{pile}(a_{10}b_{10}))$$

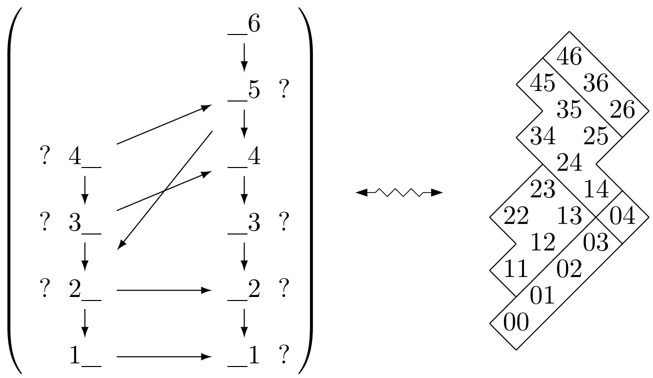
that is actually a sequence of open sets in $\mathcal{O}_A(P)$ in which the first set is $\mathsf{pile}(a_0b_0) = \mathsf{pile}(00) = \emptyset$, the last set is P, and the difference between each set and the next one is exactly one element — for example:

$$\begin{array}{rcl} \mathsf{pile}(34)\backslash\mathsf{pile}(24) &=& \{3\_\} \\ \mathsf{pile}(35)\backslash\mathsf{pile}(34) &=& \{\_5\} \end{array}$$

Note that we have two different cases: 1) the step from a_ib_i to $a_{i+1}b_{i+1}$ is a movement *northwest* in the ZHA, as in from 24 to 34; in this case $a_{i+1}b_{i+1} = (a_i + 1)b_i$, and the difference $\mathsf{pile}(a_{i+1}b_{i+1})\backslash\mathsf{pile}(a_ib_i)$ is $\{a_{i+1}\_\}$, an element of the left column of P; 2) the step from a_ib_i to $a_{i+1}b_{i+1}$ is a movement *northeast* in the ZHA, as in from 34 to 35; here $a_{i+1}b_{i+1} = a_i(b_i + 1)$, and the difference $\mathsf{pile}(a_{i+1}b_{i+1})\backslash\mathsf{pile}(a_ib_i)$ is $\{\_b_{i+1}\}$, an element of the right column of P.

The easiest way to see how to convert from a set of question marks to its associated slashing and vice-versa is by looking at an example. Let's take the structure $((P, A), Q) \leftrightsquigarrow (H, S)$ on which we've been working and build a table that shows how each step of the path $(a_0b_0, a_1b_1, \ldots a_{10}b_{10})$ is "seen" by the set Q, by the equivalence relations $\sim_Q$, $\sim_S$, $\sim_L$, $\sim_R$, and by the slashing S

written in short form. We get:



$$(a_0b_0, \ldots a_{10}b_{10}) = (00, 01, 02, 03, 04, 14, 24, 34, 35, 36, 46)$$

$\mathsf{pile}(46)\backslash\mathsf{pile}(36) = \{4\_\}$	$4\_ \in Q$	$36 \sim_Q 46$	$3 \sim_L 4$		43	
$\mathsf{pile}(36)\backslash\mathsf{pile}(35) = \{\_6\}$	$\_6 \notin Q$	$35 \not\sim_Q 36$		$5 \not\sim_R 6$		$5\backslash 6$
$\mathsf{pile}(35)\backslash\mathsf{pile}(34) = \{\_5\}$	$\_5 \in Q$	$34 \sim_Q 35$		$4 \sim_R 5$		45
$\mathsf{pile}(34)\backslash\mathsf{pile}(24) = \{3\_\}$	$3\_ \in Q$	$24 \sim_Q 34$	$2 \sim_L 3$		32	
$\mathsf{pile}(24)\backslash\mathsf{pile}(14) = \{2\_\}$	$2\_ \in Q$	$14 \sim_Q 24$	$1 \sim_L 2$		21	
$\mathsf{pile}(14)\backslash\mathsf{pile}(04) = \{1\_\}$	$1\_ \notin Q$	$04 \not\sim_Q 14$	$0 \not\sim_L 1$		1/0	
$\mathsf{pile}(04)\backslash\mathsf{pile}(03) = \{\_4\}$	$\_4 \notin Q$	$03 \not\sim_Q 04$		$3 \not\sim_R 4$		$3\backslash 4$
$\mathsf{pile}(03)\backslash\mathsf{pile}(02) = \{\_3\}$	$\_3 \in Q$	$02 \sim_Q 03$		$2 \sim_R 3$		23
$\mathsf{pile}(02)\backslash\mathsf{pile}(01) = \{\_2\}$	$\_2 \in Q$	$01 \sim_Q 02$		$1 \sim_R 2$		12
$\mathsf{pile}(01)\backslash\mathsf{pile}(00) = \{\_1\}$	$\_1 \in Q$	$00 \sim_Q 01$		$0 \sim_R 1$		01

There is an obvious correspondence between the elements of P that are *not* in Q and the '/'s and '\' in S that indicate changes of equivalence class: $P\backslash Q = \{1\_,\ \_4, \_5\}$ corresponds to 1/0, $3\backslash 4$, $5\backslash 6$.

2 J-operators

A *J-operator* on a Heyting Algebra $H \equiv (H, \leq, \top, \bot, \land, \lor, \to, \leftrightarrow, \neg)$ is a function $J : H \to H$ that obeys the axioms $\mathsf{J1}$, $\mathsf{J2}$, $\mathsf{J3}$ below; we usually write J as $\cdot^* : H \to H$, and write the axioms as rules.

$$\frac{}{P \leq P^*}\,\mathsf{J1} \qquad \frac{}{P^* = P^{**}}\,\mathsf{J2} \qquad \frac{}{(P \land Q)^* = P^* \land Q^*}\,\mathsf{J3}$$

$\mathsf{J1}$ says that the operation $\cdot^*$ is non-decreasing.
$\mathsf{J2}$ says that the operation $\cdot^*$ is idempotent.
$\mathsf{J3}$ is a bit mysterious but will have interesting consequences.

A J-operator induces an equivalence relation and equivalence classes on H, like slashings do:

$$\begin{array}{rcl} P \sim_J Q & \text{iff} & P^* = Q^* \\ [P]^J & := & \{\, Q \in H \mid P^* = Q^* \,\} \end{array}$$

The equivalence classes of a J-operator J are called *J-regions*.

The axioms $\mathsf{J1}$, $\mathsf{J2}$, $\mathsf{J3}$ have many consequences. The first ones are listed in Figure 1 as derived rules, whose names mean:
Mop (monotonicity for products): a lemma used to prove Mo,
Mo (monotonicity): $P \leq Q$ implies $P^* \leq Q^*$,
Sand (sandwiching): all truth values between P and P^* are equivalent,
$\mathsf{EC}\land$: equivalence classes are closed by '&',
$\mathsf{EC}\lor$: equivalence classes are closed by '$\lor$',
ECS: equivalence classes are closed by sandwiching,

Take a J-equivalence class, $[P]^J$, and list its elements: $[P]^J = \{P_1, \ldots, P_n\}$. Let $P_\land := ((P_1 \land P_2) \land \ldots) \land P_n$ and $P_\lor := ((P_1 \lor P_2) \lor \ldots) \lor P_n$. Clearly $P_\land \leq P_i \leq P_\lor$ for each i, so $[P]^J \subseteq [P_\land, P_\lor]$. We will use the interval notation $[P, R]$ to mean the set of all elements of H obeying $P \leq Q \leq R$:

$$[P, R] \;=\; \{\, Q \in H \mid P \leq Q \leq R \,\}.$$

Using $\mathsf{EC}\land$ and $\mathsf{EC}\lor$ several times we see that:

$$\begin{array}{rr} P_1 \land P_2 \sim_J P & P_1 \lor P_2 \sim_J P \\ (P_1 \land P_2) \land P_3 \sim_J P & (P_1 \lor P_2) \lor P_3 \sim_J P \\ \vdots & \vdots \\ ((P_1 \land P_2) \land \ldots) \land P_n \sim_J P & ((P_1 \lor P_2) \lor \ldots) \lor P_n \sim_J P \\ P_\land \sim_J P & P_\lor \sim_J P \\ P_\land \in [P]^J & P_\lor \in [P]^J \end{array}$$

$$\dfrac{}{(P \wedge Q)^* \le Q^*}\,\mathsf{Mop} \quad := \quad \dfrac{\dfrac{}{(P \wedge Q)^* = P^* \wedge Q^*}\,\mathsf{J3} \qquad \dfrac{}{P^* \wedge Q^* \le Q^*}}{(P \wedge Q)^* \le Q^*}$$

$$\dfrac{P \le Q}{P^* \le Q^*}\,\mathsf{Mo} \quad := \quad \dfrac{\dfrac{\dfrac{P \le Q}{P = P \wedge Q}}{P^* = (P \wedge Q)^*} \qquad \dfrac{}{(P \wedge Q)^* \le Q^*}\,\mathsf{Mop}}{P^* \le Q^*}$$

$$\dfrac{P \le Q \le P^*}{P^* = Q^*}\,\mathsf{Sand} \quad := \quad \dfrac{\dfrac{P \le Q}{P^* \le Q^*}\,\mathsf{Mo} \qquad \dfrac{\dfrac{Q \le P^*}{Q^* \le P^{**}}\,\mathsf{Mo} \qquad \dfrac{}{P^{**} = P^*}\,\mathsf{J2}}{Q^* \le P^*}}{P^* = Q^*}$$

$$\dfrac{P^* = Q^*}{P^* = Q^* = (P \wedge Q)^*}\,\mathsf{EC}\wedge \quad := \quad \dfrac{\dfrac{P^* = Q^*}{P^* = Q^* = P^* \wedge Q^*} \qquad \dfrac{}{P^* \wedge Q^* = (P \wedge Q)^*}\,\mathsf{J3}}{P^* = Q^* = (P \wedge Q)^*}$$

$$\dfrac{P^* = Q^*}{P^* = Q^* = (P \vee Q)^*}\,\mathsf{EC}\vee \quad := \quad \dfrac{P^* = Q^* \qquad \dfrac{\dfrac{\dfrac{}{P \le P \vee Q} \qquad \dfrac{\dfrac{}{P \le P^*}\,\mathsf{J1} \qquad \dfrac{\dfrac{}{Q \le Q^*}\,\mathsf{J1} \qquad \dfrac{P^* = Q^*}{Q^* = P^*}}{Q \le P^*}}{P \vee Q \le P^*}}{P \le P \vee Q \le P^*}}{P^* = (P \vee Q)^*}\,\mathsf{Sand}}{P^* = Q^* = (P \vee Q)^*}$$

$$\dfrac{P \le Q \le R \quad P^* = R^*}{P^* = Q^* = R^*}\,\mathsf{ECS} \quad := \quad \dfrac{\dfrac{\dfrac{P \le Q \le R \qquad \dfrac{}{R \le R^*}\,\mathsf{J1} \qquad \dfrac{P^* = R^*}{R^* = P^*}}{P \le Q \le P^*}}{P^* = Q^*}\,\mathsf{Sand} \qquad P^* = R^*}{P^* = Q^* = R^*}$$

Figure 1: J-operators: basic derived rules

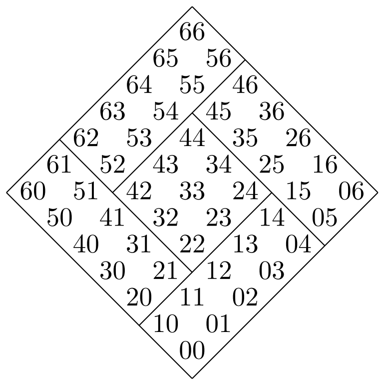
and using ECS we can see that all elements between $P_\wedge$ and $P_\vee$ are J-equivalent to P:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P_\wedge \sim_J P}{{P_\wedge}^* = P^*} \quad \dfrac{P_\vee \sim_J P}{{P_\vee}^* = P^*}}{{P_\wedge}^* = {P_\vee}^*} \quad P_\wedge \leq Q \leq P_\vee}{{P_\wedge}^* = Q^* = {P_\vee}^*}\ \mathsf{ECS} \quad {P_\vee}^* = P^*}{Q^* = P^*}}{Q \sim_J P}$$

so $[P_\wedge, P_\vee] \subseteq [P]^J$. This means that *J-regions are intervals*.

3 Cuts stopping midway

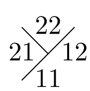
Look at the figure below, that shows a partition of a ZHA $A = [00, 66]$ into five regions, each region being an interval; this partition does not come from a slashing, as it has cuts that stop midway. Define an operation '$\cdot^*$' on A, that works by taking each truth-value P in it to the top element of its region; for example, $30^* = 61$.



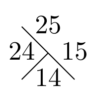
It is easy to see that '$\cdot^*$' obeys J1 and J2; however, it does *not* obey J3 — we will prove that in sec.3.1. As we will see, *the partitions of a ZHA into intervals that obey* J1, J2, J3 *ae exactly the slashings;* or, in other words, *every J-operator comes from a slashing*.

3.1 The are no Y-cuts and no λ-cuts

We want to see that if a partition of a ZHA H into intervals has "Y-cuts" or "λ-cuts", like these parts of the last diagram in sec.3,



$\Leftarrow$ this is a Y-cut



$\Leftarrow$ this is a λ-cut

then the operation J that takes each element to the top of its equivalence class cannot obey J1, J2 and J3 at the same time. We will prove that by deriving rules that say that if $11 \sim_J 12$ then $21 \sim_J 22$, and that if $15 \sim_J 25$ then $14 \sim_J 24$; actually, our rules will say that if $11^* = 12^*$ then $(11 \vee 21)^* = (12 \vee 21)^*$, and that if $15^* = 25^*$ then $(15 \wedge 24)^* = (25 \wedge 24)^*$. The rules are:

$$\dfrac{P^* = Q^*}{(P \vee R)^* = (Q \vee R)^*}\ \mathsf{NoYcuts} \quad := \quad \dfrac{\dfrac{\dfrac{P^* = Q^*}{P \vee R^* = Q \vee R^*}}{(P \vee R^*)^* = (Q \vee R^*)^*}}{(P \vee R)^* = (Q \vee R)^*}\ \circledvee_6 = \circledvee_4$$

$$\dfrac{P^* = Q^*}{(P \wedge R)^* = (Q \wedge R)^*}\ \mathsf{No\lambda cuts} \quad := \quad \dfrac{\dfrac{P^* = Q^*}{P^* \wedge R^* = Q^* \wedge R^*}}{(P \wedge R)^* = (Q \wedge R)^*}\ \mathsf{J3}$$

The expansion of double bar labeled '$\circledvee_6 = \circledvee_4$' in the top derivation uses twice the derived rule $\circledvee_6 = \circledvee_4$, that is easy to prove using the cubes of sec.4.

4 How J-operators interact with connectives

The axiom $\mathsf{J3}$ says that $(P \land Q)^* = P^* \land Q^*$ — it says something about how '$\cdot^*$' interacts with '$\land$'. Let's introduce a shorter notation. There are eight ways to replace each of the '?'s in $(P^? \land Q^?)^?$ by either nothing or a star. We establish that the three '?'s in $(P^? \land Q^?)^?$ are "worth" 1, 2 and 4 respectively, and we use $P \owedge_n Q$ to denote $(P^? \land Q^?)^?$ with the bits "that belong to n" replaced by stars. So:

$$\begin{array}{rclcrcl}
\owedge_0 & = & P \land Q, & & \owedge_4 & = & (P \land Q)^*, \\
\owedge_1 & = & P^* \land Q, & & \owedge_5 & = & (P^* \land Q)^*, \\
\owedge_2 & = & P \land Q^*, & & \owedge_6 & = & (P \land Q^*)^*, \\
\owedge_3 & = & P^* \land Q^*, & & \owedge_7 & = & (P^* \land Q^*)^*.
\end{array}$$

We omit the arguments of $\owedge_n$ when they are P and Q — so we can rewrite $(P \land Q)^* = P^* \land Q^*$ as $\owedge_4 = \owedge_3$. These conventions also hold for $\ovee$ and $\ominus\!\!\!\to$.

It is easy to prove each one of the arrows in the cubes below ($A \longrightarrow B$ means $A \leq B$):

$$\begin{tikzcd}[column sep=tiny]
& & (P^*{\land}Q^*)^* & & \\
(P^*{\land}Q)^* \arrow[urr] & & P^*{\land}Q^* \arrow[u] & & (P{\land}Q^*)^* \arrow[ull] \\
P^*{\land}Q \arrow[u] \arrow[urr] & & (P{\land}Q)^* \arrow[ull] \arrow[urr] & & P{\land}Q^* \arrow[ull] \arrow[u] \\
& & P{\land}Q \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& & (P^*{\lor}Q^*)^* & & \\
(P^*{\lor}Q)^* \arrow[urr] & & P^*{\lor}Q^* \arrow[u] & & (P{\lor}Q^*)^* \arrow[ull] \\
P^*{\lor}Q \arrow[u] \arrow[urr] & & (P{\lor}Q)^* \arrow[ull] \arrow[urr] & & P{\lor}Q^* \arrow[ull] \arrow[u] \\
& & P{\lor}Q \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& & (P^*{\to}Q^*)^* \arrow[drr] & & \\
(P^*{\to}Q)^* \arrow[urr] \arrow[drr] & & P^*{\to}Q^* \arrow[u] \arrow[drr] & & (P{\to}Q^*)^* \\
P^*{\to}Q \arrow[u] \arrow[urr] \arrow[drr] & & (P{\to}Q)^* \arrow[urr] & & P{\to}Q^* \arrow[u] \\
& & P{\to}Q \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

Let's write their sets of elements as $\owedge_{0\ldots7} := \{\owedge_0, \ldots, \owedge_7\}$, $\ovee_{0\ldots7} := \{\ovee_0, \ldots, \ovee_7\}$, and $\ominus\!\!\!\to_{0\ldots7} := \{\ominus\!\!\!\to_0, \ldots, \ominus\!\!\!\to_7\}$. The cubes above — we will call them the "obvious and-cube", the "obvious or-cube", and the "obvious implication-cube" — can be interpreted as directed graphs $(\owedge_{0\ldots7}, \mathsf{OCube}_\land)$, $(\ovee_{0\ldots7}, \mathsf{OCube}_\lor)$, $(\ominus\!\!\!\to_{0\ldots7}, \mathsf{OCube}_\to)$.

The "extended cubes" will be the directed graphs with the arrows above plus the ones coming from these derived rules:

$$\dfrac{}{(P^* \wedge Q^*)^* = P^* \wedge Q^* = (P \wedge Q)^*}\ \owedge_7 = \owedge_3 = \owedge_4 \quad := \quad \dfrac{\dfrac{\dfrac{}{P^{**} = P^*}\ \mathsf{J2} \quad \dfrac{}{Q^{**} = Q^*}\ \mathsf{J2}}{(P^* \wedge Q^*)^* = P^{**} \wedge Q^{**} = P^* \wedge Q^* = (P \wedge Q)^*}\ \mathsf{J3}}{(P^* \wedge Q^*)^* = P^* \wedge Q^* = (P \wedge Q)^*}$$

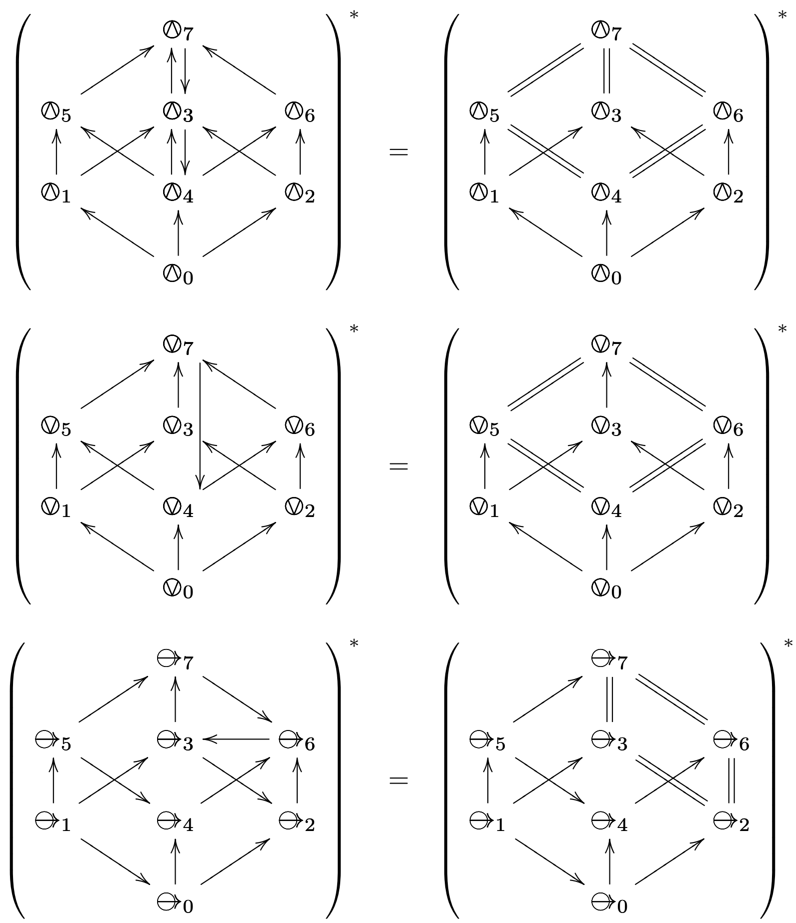
$$\dfrac{}{(P^* \vee Q^*)^* \leq (P \vee Q)^*}\ \ovee_7 \leq \ovee_3 \quad := \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \leq P \vee Q}}{P^* \leq (P \vee Q)^*}\ \mathsf{Mo} \quad \dfrac{\dfrac{}{Q \leq P \vee Q}}{Q^* \leq (P \vee Q)^*}\ \mathsf{Mo}}{P^* \vee Q^* \leq (P \vee Q)^*}}{(P^* \vee Q^*)^* \leq (P \vee Q)^{**}}\ \mathsf{Mo}}{(P^* \vee Q^*)^* \leq (P \vee Q)^*}\ \mathsf{J2}$$

$$\dfrac{}{(P \to Q^*)^* \leq P^* \to Q^*}\ \orightarrow_6 \leq \orightarrow_3 \quad := \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \to Q^* \leq P \to Q^*}}{(P \to Q^*) \wedge P \leq Q^*}}{((P \to Q^*) \wedge P)^* \leq Q^{**}}\ \mathsf{Mo}}{((P \to Q^*) \wedge P)^* \leq Q^*}\ \mathsf{J2}}{(P \to Q^*)^* \wedge P^* \leq Q^*}\ \mathsf{J3}}{(P \to Q^*)^* \leq P^* \to Q^*}$$

where $\owedge_7 = \owedge_3 = \owedge_4$ will be interpreted as these arrows:

$$(P^* \wedge Q^*)^* \leftrightarrows P^* \wedge Q^* \leftrightarrows (P \wedge Q)^*$$

The directed graphs of these "extended cubes" will be called $(\owedge_{0\ldots7}, \mathsf{ECube}_\wedge)$, $(\ovee_{0\ldots7}, \mathsf{ECube}_\vee)$, $(\orightarrow_{0\ldots7}, \mathsf{ECube}_\to)$. We are interested in the (non-strict) partial orders that they generate, and we want an easy way to remember these partial orders. The figure below shows these extended cubes at the left, and at the right the "simplified cubes", $\mathsf{SCube}_\wedge$, $\mathsf{SCube}_\vee$, and $\mathsf{SCube}_\to$, that generate the same partial orders that the extended cubes.



From these cubes it is easy to see, for example, that we can prove $\circledvee_5 = \circledvee_6$ (as a derived rule).

5 Valuations

Let $H_\odot$ and $J_\odot$ be a ZHA and a J-operator on it, and let $v_\odot$ be a function from the set $\{P, Q\}$ to H. By an abuse of language $v_\odot$ will also denote the triple $(H_\odot, J_\odot, v_\odot)$ — and by a second abuse of language $v_\odot$ will also denote the obvious extension of $v_\odot : \{P, Q\} \to H$ to the set of all valid expressions formed from P, Q, $\cdot^*$, $\top$, $\bot$, and the connectives.

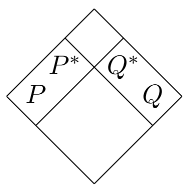
Let $i, j \in \{0, \ldots, 7\}$. Then $(\owedge_i, \owedge_j) \in \mathsf{SCube}^*_\wedge$ means that $\owedge_i \le \owedge_j$ is a theorem, and so $v_\odot(\owedge_i) \le v_\odot(\owedge_j)$ holds; i.e.,

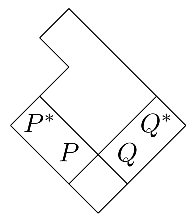
$$\mathsf{SCube}^*_\wedge \subseteq \{\, (\owedge_i, \owedge_j) \mid i, j \in \{0, \ldots, 7\},\ v_\odot(\owedge_i) \le v_\odot(\owedge_j) \,\}$$

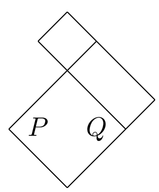
and the same for:

$$\begin{array}{r} \mathsf{SCube}^*_\vee \subseteq \{\, (\ovee_i, \ovee_j) \mid i, j \in \{0, \ldots, 7\},\ v_\odot(\ovee_i) \le v_\odot(\ovee_j) \,\} \\ \mathsf{SCube}^*_\to \subseteq \{\, (\oto_i, \oto_j) \mid i, j \in \{0, \ldots, 7\},\ v_\odot(\oto_i) \le v_\odot(\oto_j) \,\} \end{array}$$

Some valuations that turn these '$\subseteq$'s into '$=$'. Let

$(H_\wedge, J_\wedge, v_\wedge) =$ 

$(H_\vee, J_\vee, v_\vee) =$ 

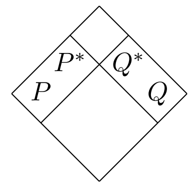
$(H_\to, J_\to, v_\to) =$ 

then

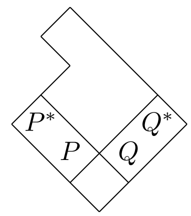
$$\begin{array}{r} \mathsf{SCube}^*_\wedge = \{\, (\owedge_i, \owedge_j) \mid i, j \in \{0, \ldots, 7\},\ v_\wedge(\owedge_i) \le v_\wedge(\owedge_j) \,\} \\ \mathsf{SCube}^*_\vee = \{\, (\ovee_i, \ovee_j) \mid i, j \in \{0, \ldots, 7\},\ v_\vee(\ovee_i) \le v_\vee(\ovee_j) \,\} \\ \mathsf{SCube}^*_\to = \{\, (\oto_i, \oto_j) \mid i, j \in \{0, \ldots, 7\},\ v_\to(\oto_i) \le v_\to(\oto_j) \,\} \end{array}$$

or, in more elementary terms:

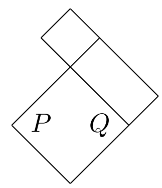
A very important fact. For any i and j,

$\owedge_i \leq \owedge_j$ is a theorem iff it is true in 

,

$\ovee_i \leq \ovee_j$ is a theorem iff it is true in 

,

$\ominus\!\!\to_i \leq \ominus\!\!\to_j$ is a theorem iff it is true in 

.

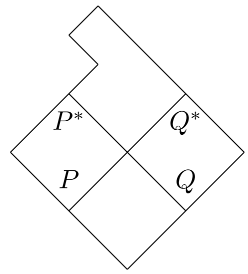
The very important fact, and the valuations $v_\wedge$, $v_\vee$, $v_\to$, give us:

- a way to *remember* which sentences of the forms $\owedge_i \leq \owedge_j$, $\ovee_i \leq \ovee_j$, $\ominus\!\!\to_i \leq \ominus\!\!\to_j$ are theorems;
- countermodels for all the sentences of these forms not in $\mathsf{SCube}_\wedge$, $\mathsf{SCube}_\vee$, $\mathsf{SCube}_\to$. For example, $\ovee_7 \leq \ovee_4$ is not in $\mathsf{SCube}_\vee$; and $v_\vee(\ovee_7) \leq v_\vee(\ovee_4)$, which shows that $\ovee_7 \leq \ovee_4$ can't be a theorem.

An observation. I arrived at the cubes $\mathsf{ECube}^*_\wedge$, $\mathsf{ECube}^*_\vee$, $\mathsf{ECube}^*_\to$ by taking the material in the corollary 5.3 of chapter 5 in [Bel88] and trying to make it fit into less mental space (as discussed in [Och13]); after that I wanted to be sure that each arrow that is not in the extended cubes has a countermodel, and I found the countermodels one by one; then I wondered if I could find a single countermodel for all non-theorems in $\mathsf{ECube}^*_\wedge$ (and the same for $\mathsf{ECube}^*_\vee$ and $\mathsf{ECube}^*_\to$), and I tried to start with a valuation that distinguished *some* equivalence classes in $\mathsf{ECube}^*_\wedge$, and change it bit by bit, getting valuations that distinguished more equivalence classes at every step. Eventually I arrived at $v_\wedge$, $v_\vee$ and at $v_\to$, and at the — surprisingly nice — "very important fact" above.

Note that this valuation

$$(H_{\wedge\vee}, J_{\wedge\vee}, v_{\wedge\vee}) \quad =$$



distinguishes all equivalence classes in $\mathsf{ECube}^*_\wedge$ and in $\mathsf{ECube}^*_\vee$, but not in $\mathsf{ECube}^*_\to$... it "thinks" that $P \to Q$ and $P^* \to Q$ are equal.

6 Polynomial J-operators

It is not hard to check that for any Heyting Algebra H and any $Q, R \in H$ the operations $(\neg\neg)$, …, $(\vee Q \wedge {\to} R)$ below are J-operators:

$$\begin{array}{rcl} (\neg\neg)(P) & = & \neg\neg P \\ ({\to}{\to}R)(P) & = & (P{\to}R){\to}R \\ (\vee Q)(P) & = & P \vee Q \\ ({\to}R)(P) & = & P{\to}R \\ (\vee Q \wedge {\to}R)(P) & = & (P{\vee}Q) \wedge (P{\to}R) \end{array}$$

Checking that they are J-operators means checking that each of them obeys J1, J2, J3. Let's define formally what are J1, J2 and J3 "for a given $F : H \to H$":

$$\begin{array}{rcc} \mathsf{J1}_F & := & (P \leq F(P)) \\ \mathsf{J2}_F & := & (F(P) = F(F(P)) \\ \mathsf{J3}_F & := & (F(P \wedge P') = F(P) \wedge F(P')) \end{array}$$

and:

$$\mathsf{J123}_F \quad := \quad \mathsf{J1}_F \wedge \mathsf{J2}_F \wedge \mathsf{J3}_F.$$

Checking that $(\neg\neg)$ obeys J1, J2, J3 means proving $\mathsf{J123}_{(\neg\neg)}$ using only the rules from intuitionist logic from section 10 of [PH1]; we will leave the proof of this, of and $\mathsf{J123}_{({\to}{\to}R)}$, $\mathsf{J123}_{(\vee Q)}$, and so on, to the reader.

The J-operator $(\vee Q \wedge {\to}R)$ is a particular case of building more complex J-operators from simpler ones. If $J, K : H \to H$, we define:

$$(J \wedge K) := \lambda P{:}H.(J(P){\wedge}K(P))$$

it not hard to prove $\mathsf{J123}_{(J \wedge K)}$ from $\mathsf{J123}_J$ and $\mathsf{J123}_K$ using only the rules from intuitionistic logic.

The J-operators above are the first examples of J-operators in Fourman and Scott's "Sheaves and Logic" ([FS79]); they appear in pages 329–331, but with these names (our notation for them is at the right):

(i) *The closed quotient,*

$$J_a p = a \vee p \qquad J_Q = (\vee Q).$$

(ii) *The open quotient,*

$$J^a p = a \to p \qquad J^R = ({\to}R).$$

(iii) *The Boolean quotient.*

$$B_a p = (p \to a) \to a \qquad B_R = ({\to}{\to}R).$$

(iv) *The forcing quotient.*

$$(J_a \wedge J^b)p = (a \vee p) \wedge (b \to p) \qquad (J_Q \wedge J^R) = (\vee Q \wedge {\to}R).$$

(vi) *A mixed quotient.*

$$(B_a \wedge J^a)p = (p \to a) \to p \qquad (B_Q \wedge J^Q) = (\to\to Q \wedge \to Q).$$

The last one is tricky. From the definition of B_a and J^a what we have is

$$(B_a \wedge J^a)p = ((p \to a) \to a) \wedge (a \to p),$$

but it is possible to prove

$$((p \to a) \to a) \wedge (a \to p) \quad \leftrightarrow \quad ((p \to a) \to p)$$

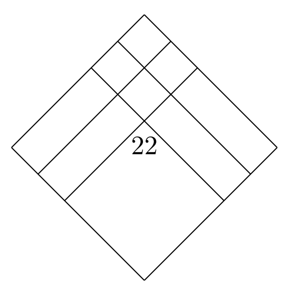
intuitionistically.

The operators above are "polynomials on $P, Q, R, \to, \wedge, \vee, \bot$" in the terminology of Fourman/Scott: "If we take a polynomial in $\to, \wedge, \vee, \bot$, say, $f(p, a, b, \ldots)$, it is a decidable question whether for all $a, b, \ldots$ it defines a J-operator" (p.331).

When I started studying sheaves I spent several years without any visual intuition about the J-operators above. I was saved by ZHAs and brute force — and the brute force method also helps in testing if a polynomial (in the sense above) is a J-operator in a particular case. For example, take the operators $\lambda P{:}H.(P \wedge 22)$ and $(\vee 22)$ on $H = [00, 44]$:

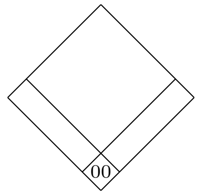
$$\lambda P{:}H.(P \wedge 22) \quad = \quad \begin{array}{ccccccccc}
 & & & & 22 & & & & \\
 & & & 22 & & 22 & & & \\
 & & 22 & & 22 & & 22 & & \\
 & 21 & & 22 & & 22 & & 12 & \\
20 & & 21 & & 22 & & 12 & & 02 \\
 & 20 & & 21 & & 12 & & 02 & \\
 & & 20 & & 11 & & 02 & & \\
 & & & 10 & & 01 & & & \\
 & & & & 00 & & & &
\end{array}$$

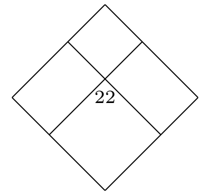
$$(\vee 22) \quad = \quad \begin{array}{ccccccccc}
 & & & & 44 & & & & \\
 & & & 43 & & 34 & & & \\
 & & 42 & & 33 & & 24 & & \\
 & 42 & & 32 & & 23 & & 24 & \\
42 & & 32 & & 22 & & 23 & & 24 \\
 & 32 & & 22 & & 22 & & 23 & \\
 & & 22 & & 22 & & 22 & & \\
 & & & 22 & & 22 & & & \\
 & & & & 22 & & & &
\end{array} \quad =$$

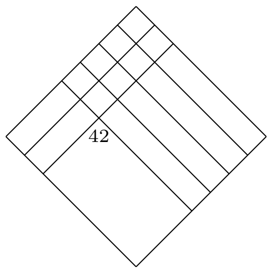


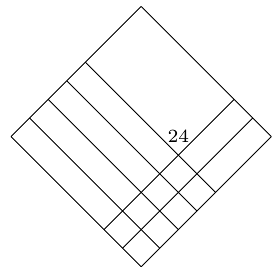
The first one, $\lambda P{:}H.(P \wedge 22)$, is not a J-operator; one easy way to see that is to look at the region in which the result is 22 — its top element is 44, and this violates the conditions on slash-operators in sec.1.3. The second operator, $(\vee 22)$, is a slash operator and a J-operator; at the right we introduce a convenient notation for visualizing the action of a polynomial slash-operator, in which we draw only the contours of the equivalence classes and the constants that appear in the polynomial.

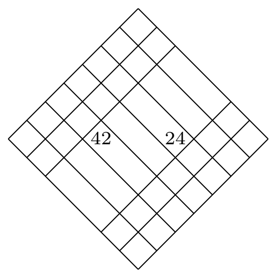
Using this new notation, we have:

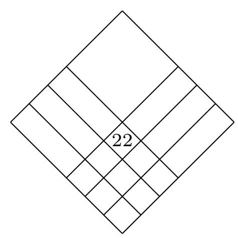
$(\neg\neg) \;=\; (\to\to 00) \;=$ 

$(\to\to 22) \;=$ 

$(\vee 42) \;=$ 

$(\to 24) \;=$ 

$(\vee 42 \land \to 24) \;=$ 

$(\to\to 22 \land \to 22) \;=$ 

Note that the slashing for $(\vee 42 \land \to 24)$ has all the cuts for $(\vee 42)$ plus all the cuts for $(\to 24)$, and $(\vee 42 \land \to 24)$ "forces $42 \leq 24$" in the following sense: if $P^* = (\vee 42 \land \to 24)(P)$ then $42^* \leq 24^*$.

6.1 An algebra of piccs

We saw in the last section a case in which $(J \land K)$ has all the cuts from J plus all the cuts from K; this suggests that we *may* have an operation dual to that, that behaves as this: $(J \lor K)$ has exactly the cuts that are both in J and in K:

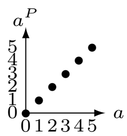
$$\begin{array}{rcl}\mathsf{Cuts}(J \land K) &=& \mathsf{Cuts}(J) \cup \mathsf{Cuts}(K) \\ \mathsf{Cuts}(J \lor K) &=& \mathsf{Cuts}(J) \cap \mathsf{Cuts}(K)\end{array}$$

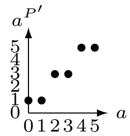
And it $J_1, \ldots, J_n$ are all the slash-operators on a given ZHA, then

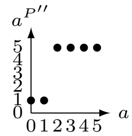
$$\begin{array}{rclcl}
\mathsf{Cuts}(J_1 \wedge \ldots \wedge J_n) & = & \mathsf{Cuts}(J_1) \cup \ldots \cup \mathsf{Cuts}(J_k) & = & \text{(all cuts)} \\
\mathsf{Cuts}(J_1 \vee \ldots \vee J_n) & = & \mathsf{Cuts}(J_1) \cap \ldots \cap \mathsf{Cuts}(J_k) & = & \text{(no cuts)}
\end{array}$$

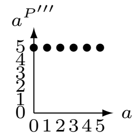
yield the minimal element and the maximal element, respectively, of an algebra of slash-operators; note that the slash-operator with "all cuts" is the identity map $\lambda P\colon H.P$, and the slash-operator with "no cuts" is the one that takes all elements to $\top$: $\lambda P\colon H.\top$. This yields a lattice of slash-operators, in which the partial order is $J \leq K$ iff $\mathsf{Cuts}(J) \supseteq \mathsf{Cuts}(K)$. This is somewhat counterintuitive if we think in terms of cuts — the order seems to be reversed — but it makes a lot of sense if we think in terms of piccs (sec.1.2) instead.

Each picc P on $\{0, \ldots, n\}$ has an associated function $\cdot^P$ that takes each element to the top element of its equivalence class. If we define $P \leq P'$ to mean $\forall a \in \{0, \ldots, n\}.\, a^P \leq a^{P'}$, then we have this:



$\leq$


$\leq$


$\leq$


$$\begin{array}{ccccccc}
0|1|2|3|4|5 & \leq & 01|23|45 & \leq & 01|2345 & \leq & 012345 \\
P & \leq & P' & \leq & P'' & \leq & P'''
\end{array}$$

This yields a partial order on piccs, whose bottom element is the identity function $0|1|2|\ldots|n$, and the top element is $012\ldots n$, that takes all elements to n.

The piccs on $\{0, \ldots, n\}$ form a Heyting Algebra, where $\top = 01\ldots n$, $\bot = 0|1|\ldots|n$, and '$\wedge$' and '$\vee$' are the operations that we have discussed above; it is possible to define a '$\to$' there, but this '$\to$' is not going to be useful for us and we are mentioning it just as a curiosity. We have, for example:

$$\begin{tikzcd}
& 01234 & \\
& 01|234 \arrow[u] & \\
0|1|234 \arrow[ur] & & 01|2|34 \arrow[ul] \\
& 0|1|2|34 \arrow[ul] \arrow[ur] & \\
& 0|1|2|3|4 \arrow[u] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \top & \\
& P \vee Q \arrow[u] & \\
P \arrow[ur] & & Q \arrow[ul] \\
& P \wedge Q \arrow[ul] \arrow[ur] & \\
& \bot \arrow[u] &
\end{tikzcd}$$

6.2 An algebra of J-operators

Fourman and Scott define the operations $\wedge$ and $\vee$ on J-operators in pages 325 and 329 ([FS79]), and in page 331 they list ten properties of the algebra of

J-operators:

(i)	$J_a \vee J_b$	$=$	$J_{a\vee b}$	$(\vee 21) \vee (\vee 12) = (\vee 22)$
(ii)	$J^a \vee J^b$	$=$	$J^{a\wedge b}$	$(\to 32) \vee (\to 23) = (\to 22)$
(iii)	$J_a \wedge J_b$	$=$	$J_{a\wedge b}$	$(\vee 21) \wedge (\vee 12) = (\vee 11)$
(iv)	$J^a \wedge J^b$	$=$	$J^{a\vee b}$	$(\to 32) \wedge (\to 23) = (\to 33)$
(v)	$J_a \wedge J^a$	$=$	$\bot$	$(\vee 22) \wedge (\to 22) = (\bot)$
(vi)	$J_a \vee J^a$	$=$	$\top$	$(\vee 22) \vee (\to 22) = (\top)$
(vii)	$J_a \vee K$	$=$	$K \circ J_a$	
(viii)	$J^a \vee K$	$=$	$J^a \circ K$	
(ix)	$J_a \vee B_a$	$=$	B_a	
(x)	$J^a \vee B_b$	$=$	$B_{a\to b}$	

The first six are easy to visualize; we won't treat the four last ones. In the right column of the table above we've put a particular case of (i), ..., (vi) in our notation, and the figures below put all together.

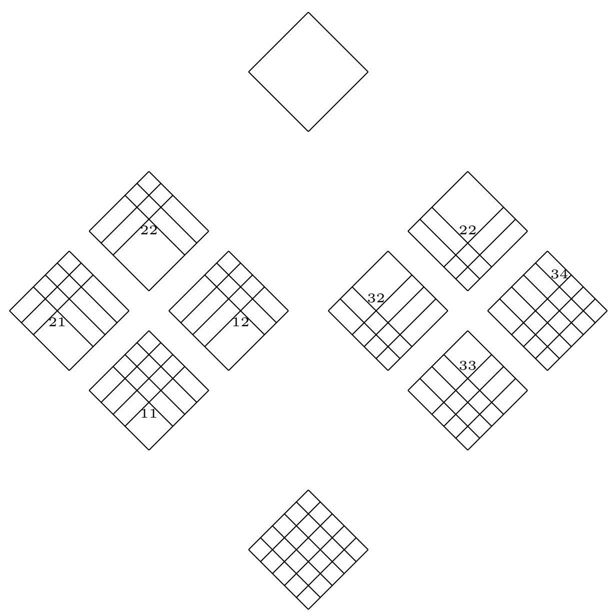
In Fourman and Scott's notation,

$$
\begin{tikzcd}
 & & & J_\top = \top = J^\bot & & & \\
 & J_{22} \arrow[urr] & & & & J^{22} \arrow[ull] & \\
J_{21} \arrow[ur] & & J_{12} \arrow[ul] & & J^{32} \arrow[ur] & & J^{23} \arrow[ul] \\
 & J_{11} \arrow[ul] \arrow[ur] & & & & J^{11} \arrow[ul] \arrow[ur] & \\
 & & & J_\bot = \bot = J^\top \arrow[ull] \arrow[urr] & & &
\end{tikzcd}
$$

in our notation,

$$
\begin{tikzcd}[column sep=small]
 & & & (\top\vee)=(\lambda P.\top)=(\bot\to) & & & \\
 & (22\vee) \arrow[urr] & & & & (22\to) \arrow[ull] & \\
(21\vee) \arrow[ur] & & (12\vee) \arrow[ul] & & (32\to) \arrow[ur] & & (23\to) \arrow[ul] \\
 & (11\vee) \arrow[ul] \arrow[ur] & & & & (33\to) \arrow[ul] \arrow[ur] & \\
 & & & (\bot\vee)=(\lambda P.P)=(\top\to) \arrow[ull] \arrow[urr] & & &
\end{tikzcd}
$$

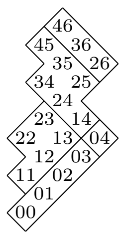
and drawing the polynomial J-operators as in sec.6:

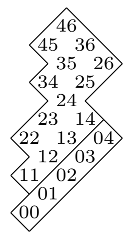


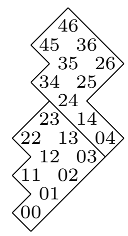
6.3 All slash-operators are polynomial

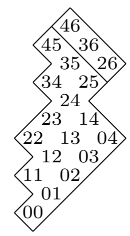
Here is an easy way to see that all slashings — i.e., J-operators on ZHAs — are polynomial. Every slashing J has only a finite number of cuts; call them

$J_1, \ldots, J_n$. For example:

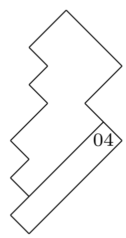
$J =$ 

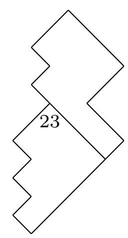
$J_1 =$ 

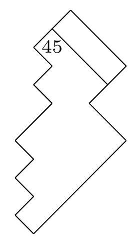
$J_2 =$ 

$J_3 =$ 

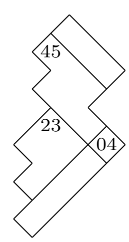
Each cut J_i divides the ZHA into an upper region and a lower region, and $J_i(00)$ yields the top element of the lower region. Also, $(\to\!\to J_i(00))$ is a polynomial way of expressing that cut:

$J_1 =$ $(\to\!\to 04) =$ 

$J_2 =$ $(\to\!\to 23) =$ 

$J_3 =$ $(\to\!\to 45) =$ 

The conjunction of these '$(\to\!\to J_i(00))$'s yields the original slashing:

$(\to\!\to 04) \wedge (\to\!\to 23) \wedge (\to\!\to 45) =$  $= J$

7 Categories, toposes, sheaves

In this section I will explain *very, very briefly* how to adapt what we saw about J-operators to toposes. The first big diagram that we will try to understand is the in one in Figure 2 below, that shows in its upper part a structure $((P, A), Q) \leftrightsquigarrow (H, J)$ with a 2CG with question marks and its associated ZHA with J-operator, and in its lower part the classifier Ω of the topos $\mathbf{Set}^{(P,A)}$ and the local operator $j : \Omega \to \Omega$ that is associated to Q and J. This big diagram shows how to define the classifier and the local operator, and in sec.7.8 we will see another big diagram that shows how to define sheaves and sheafification.

I will omit some technical details — a very readable reference for them is [McL92], chapters 13 and 22. I learned most of them from [Bel88], though.

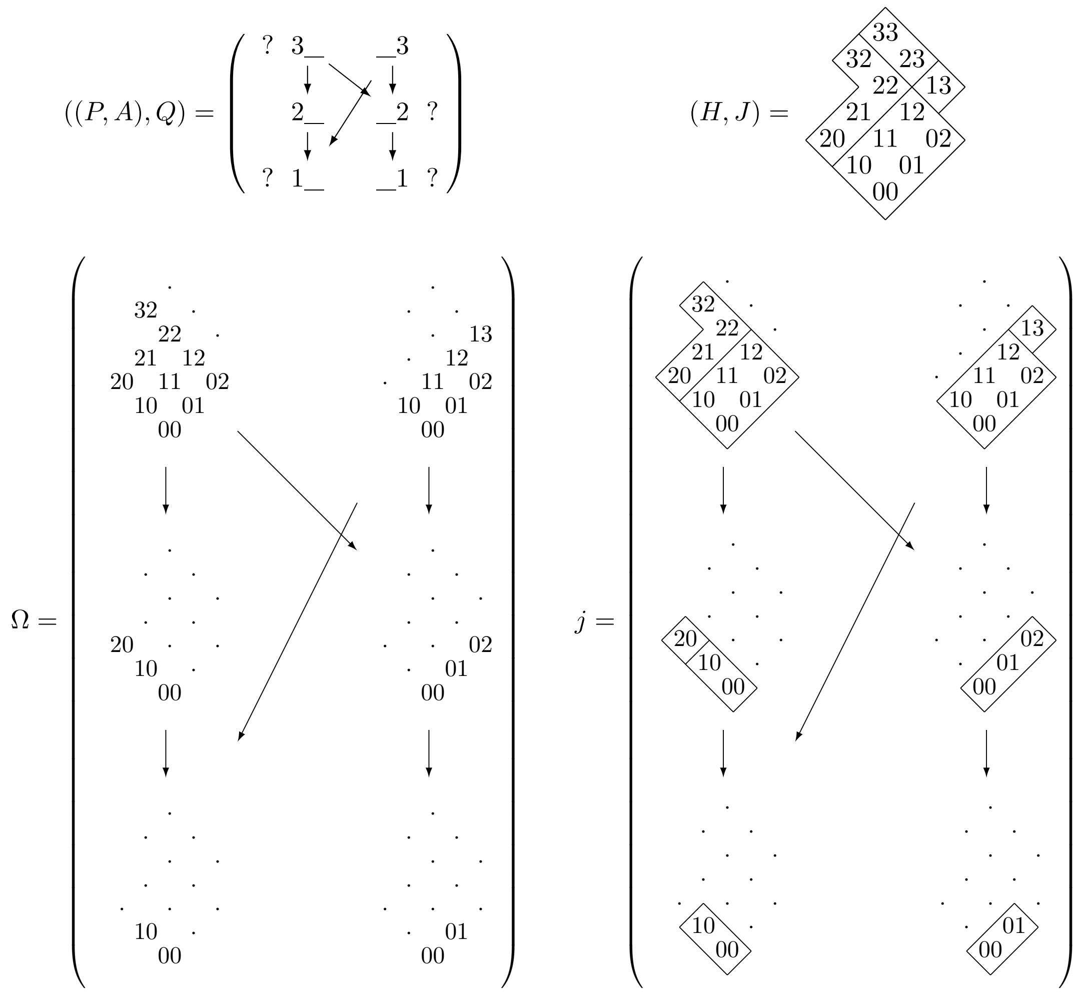
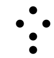


Figure 2: The classifier and a local operator in a particular case

7.1 Toposes of the form $\mathbf{Set}^{(P,A)}$

In sec.2 of [PH1] we established that the same bullet diagram — say,  — could be intepreted as subset of $\mathbb{Z}^2$, as a DAG, or as poset, depending on the context. Now we will do something similar for graphs whose nodes are labeled. A diagram like this

$$\begin{array}{ccccc} & 1 & & & \\ \swarrow & & \searrow & & \\ 2 & & 3 & & \\ \searrow & \swarrow & & \searrow & \\ & 4 & & 5 \\ & \searrow & \swarrow & & \\ & & 6 & & \end{array}$$

is interpreted as a DAG by default, but in this section it will be also be interpreted as a (posetal) category in some contexts. We will keep the same notation: if (P, A) is a DAG then we will denote (P, A) “regarded as a category” by (P, A).

A functor F from a category (P, A) to **Set** can be drawn as a diagram with the same shape as (P, A). If we draw the internal view of $F : (P, A) \to \mathbf{Set}$ over its internal view as in the introduction of [PH1] we get this diagram:

$$\left(\begin{array}{ccccc} & 1 & & & \\ \swarrow & & \searrow & & \\ 2 & & 3 & & \\ \searrow & \swarrow & & \searrow & \\ & 4 & & 5 \\ & \searrow & \swarrow & & \\ & & 6 & & \end{array}\right) \longmapsto \left(\begin{array}{ccccc} & F_1 & & & \\ \swarrow & & \searrow & & \\ F_2 & & F_3 & & \\ \searrow & \swarrow & & \searrow & \\ & F_4 & & F_5 \\ & \searrow & \swarrow & & \\ & & F_6 & & \end{array}\right)$$

$$(P, A) \xrightarrow{\quad\quad F \quad\quad} \mathbf{Set}$$

The ‘$\longmapsto$’ in it stands for a bunch of ‘$\longmapsto$’s, one for each object and one for each morphism.

We will only draw the upper-right part of diagrams like the one above. With this convention, an object $F \in \mathbf{Set}^{(P,A)}$ can be drawn as:

$$F \quad = \quad \left(\begin{array}{ccccc} & F_1 & & & \\ \swarrow & & \searrow & & \\ F_2 & & F_3 & & \\ \searrow & \swarrow & & \searrow & \\ & F_4 & & F_5 \\ & \searrow & \swarrow & & \\ & & F_6 & & \end{array}\right)$$

Every category of the form $\mathbf{Set}^{(P,A)}$ where (P, A) is a finite graph is a topos — see [E], example A2.1.3 — so categories of the form $\mathbf{Set}^{(P,A)}$ are toposes whose objects can be drawn as (P, A)-shaped diagrams.

7.2 The logic of toposes of the form $\mathbf{Set}^{(P,A)}$

The terminal object $1 \in \mathbf{Set}^{(P,A)}$ is:

$$1 \quad = \quad \left(\begin{tikzcd}[column sep=small, row sep=small] & \{*\} \arrow[dl] \arrow[dr] & & \\ \{*\} \arrow[dr] & & \{*\} \arrow[dl] \arrow[dr] & \\ & \{*\} \arrow[dr] & & \{*\} \arrow[dl] \\ & & \{*\} & \end{tikzcd} \right)$$

and we can obtain all its subobjects by replacing some of the '$\{*\}$'s in it by empty sets. If we rewrite each $\{*\}$ as 1 and each $\emptyset$ as 0 and use a more compact notation, then $1 = \begin{smallmatrix} &1&&\\ 1&&1&\\ &1&&1\\ &&1& \end{smallmatrix}$ and:

$$\mathrm{Sub}(1) = \left\{ \begin{smallmatrix} &0&&\\ 0&&0&\\ &0&&0\\ &&0& \end{smallmatrix},\ \begin{smallmatrix} &0&&\\ 0&&0&\\ &0&&0\\ &&1& \end{smallmatrix},\ \begin{smallmatrix} &0&&\\ 0&&0&\\ &0&&1\\ &&1& \end{smallmatrix},\ \begin{smallmatrix} &0&&\\ 0&&0&\\ &1&&0\\ &&1& \end{smallmatrix},\ \begin{smallmatrix} &0&&\\ 0&&0&\\ &1&&1\\ &&1& \end{smallmatrix},\ \begin{smallmatrix} &0&&\\ 0&&1&\\ &1&&1\\ &&1& \end{smallmatrix},\ \begin{smallmatrix} &0&&\\ 1&&0&\\ &1&&0\\ &&1& \end{smallmatrix},\ \begin{smallmatrix} &0&&\\ 1&&0&\\ &1&&1\\ &&1& \end{smallmatrix},\ \begin{smallmatrix} &0&&\\ 1&&1&\\ &1&&1\\ &&1& \end{smallmatrix},\ \begin{smallmatrix} &1&&\\ 1&&1&\\ &1&&1\\ &&1& \end{smallmatrix} \right\}$$

The Heyting Algebra of subobjects of 1 when $(P,A) = \begin{smallmatrix} &\bullet&&\\ \bullet&&\bullet&\\ &\bullet&&\bullet\\ &&\bullet& \end{smallmatrix}$ is essentially the same as the order topology $\mathcal{O}_A(P)$ that we saw in sec.12 of [PH1]! This holds for all graphs, and when (P,A) is a 2CG — for example, when

$$(P,A) \quad = \quad \left(\begin{tikzcd}[column sep=small, row sep=small] 3\_ \arrow[d] \arrow[rd] & \_3 \arrow[d] \\ 2\_ \arrow[d] & \_2 \arrow[d] \arrow[ld] \\ 1\_ & \_1 \end{tikzcd} \right)$$

we can abbreviate the result further using the ideas is sec.15 of [PH1]:

$$\mathrm{Sub}(1) \quad = \quad \mathcal{O} \left(\begin{tikzcd}[column sep=small, row sep=small] 3\_ \arrow[d] \arrow[rd] & \_3 \arrow[d] \\ 2\_ \arrow[d] & \_2 \arrow[d] \arrow[ld] \\ 1\_ & \_1 \end{tikzcd} \right) \quad = \quad \begin{array}{ccccc} & & 33 & & \\ & 32 & & 23 & \\ & & 22 & & 13 \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{array}$$

So: the "logic" of a topos of the form $\mathbf{Set}^{(P,A)}$ — i.e., its Heyting Algebra of subobjects of the terminal — is exactly the topology $\mathcal{O}_A(P)$.

7.3 Morphisms as natural transformations

If F and G are objects of a category $\mathbf{Set}^{\mathbf{A}}$ and $T : F \to G$ is a morphism between them then F and G are functors and $T : F \to G$ is a natural transformation, and T has to obey a "naturalness condition" that says that for every morphism $v : B \to C$ in $\mathbf{A}$ a certain "obvious" square must commute. We can draw that condition as the commutativity of the middle square below,

$$
\begin{tikzcd}
B \arrow[d, "v"'] \\
C
\end{tikzcd}
\qquad
\begin{tikzcd}
FB \arrow[r, "TB"] \arrow[d, "Fv"'] & GB \arrow[d, "Gv"] \\
FC \arrow[r, "TC"] & GC
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, mapsto] \arrow[dd, mapsto] & (TB)(x) \arrow[d, mapsto] \\
& (Gv \circ TB)(x) \\
(Fv)(x) \arrow[r, mapsto] & (TC \circ Fv)(x)
\end{tikzcd}
$$

$$
\begin{tikzcd}
F \arrow[r, "T"] & G
\end{tikzcd}
$$

and as the domain of F and G is $\mathbf{Set}$ we can express that naturality as

$$\forall (v : B \to C). \forall x \in FB. (Gv \circ TB)(x) = (TC \circ Fv)(x)$$

and represent that as the square at the right above.

We will often draw these morphisms/natural transformations like this,

$$
\left(
\begin{tikzcd}[column sep=small, row sep=small]
& F_1 \arrow[dl] \arrow[dr] & & \\
F_2 \arrow[dr] & & F_3 \arrow[dl] \arrow[dr] & \\
& F_4 \arrow[dr] & & F_5 \arrow[dl] \\
& & F_6 &
\end{tikzcd}
\right)
\xrightarrow{T}
\left(
\begin{tikzcd}[column sep=small, row sep=small]
& F_1 \arrow[dl] \arrow[dr] & & \\
F_2 \arrow[dr] & & F_3 \arrow[dl] \arrow[dr] & \\
& F_4 \arrow[dr] & & F_5 \arrow[dl] \\
& & F_6 &
\end{tikzcd}
\right)
$$

leaving the category $\mathbf{A}$ implicit. The '$\xrightarrow{T}$' is a pack of six functions between sets, $T_1 : F_1 \to G_1$, ..., $T_6 : F_6 \to G_6$ — compare with the meaning of the '$\longmapsto$' in sec.7.1.

The definition of the local operator $j : \Omega \to \Omega$ in Figure 2 is a natural transformation written in a very compact form. In that example $j_3\_(21) = 32$.

7.4 The classifier

Take a map $t : 1 \to C$ in a topos. Choose a map $g : B \to C$ and form the pullback with t, obtaining maps $f : A \to B$ and $h : A \to 1$. We can prove that any map from the terminal is monic, and this implies that t is monic, and so, by a property of pullbacks, f is a monic too; and h is the unique map from A to the terminal. In a diagram:

$$
\begin{tikzcd} & 1 \arrow[d, "t"] \\ B \arrow[r, "g"'] & C \end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "t"] \\ B \arrow[r, "g"'] & C \end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd} A \arrow[r, "!"] \arrow[d, tail, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "t"] \\ B \arrow[r, "g"'] & C \end{tikzcd}
$$

We can consider that the operation "form the pullback with $t : 1 \rightarrowtail C$" receives a map $g : B \to C$ and returns a monic $f : A \rightarrowtail B$ "completing the pullback".

Every topos has a classifier object Ω and a "true" map $\top : 1 \rightarrowtail \Omega$ with the property that for every monic $f : A \rightarrowtail B$ there is a unique map $\chi : B \to \Omega$ "completing the pullback". In a diagram:

$$
\begin{tikzcd} A \arrow[d, tail, "f"'] \\ B \end{tikzcd}
\quad
\begin{tikzcd} 1 \arrow[d, tail, "\top"] \\ \Omega \end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd} A \arrow[r, "!"] \arrow[d, tail, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "\top"] \\ B \arrow[r, "\chi"'] & \Omega \end{tikzcd}
\qquad
\begin{tikzcd} & 1 \arrow[d, tail, "\top"] \\ B \arrow[r, "\chi"'] & \Omega \end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd} A' \arrow[r, "!"] \arrow[d, tail, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "\top"] \\ B \arrow[r, "\chi"'] & \Omega \end{tikzcd}
$$

These two operations, $f \mapsto \chi$ and $\chi \mapsto f'$, are not exactly inverse to one another: if we apply them in the order $f \mapsto \chi \mapsto f'$ we may obtain an f' that is isomorphic to f in the sense that there is an iso $A \leftrightarrow A'$ such that the triangle below commutes:

$$
\begin{tikzcd} A \arrow[rr, leftrightarrow] \arrow[dr, tail, "f"'] & & A' \arrow[dl, tail, "f'"] \\ & \Omega & \end{tikzcd}
$$

This is explained in [LS86, p.139]

7.5 The classifier and the local operator

We know that every category $\mathbf{Set}^{(P,A)}$ is a topos, but how do we calculate and visualize its classifier object Ω and the map $\top : 1 \to \Omega$? And what is the local operator $j : \Omega \to \Omega$ "associated to" our J-operator $J : \mathrm{Sub}(1) \to \mathrm{Sub}(1)$?

We need to start by understanding two pullbacks. Remember that:

- $\top : 1 \to \Omega$ has a property can be expressed in two equivalent ways: 1) for each object C we have $\mathrm{Sub}(C) \cong \mathrm{Hom}(C, \Omega)$, and 2) for every monic

$B \rightarrowtail C$ there is exactly one map $\chi_B : C \to \Omega$ making the square below — "the Q-shaped diagram" — a pullback:

$$\begin{tikzcd}
B \arrow[r, "!"] \arrow[d, tail, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "\top"] & \\
C \arrow[r, "\chi_B"] & \Omega \arrow[r, "j"] & \Omega
\end{tikzcd}$$

- a local operator (also called a "modality", a "Lawvere-Tierney topology", or a "topology") is a map $j : \Omega \to \Omega$ obeying $j \circ \top = \top$, $j \circ j = j$ and $j \circ \land = \land \circ (j \times j)$,
- a local operator j induces a j-closure operator — see chapter 21 of [McL92] or chapter 5 of [Bel88] —, and this j-closure operator can be seen as a map from each $\mathrm{Sub}(C)$ to itself. The closure of a subobject $i : B \rightarrowtail C$ is the subobject $\overline{1} : \overline{B} \rightarrowtail C$ obtained by pullback in the diagram below ("the rectangle"):

$$\begin{tikzcd}
\overline{B} \arrow[rr, "!"] \arrow[d, tail, "\overline{i}"'] \arrow[dr, phantom, "\lrcorner", very near start] & & 1 \arrow[d, tail, "\top"] \\
C \arrow[r, "\chi_B"] & \Omega \arrow[r, "j"] & \Omega
\end{tikzcd}$$

We will write the restriction of a local operator j to $\mathrm{Sub}(1)$ as $\mathsf{J}(j)$ and we will say that a j is "associated to" a J when $\mathsf{J}(j) = J$.

There are two ways to "understand" the pullbacks above: the first one is by doing the calculations formally and checking that everything works, the second one is by checking some particular cases and developing visual intuition from that.

7.6 Understanding the pullbacks formally

The calculations are routine if we know the right language, and if we suppose — without loss of generality — that the monix $i : B \rightarrowtail C$ is a "canonical subobject" in the sense that each $B(p) \subseteq C(p)$ and each function $B(p \xrightarrow{!} q) : B(p) \to B(q)$ is a restriction of the corresponding function $C(p \xrightarrow{!} q) : C(p) \to C(q)$.

We need some definitions:

$$\begin{array}{rcl}
1(p) & = & \{*\} \\
1(p \xrightarrow{!} q)(*) & = & * \\
\Omega(p) & = & \mathrm{Sub}(\downarrow p) \\
\Omega(p \xrightarrow{!} q)(R) & = & R \wedge \downarrow q \\
\top(p)(*) & = & \downarrow p \\
j(p)(R) & = & R^* \wedge \downarrow p \\
\chi_B(p)(R) & = & \{\, r \in \downarrow p \mid C(p \xrightarrow{!} r)(c) \in B(r) \,\}
\end{array}$$

The first step is to check the five naturality conditions in the next page — we leave the rest to the reader. The main exercise is to check that if the monic $i : B \rightarrowtail C$ is $i : P \rightarrowtail 1$ for a truth-value P then its closure is $i : \overline{P} \rightarrowtail 1$ with $\overline{P}$ being exactly $J(P)$, i.e., P^*.

$$
\begin{tikzcd}
p \arrow[d, "!"'] & B(p) \arrow[r, "!"] \arrow[d, "B(p \xrightarrow{!} q)"'] & \{*\} \arrow[d, "!"] & b \arrow[r, mapsto] \arrow[d, mapsto] & * \arrow[d, mapsto] \\
q & B(q) \arrow[r, "!"] & \{*\} & B(p \xrightarrow{!} q)(b) \arrow[r, mapsto] & \begin{array}{c} * \\ * \end{array}
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \arrow[r] & 1
\end{tikzcd}
$$

$$
\begin{tikzcd}
p \arrow[d, "!"'] & B(p) \arrow[r, hook, "ip"] \arrow[d, "B(p \xrightarrow{!} q)"'] & C(p) \arrow[d, "C(p \xrightarrow{!} q)"] & b \arrow[r, mapsto] \arrow[d, mapsto] & b \arrow[d, mapsto] \\
q & B(q) \arrow[r, hook, "iq"] & C(q) & B(p \xrightarrow{!} q)(b) \arrow[r, mapsto] & \begin{array}{c} C(p \xrightarrow{!} q)(b) \\ B(p \xrightarrow{!} q)(b) \end{array}
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \arrow[r, hook, "i"] & C
\end{tikzcd}
$$

$$
\begin{tikzcd}
p \arrow[d, "!"'] & \{*\} \arrow[r, "\top p"] \arrow[d, "!"'] & \mathrm{Sub}(\downarrow p) \arrow[d, "!"] & * \arrow[r, mapsto] \arrow[d, mapsto] & \downarrow p \arrow[d, mapsto] \\
q & \{*\} \arrow[r, "\top q"] & \mathrm{Sub}(\downarrow q) & * \arrow[r, mapsto] & \begin{array}{c} \downarrow p \wedge \downarrow q \\ \downarrow q \end{array}
\end{tikzcd}
$$

$$
\begin{tikzcd}
1 \arrow[r, "\top"] & \Omega
\end{tikzcd}
$$

$$
\begin{tikzcd}
p \arrow[d, "!"'] & C(p) \arrow[r, "\chi_B(p)"] \arrow[d, "C(p \xrightarrow{!} q)"'] & \mathrm{Sub}(\downarrow p) \arrow[d, "\Omega(p \xrightarrow{!} q)"] & c \arrow[r, mapsto] \arrow[d, mapsto] & \{\, r \in \downarrow p \mid C(p \xrightarrow{!} r)(c) \in B(r) \,\} \arrow[d, mapsto] \\
q & C(q) \arrow[r, "\chi_B(q)"] & \mathrm{Sub}(\downarrow q) & C(p \xrightarrow{!} q)(c) \arrow[r, mapsto] & \begin{array}{c} \{\, r \in \downarrow p \mid C(p \xrightarrow{!} r)(c) \in B(r) \,\} \wedge \downarrow q \\ \{\, s \in \downarrow q \mid C(q \xrightarrow{!} s)(C(p \xrightarrow{!} q)(c)) \in B(s) \,\} \end{array}
\end{tikzcd}
$$

$$
\begin{tikzcd}
C \arrow[r, "\chi_B"] & \Omega
\end{tikzcd}
$$

$$
\begin{tikzcd}
p \arrow[d, "!"'] & \mathrm{Sub}(\downarrow p) \arrow[r, "j(p)"] \arrow[d, "\Omega(p \xrightarrow{!} q)"'] & \mathrm{Sub}(\downarrow p) \arrow[d, "\Omega(p \xrightarrow{!} q)"] & R \arrow[r, mapsto] \arrow[d, mapsto] & R^* \wedge \downarrow p \arrow[d, mapsto] \\
q & \mathrm{Sub}(\downarrow q) \arrow[r, "j(q)"] & \mathrm{Sub}(\downarrow q) & R \wedge \downarrow q \arrow[r, mapsto] & \begin{array}{c} (R^* \wedge \downarrow p) \wedge \downarrow q \\ (R \wedge \downarrow q)^* \wedge \downarrow q \end{array}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\Omega \arrow[r, "j"] & \Omega
\end{tikzcd}
$$

Figure 3: The five square conditions in the Q-shaped diagram

7.7 Understanding the pullbacks visually

The best way to develop visual intuition about the Ω and the j associated to a $((P,A),Q)$ is to try to work out the details in some particular cases — I've chosen two, presented as execises below. They both use the $((P,A),Q)$, the Ω and the j from Figure 2.

Exercise 1. In the case

$$\begin{tikzcd}
11 \arrow[r, "!"] \arrow[d, tail, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "\top"] & \\
33 \arrow[r, "\chi_B"] & \Omega \arrow[r, "j"] & \Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{11} \arrow[rr, "!"] \arrow[d, tail, "\bar{i}"'] \arrow[dr, phantom, "\lrcorner", very near start] & & 1 \arrow[d, tail, "\top"] \\
33 \arrow[r, "\chi_B"] & \Omega \arrow[r, "j"] & \Omega
\end{tikzcd}$$

what is χ_B? And what is $\overline{11}$?

Exercise 2. In the case

$$\begin{tikzcd}
11 \arrow[r, "!"] \arrow[d, tail, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "\top"] & \\
23 \arrow[r, "\chi_B"] & \Omega \arrow[r, "j"] & \Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{11} \arrow[rr, "!"] \arrow[d, tail, "\bar{i}"'] \arrow[dr, phantom, "\lrcorner", very near start] & & 1 \arrow[d, tail, "\top"] \\
23 \arrow[r, "\chi_B"] & \Omega \arrow[r, "j"] & \Omega
\end{tikzcd}$$

what is χ_B? And what is $\overline{11}$?

7.8 Kan extensions

In [Rie16], sec.6.1, right Kan extensions are explained using the two diagrams below. The notation of cells is explained in sec.1.7 of the book, and modulo the types — that can be inferred from the diagrams — a right Kan extension of K along K is a pair $(\mathrm{Ran}_K F, \epsilon)$ such that for all (G, α) there is a unique β making everything commute.

$$
\begin{tikzcd}
\mathsf{C} \arrow[rr, "F"] \arrow[dr, "K"'] & & \mathsf{E} \\
& \mathsf{D} \arrow[ur, bend right, "G"'] \arrow[u, phantom, shift left=6ex, "\Uparrow\delta"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\epsilon\Uparrow"] & \mathsf{E} \\
& \mathsf{D} \arrow[ur, "\mathrm{Ran}_K F" description] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, bend right=20, "\Uparrow\beta"] &
\end{tikzcd}
$$

If we specialize E to $\mathbf{Set}$ and do some renamings, the diagram becomes:

$$
\begin{tikzcd}
\mathbf{A} \arrow[rr, "D"] \arrow[dr, "f"'] & & \mathbf{Set} \\
& \mathbf{B} \arrow[ur, bend right, "C"'] \arrow[u, phantom, shift left=6ex, "\Uparrow\alpha"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{A} \arrow[rr, "D"] \arrow[dr, "f"'] & \arrow[d, phantom, "\epsilon\Uparrow"] & \mathbf{Set} \\
& \mathbf{B} \arrow[ur, "\mathrm{Ran}_f D" description] \arrow[ur, bend right, "C"'] \arrow[ur, phantom, bend right=20, "\Uparrow\beta"] &
\end{tikzcd}
$$

and if we change its *shape* to stress that ϵ "looks like" a counit map and Ran_f "looks like" the right adjoint to the functor f^*, we get this:

$$
\begin{tikzcd}
f^*\mathrm{Ran}_f D \arrow[d, "\epsilon"] & f^*C \arrow[d, "\beta^\flat"', "\alpha"'] \arrow[r, phantom, "\leftrightarrows"] & C \arrow[l, mapsto] \arrow[d, "\beta", "\alpha^\sharp"] & C \arrow[d, "d"] \\
D & D \arrow[r, mapsto] & \mathrm{Ran}_f D & \mathrm{Ran}_f f^*C
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbf{Set}^{\mathbf{A}} \arrow[r, shift right, "\mathrm{Ran}_f"'] & \mathbf{Set}^{\mathbf{B}} \arrow[l, shift right, "f^*"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbf{A} \arrow[r, "f"] & \mathbf{B}
\end{tikzcd}
$$

When the categories $\mathbf{A}$ and $\mathbf{B}$ are finite posets we get:

- $\mathbf{Set}^{\mathbf{A}}$ and $\mathbf{Set}^{\mathbf{B}}$ are toposes (we saw this in sec.7.1),
- the functor f^* is "precomposition with f", in this sense: if C is an object of $\mathbf{Set}^B$ and $A \in \mathbf{A}$ then $(f^*C)(A)$ is $C(f(A))$,
- the left and right Kan extensions Lan_f and Ran_f and can be defined and calculated by the formulas in sec.6.2 of [Rie16],

- we have adjunctions $\mathrm{Lan}_f \dashv f^* \dashv \mathrm{Ran}_f$, and so the structure $(\mathrm{Lan}_f \dashv f^* \dashv \mathrm{Ran}_f)$ can be seen as an essential geometric morphism $f : \mathbf{Set}^{\mathbf{A}} \to \mathbf{Set}^{\mathbf{B}}$ ([E], A4.1.4); as f^* is a right adjoint it preserves limits ([Rie16], sec.4.5, and [Awo06], sec.9.6), and so $(f^* \dashv \mathrm{Ran}_f)$ is a geometric morphism $f : \mathbf{Set}^{\mathbf{A}} \to \mathbf{Set}^{\mathbf{B}}$. We usually rename $(\mathrm{Lan}_f \dashv f^* \dashv \mathrm{Ran}_f)$ to $(f^! \dashv f^* \dashv f_*)$
- when $f : \mathbf{A} \to \mathbf{B}$ is something very simple we can find $\mathrm{Ran}_f D$ "by hand" — for example, in the example below, discussed in [Och19]:

$$
\begin{array}{ccccccc}
\begin{pmatrix} C_2 & & & & C_3 & & \\ & \searrow & & \swarrow & & \searrow & \\ & & C_4 & & & & C_5 \end{pmatrix} & &
\begin{pmatrix} C_2 & & & & C_3 & & \\ & \searrow & & \swarrow & & \searrow & \\ & & C_4 & & & & C_5 \end{pmatrix} & \longmapsfrom &
\begin{pmatrix} & & C_1 & & & & \\ & \swarrow & & \searrow & & & \\ C_2 & & & & C_3 & & \\ & \searrow & & \swarrow & & \searrow & \\ & & C_4 & & & & C_5 \\ & & & \searrow & & \swarrow & \\ & & & & C_6 & & \end{pmatrix} & &
\begin{pmatrix} & & C_1 & & & & \\ & \swarrow & & \searrow & & & \\ C_2 & & & & C_3 & & \\ & \searrow & & \swarrow & & \searrow & \\ & & C_4 & & & & C_5 \\ & & & \searrow & & \swarrow & \\ & & & & C_6 & & \end{pmatrix} \\
{\scriptstyle \epsilon D}\downarrow & & \downarrow & \longleftrightarrow & \downarrow & & \downarrow{\scriptstyle \eta C} \\
\begin{pmatrix} D_2 & & & & D_3 & & \\ & \searrow & & \swarrow & & \searrow & \\ & & D_4 & & & & D_5 \end{pmatrix} & &
\begin{pmatrix} D_2 & & & & D_3 & & \\ & \searrow & & \swarrow & & \searrow & \\ & & D_4 & & & & D_5 \end{pmatrix} & \longmapsto &
\begin{pmatrix} & & D_2 \times_{D_4} D_3 & & & & \\ & \swarrow & & \searrow & & & \\ D_2 & & & & D_3 & & \\ & \searrow & & \swarrow & & \searrow & \\ & & D_4 & & & & D_5 \\ & & & \searrow & & \swarrow & \\ & & & & 1 & & \end{pmatrix} & &
\begin{pmatrix} & & C_2 \times_{C_4} C_3 & & & & \\ & \swarrow & & \searrow & & & \\ C_2 & & & & C_3 & & \\ & \searrow & & \swarrow & & \searrow & \\ & & C_4 & & & & C_5 \\ & & & \searrow & & \swarrow & \\ & & & & 1 & & \end{pmatrix}
\end{array}
$$

$$
\begin{tikzcd}
\mathbf{Set}^{\mathbf{A}} \arrow[r, shift right, "f_*"'] & \mathbf{Set}^{\mathbf{B}} \arrow[l, shift right, "f^*"']
\end{tikzcd}
$$

$$
\mathbf{Set}^{\mathbf{A}} \xrightarrow{\quad f \quad} \mathbf{Set}^{\mathbf{B}}
$$

$$
\begin{pmatrix} 2 & & & & 3 & & \\ & \searrow & & \swarrow & & \searrow & \\ & & 4 & & & & 5 \end{pmatrix} \xrightarrow{f}
\begin{pmatrix} & & 1 & & & & \\ & \swarrow & & \searrow & & & \\ 2 & & & & 3 & & \\ & \searrow & & \swarrow & & \searrow & \\ & & 4 & & & & 5 \\ & & & \searrow & & \swarrow & \\ & & & & 6 & & \end{pmatrix}
$$

Every situation in which the category $\mathbf{B}$ is a (P, A) and the category $\mathbf{A}$ is the full subcategory of (P, A) whose objects are $P\backslash Q$ yields a situation like the one in the diagram above, in which the maps ϵD are isos, the geometric morphism f is an "inclusion" and the functor that takes each C to $f_* f^* C$ is a sheafification functor. A diagram with an example fully worked out will be included in the next version of this paper at the Arxiv.

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