

Notes on Michael Barr and Charles Wells's

"Category Theory for Computing Science":

<http://www.tac.mta.ca/tac/reprints/articles/22/tr22.pdf>

<http://www.tac.mta.ca/tac/reprints/articles/22/tr22abs.html>

These notes are at:

<http://angg.twu.net/LATEX/2020barrwellsctcs.pdf>

12. Fibrations

12.1.1. Fibrations and opfibrations (page 331):

u is cartesian (for f and Y) when:

$$\begin{array}{ccc}
 & \forall Z & \\
 & \swarrow \exists! w & \searrow \forall v \\
 \mathcal{E} & & X \xrightarrow{u} Y \\
 \downarrow P & & \\
 \mathcal{C} & P(Z) & \\
 & \swarrow h & \searrow P(v) \\
 & C \xrightarrow{f} D &
 \end{array}$$

u is opcartesian (for f and X):

$$\begin{array}{ccc}
 & & \forall Z \\
 & \swarrow \forall v & \nearrow \exists! w \\
 \mathcal{E} & X \xrightarrow{u} Y & \\
 \downarrow P & & \\
 \mathcal{C} & C \xrightarrow{f} D & P(Z) \\
 & \swarrow P(v) & \nearrow \forall k
 \end{array}$$

12.1.4. Example:

$$\begin{array}{ccccc}
 & \forall(A', C'') & & \xrightarrow{\forall(g,h)} & \\
 & \searrow & & & \\
 A \times C & \xrightarrow{(g,u)} & (A, C) & \xrightarrow{\gamma(f,v)=(id_A, f)} & Y = (A', C) \\
 \downarrow P & & & & \\
 C & & C'' & \xrightarrow{h} & C' \\
 & & \searrow \forall u & & \\
 & & C & \xrightarrow{f} & C'
 \end{array}$$

12.1.7 Cleavages induce functors (page 334)

Let $P : \mathcal{E} \rightarrow \mathcal{C}$ be an opfibration with opcleavage κ . Define $F : \mathcal{C} \rightarrow \mathbf{Cat}$ by...

$$\begin{array}{ccc}
 X & \xrightarrow{\kappa(f, X)} & Ff(X) \\
 u \downarrow & \swarrow \lrcorner & \downarrow \exists! Ff(u) \\
 X' & \xrightarrow{\kappa(f, X')} & Ff(X') \\
 v \downarrow & \swarrow \lrcorner & \downarrow \exists! Ff(v) \\
 \mathcal{E} & X'' \xrightarrow{\kappa(f, X'')} Ff(X'') & \\
 P \downarrow & C \xrightarrow{f} D & \\
 \mathcal{C} & &
 \end{array}$$

In a similar way, split fibrations give functors $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$. Let $P : \mathcal{E} \rightarrow \mathcal{C}$ be a fibration with cleavage γ . Define $F : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$ by

$$\begin{array}{ccc}
 Ff(Y) & \xrightarrow{\gamma(f, Y)} & Y \\
 \exists! Ff(u) \downarrow & \swarrow \lrcorner & \downarrow u \\
 Ff(Y') & \xrightarrow{\gamma(f, Y')} & Y' \\
 \exists! Ff(v) \downarrow & \swarrow \lrcorner & \downarrow v \\
 \mathcal{E} & Ff(Y'') \xrightarrow{\gamma(f, Y'')} Y'' & \\
 P \downarrow & C \xrightarrow{f} D & \\
 \mathcal{C} & &
 \end{array}$$

12.2 The Grothendieck construction

12.2.8 and **12.2.9:** Given a functor $F : \mathcal{C} \rightarrow \mathbf{Cat}$ (...) the Grothendieck construction in this more general setting constructs the opfibration induced by F , a category $\mathbf{G}(\mathcal{C}, F)$ defined as follows:

$$\begin{array}{ccc}
 x \mapsto (Ff)(x) \mapsto (Fg)((Ff)(x)) & & \\
 \downarrow u & & \downarrow (Fg)(u) \\
 x' \mapsto (Fg)(x') & & \\
 & & \downarrow v \\
 & & x'' \\
 \mathbf{Cat} & F(C) \xrightarrow{Ff} F(C') \xrightarrow{Fg} F(C'') & \mathbf{G}(\mathcal{C}, F) & (x, C) \xrightarrow{(u,f)} (x', C') \xrightarrow{(v,g)} (x'', C'') \\
 \uparrow F & & \downarrow P & \\
 \mathcal{C} & C \xrightarrow{f} C' \xrightarrow{g} C'' & \mathcal{C} & C \xrightarrow{f} C' \xrightarrow{g} C''
 \end{array}$$

12.2.10 An analogous construction, also called the Grothendieck construction (in fact this is the original one), produces a split fibration $\mathbf{F}(\mathcal{C}, G)$ given a functor $G : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$:

$$\begin{array}{ccc}
 x & & \\
 \downarrow u & & \\
 (Gf)(x') \longleftarrow x' & & \\
 \downarrow (Gf)(v) & & \downarrow v \\
 (Gf)((Gg)(x'')) \longleftarrow (Gg)(x'') \longleftarrow x'' & & \\
 \mathbf{Cat} & G(C) \xleftarrow{Gf} G(C') \xleftarrow{Gg} G(C'') & \mathbf{F}(\mathcal{C}, G) & (C, x) \xrightarrow{(f,u)} (C', x') \xrightarrow{(g,v)} (C'', x'') \\
 \uparrow G & & \downarrow P & \\
 \mathcal{C}^{\text{op}} & C \xrightarrow{f} C' \xrightarrow{g} C'' & \mathcal{C} & C \xrightarrow{f} C' \xrightarrow{g} C''
 \end{array}$$