(Page 77):

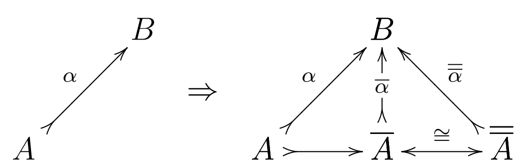
3.13 Definition. Let $\mathcal{E}$ be any category with pullbacks. A *universal closure operation* on $\mathcal{E}$ is defined by specifying, for each $X \in \mathcal{E}$, a closure operation (i.e., an increasing, order-preserving, idempotent map) on the poset of subobjects of X — we denote the closure of $X' \rightarrowtail X$ by $\overline{X'} \rightarrowtail X$ — in such a way that closure commutes with pullback along morphisms of $\mathcal{E}$; i.e., given $Y \xrightarrow{f} X$, we have $f^*(\overline{X'}) \cong \overline{f^*(X')}$ as subobjects of Y.

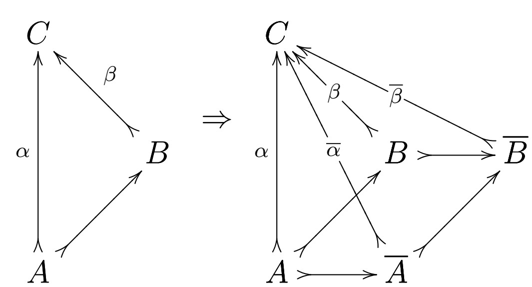
We shall use the words *dense* and *closed* with their usual meanings relative to a universal closure operation; i.e., $X' \rightarrowtail X$ is dense if $\overline{X'} \cong X'$, and closed if $\overline{X'} \cong X'$.

Here is a way to visualize those rules.

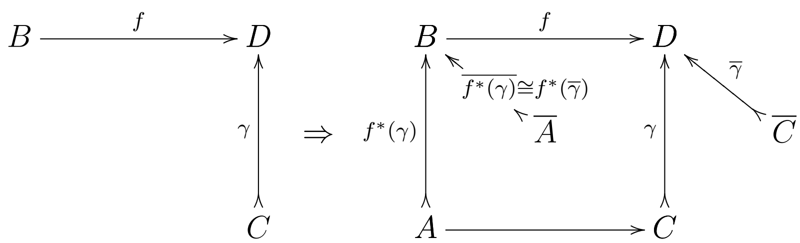
First line: a monic map $\alpha : A \rightarrowtail B$ factors through its closure $\overline{\alpha} : \overline{A} \rightarrowtail B$; the factorization arrow $A \rightarrowtail \overline{A}$ is not usually named. The closure of $\overline{\alpha} : \overline{A} \rightarrowtail B$ is a monic $\overline{\overline{\alpha}} : \overline{\overline{A}} \rightarrowtail B$ isomorphic to $\overline{\alpha} : \overline{A} \rightarrowtail B$. In a shorter notation, $A \leq \overline{A} \cong \overline{\overline{A}}$.

Second line: in the shorter notation the closure operation is order-preserving iff $A \leq B$ implies $\overline{A} \leq \overline{B}$; more formally, if $(\alpha : A \rightarrowtail C) \leq (\beta : B \rightarrowtail C)$ implies $(\overline{\alpha} : \overline{A} \rightarrowtail C) \leq (\overline{\beta} : \overline{B} \rightarrowtail C)$, where each '$\leq$'s between monics should be read as "factors through".

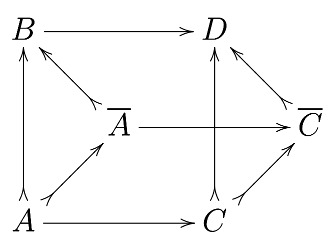




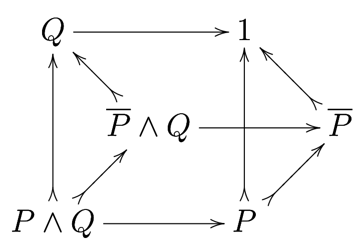
The best way to visualize the last rule is by a slight diagrammatic abuse of of language. We start with a monic $\gamma : C \to D$ and an arrow $f : B \to D$ that is not necessarily a monic, as below. We form their pullback, and we call the arrow at the left wall $f^*(\gamma) : A \rightarrowtail B$. Let $\overline{\gamma} : \overline{C} \rightarrowtail D$ and $\overline{f^*(\gamma)} : \overline{A} \rightarrowtail B$ be the closures of γ and $f^*(\gamma)$. If we draw everything as below then the natural way to draw the pullback of $\overline{\gamma} : \overline{C} \rightarrowtail D$ by f would be as an arrow $f^*(\overline{\gamma})$ in the same position as $\overline{f^*(\gamma)} : \overline{A} \rightarrowtail B$; what the rule $\overline{f^*(\gamma)} \cong f^*(\overline{\gamma})$ says is that $\overline{f^*(\gamma)}$ and $f^*(\overline{\gamma})$ are isomorphic as subobjects of B — but we will draw $\overline{f^*(\gamma)}$ and $f^*(\overline{\gamma})$ as if they were a single arrow.

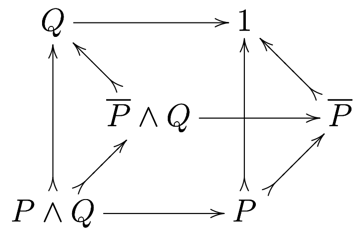


We will usually draw that diagram as this, and omit the names of most, or all, of its arrows.



A J-operator induces a universal closure




Q
1
$\overline{P} \wedge Q$
$\overline{P}$
$P \wedge Q$
P