Notes on Hyperdoctrines

Introduction

One of the main ideas that I tried to develop in my PhD thesis a looong time ago was that *constructions in Category Theory can be expressed in a language that looks like Natural Deduction*. That language turned out to be *very* difficult to formalize, but it gave rise to several spin-offs, most of them related to translating between languages... basically, between:

1. the internal language of a topos
2. the language of the categorical constructions in a topos
3. the internal language of the "archetypal" topos, **Set**
4. the language of the categorical constructions in a hyper-doctrine
5. the languages in Bart Jacobs's book
6. the language of a general case ("for adults")
7. the language of a particular case ("for children")

Why hyperdoctrines?

A hyperdoctrine is a category in which we can interpret first-order logic ("FOL").
(↑ Both Sequent Calculus and Natural Deduction!)

Hyperdoctrines are hard to define (many axioms!)
Toposes are easy to define (5 axioms!)
Every topos is a hyperdoctrine (hard to prove!)
We can interpret FOL in a topos (hard!)
We can interpret FOL in a hyperdoctrine (easy!)

Hyperdoctrines are more "modular" than toposes (in the sense of [Jac99], pages 8–11).
I have some technical and personal reasons for not liking toposes very much — see [Och13], sections 12, 19, 20.

Why hyperdoctrines? (2)
The fibration $\mathrm{Cod} : \mathbf{Set}^{\downarrow} \to \mathbf{Set}$ is an archetypal hyperdoctrine.

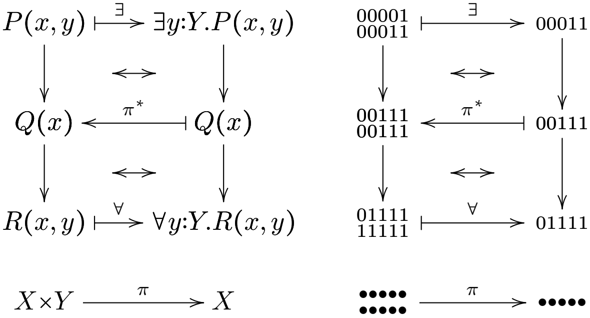
We can use hyperdoctrines to learn Natural Deduction! Two of the quantifier rules in Natural Deduction have restrictions that are very technical, and that only made sense to me when I understood their categorical versions!!! Or, more precisely, these adjunctions:

$$\exists \dashv \pi^* \dashv \forall$$

What made me understand them was the concrete example in the next page...

Quantifiers as adjoints, for children

A concrete example with
$X = \{0, 1, 2, 3, 4\}$, $Y = \{0, 1\}$,
$P = \begin{smallmatrix}00001\\00011\end{smallmatrix}$, $Q = 00111$, $R = \begin{smallmatrix}01111\\11111\end{smallmatrix}$:



Quantifiers as adjoints, for children (2)

b

a

Arrows

$$A \rightarrowtail B \quad \text{monic}$$
$$A \hookrightarrow B \quad \text{inclusion}$$

A category of "predicates"

I will use '$\hookrightarrow$' to denote inclusion maps in **Set**.

The category $\mathbf{Set}^{\hookrightarrow}$ has:

inclusions as its objects, and

commutative squares as its morphisms.

A morphism $\begin{pmatrix} A \\ \downarrow \\ B \end{pmatrix} \xrightarrow{(f,g)} \begin{pmatrix} C \\ \downarrow \\ D \end{pmatrix}$ in $\mathbf{Set}^{\hookrightarrow}$

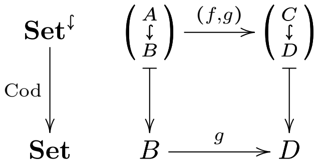
is a commutative square
$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ \downarrow & & \downarrow \\ B & \xrightarrow{g} & D \end{array}$$
in **Set**.

The codomain functor $\mathrm{Cod} : \mathbf{Set}^{\hookrightarrow} \to \mathbf{Set}$ works like this:

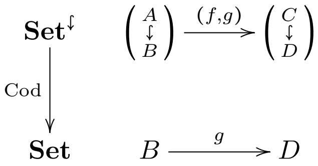
$$\mathrm{Cod}\left(\begin{pmatrix} A \\ \downarrow \\ B \end{pmatrix}\right) = B, \quad \mathrm{Cod}\left(\begin{pmatrix} A \\ \downarrow \\ B \end{pmatrix} \xrightarrow{(f,g)} \begin{pmatrix} C \\ \downarrow \\ D \end{pmatrix}\right) = (B \xrightarrow{g} D).$$

Predicates over a set

I will draw the functor $\mathrm{Cod} : \mathbf{Set}^{\hookdownarrow} \to \mathbf{Set}$ as going downwards:



I will usually omit the downward '$\mapsto$'s:



Some standard terminology

(See [Jac99], p.26, for the full definition!)
$p : \mathbb{E} \to \mathbb{B}$ is a fibration (the general case),
$\mathrm{Cod} : \mathbf{Set}^{\downarrow} \to \mathbf{Set}$ is a fibration (our archetypal case),

$$\begin{array}{lccl} \mathbf{Set}^{\downarrow} & \begin{pmatrix} A \\ \downarrow \\ B \end{pmatrix} \xrightarrow{(f,g)} \begin{pmatrix} C \\ \downarrow \\ D \end{pmatrix} & & \leftarrow \text{Entire category} \\ \Big\downarrow \mathrm{Cod} & & & \leftarrow \text{Projection functor} \\ \mathbf{Set} & B \xrightarrow{g} D & & \leftarrow \text{Base category} \end{array}$$

$\begin{pmatrix} A \\ \downarrow \\ B \end{pmatrix}$ is an object "over" B,

$\begin{pmatrix} A \\ \downarrow \\ B \end{pmatrix} \xrightarrow{(f,g)} \begin{pmatrix} C \\ \downarrow \\ D \end{pmatrix}$ is a morphism "over" $B \overset{g}{\to} D$.

$\mathrm{Cod}^{-1}(B)$ is the fiber over B (a category!)

Dummett

The rules for the quantifiers in ND, from [Dum00], p.89:

$$\frac{\Gamma : A(y)}{\Gamma : \forall x\, A(x)}\ \forall+ \qquad \frac{\Gamma : \forall x\, A(x)}{\Gamma : A(t)}\ \forall-$$

$$\frac{\Gamma : A(y)}{\Gamma : \exists x\, A(x)}\ \exists+ \qquad \frac{\Gamma : \exists x\, A(x) \quad \Delta, A(y) : C}{\Gamma, \Delta : C}\ \exists-$$

Let's specialize and change notation a bit...

$$\frac{P(x) \vdash Q(x,y)}{P(x) \vdash \forall y{:}Y.Q(x,y)}\ \forall+ \qquad \frac{P(x) \vdash \forall y{:}Y.Q(x,y)}{P(x) \vdash Q(x,f(x))}\ \forall-$$

$$\frac{P(x) \vdash Q(x,f(x))}{P(x) \vdash \exists y{:}Y.Q(x,y)}\ \exists+ \qquad \frac{P(x) \vdash \exists y{:}Y.Q(x,y) \quad R(x), Q(x,f(x)) \vdash S(x)}{P(x), R(x) \vdash S(x)}\ \exists-$$

The rule ∃− in Natural Deduction

Let's convert the hardest rule, ∃−, to ND...

$$\frac{P(x) \vdash \exists y{:}Y.Q(x,y) \quad R(x), Q(x,f(x)) \vdash S(x)}{P(x), R(x) \vdash S(x)}\ \exists-$$

It becomes:

$$\dfrac{\begin{array}{c} P(x) \\ \vdots\ \alpha \\ \exists y{:}Y.Q(x,y) \end{array} \qquad \begin{array}{c} R(x) \quad [Q(x,f(x))]^1 \\ \vdots\ \beta \\ S(x) \end{array}}{S(x)}\ \exists-;1 \qquad \Longrightarrow \qquad \begin{array}{c} P(x) \quad R(x) \\ \vdots\ \gamma \\ S(x) \end{array}$$

where:

$$\begin{array}{rcl} \alpha & = & P(x) \vdash \exists y{:}Y.Q(x,y) \\ \beta & = & R(x), Q(x,f(x)) \vdash S(x) \\ \gamma & = & P(x), R(x) \vdash S(x) \end{array}$$

A variant of the rule ∃–

Let's look at a simpler rule, called '∃ – –'.

$$\dfrac{\begin{array}{c} P(x)\\ \vdots\ \alpha \\ \exists y{:}Y.Q(x,y)\end{array} \qquad \begin{array}{c} R(x)\quad [Q(x,f(x))]^1 \\ \vdots\ \beta \\ S(x)\end{array}}{S(x)}\ \exists\text{–};1 \qquad\Longrightarrow\qquad \begin{array}{c}P(x)\quad R(x)\\ \vdots\ \gamma\\ S(x)\end{array}$$

$$\dfrac{\exists y{:}Y.Q(x,y) \qquad \begin{array}{c} [Q(x,f(x))]^1 \\ \vdots\ \delta \\ S'(x)\end{array}}{S'(x)}\ \exists - -;1 \qquad\Longrightarrow\qquad \begin{array}{c}\exists y{:}Y.Q(x,y)\\ \vdots\ \epsilon\\ S'(x)\end{array}$$

$$\dfrac{Q(x,f(x)) \vdash S'(x)}{\exists y{:}Y.Q(x,y) \vdash S'(x)}\ \exists - -$$

The rules ∃– and ∃ – – are "equivalent" (1)

$$\frac{Q(x, f(x)) \vdash S'(x)}{\exists y{:}Y.Q(x,y) \vdash S'(x)}\ \exists --$$

$$\frac{P(x) \vdash \exists y{:}Y.Q(x,y) \quad R(x), Q(x, f(x)) \vdash S(x)}{P(x), R(x) \vdash S(x)}\ \exists -$$

$$\cfrac{P(x) \vdash \exists y{:}Y.Q(x,y) \quad \cfrac{\cfrac{R(x), Q(x, f(x)) \vdash S(x)}{Q(x, f(x)) \vdash R(x) \supset S(x)}}{\exists y{:}Y.Q(x,y) \vdash R(x) \supset S(x)}\ \exists --}{\cfrac{P(x) \vdash R(x) \supset S(x)}{P(x), R(x) \vdash S(x)}}\ \text{Cut}$$

The rules ∃– and ∃ – – are "equivalent" (2)
making $\Delta := \{\}$ instead of $\Delta := \{R(x)\}$,
and after that $P(x) := \exists y{:}Y.Q(x,y)$ and $S(x) := S'(x)$,

$$\frac{Q(x,f(x)) \vdash S'(x)}{\exists y{:}Y.Q(x,y) \vdash S'(x)}\ \exists --$$

$$\frac{P(x) \vdash \exists y{:}Y.Q(x,y) \quad R(x), Q(x,f(x)) \vdash S(x)}{P(x), R(x) \vdash S(x)}\ \exists -$$

$$\frac{P(x) \vdash \exists y{:}Y.Q(x,y) \quad Q(x,f(x)) \vdash S(x)}{P(x) \vdash S(x)}\ \exists -$$

$$\frac{\overline{\exists y{:}Y.Q(x,y) \vdash \exists y{:}Y.Q(x,y)} \quad Q(x,f(x)) \vdash S'(x)}{\exists y{:}Y.Q(x,y) \vdash S'(x)}\ \exists -$$

Predicates, visually

We can use hyperdoctrines to understand:

1. the rules for $\exists$ in ND
2. some notations in [Jac99]
3. the internal language of a topos
4. polymorphism (as in Haskell; advanced)

We will do (1) and (2) in these notes.

Judgments (for children)

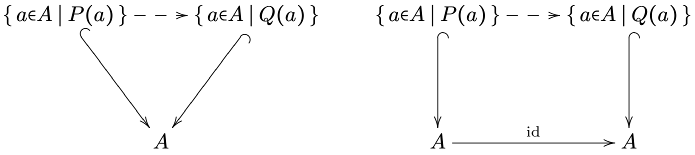
We will interpret $a : A, b : B, P(a,b) \vdash Q(a,b)$ as:
for every choice of $a \in A$,
for every choice of $b \in B$,
if $P(a,b)$ is true
then $Q(a,b)$ is true.

...or, equivalently, as:
$\{\,(a,b) \in A \times B \mid P(a,b)\,\} \subseteq \{\,(a,b) \in A \times B \mid Q(a,b)\,\}$.

For variants of this and lots of fun (yeah!), see:
`http://angg.twu.net/LATEX/2019notes-types.pdf`
(slides 14-16).

Judgments (for children, but categorically)

So $a : A, P(a) \vdash Q(a)$ means
$\{ a \in A \mid P(a) \} \subseteq \{ a \in A \mid Q(a) \}$,
and this is true if and only if
we have arrows '$- - \succ$' in:



When these arrows exist they are unique.

Predicates over a set (2)

I will say that $\begin{pmatrix} A \\ \downarrow \\ B \end{pmatrix}$ is an object "over" B, and that

$\begin{pmatrix} A \\ \downarrow \\ B \end{pmatrix} \xrightarrow{(f,g)} \begin{pmatrix} C \\ \downarrow \\ D \end{pmatrix}$ is a morphism "over" $B \overset{g}{\to} D$.

Some standard terminology: in our particular case,
$\mathrm{Cod} : \mathbf{Set}^{\downarrow} \to \mathbf{Set}$ is a fibration.
Cod is the projection functor.
$\mathbf{Set}^{\downarrow}$ is the total category (or the entire category).
Set is the base category.

General case (see [Jac99], p.26, for the full definition):
$p : \mathbb{E} \to \mathbb{B}$ is a fibration.
p is the projection functor.
$\mathbb{E}$ is the total category (or the entire category).
$\mathbb{B}$ is the base category.

For the general case and a formal definition see [Jac99], page 26; he uses the notation $p : \mathbb{E} \to \mathbb{B}$...

$$\begin{array}{c} \mathbf{Set}^{\downarrow} \\ \downarrow \mathrm{Cod} \\ \mathbf{Set} \end{array}$$

$$\left(\begin{array}{c} hi \\ \mathbf{Set}^{\downarrow} \\ \downarrow \ \mathrm{Cod} \\ \mathbf{Set} \end{array} \right)$$

$$\left(\begin{array}{c} \mathbb{E} \\ \downarrow p \\ \mathbf{Set} \end{array} \right)$$

$$\begin{array}{lll} \mathbf{Set}^{\downarrow} & \mathbb{E} & \text{(entire category)} \\ \quad\downarrow \text{Cod} & \downarrow p & \text{(projection functor)} \\ \mathbf{Set} & \mathbb{B} & \text{(base category)} \end{array}$$

The functor $\begin{pmatrix} \mathbf{Set}^{\downarrow} \\ \quad\downarrow \text{Cod} \\ \mathbf{Set} \end{pmatrix}$ is a fibration.

General case: $\begin{pmatrix} \mathbb{E} \\ \downarrow p \\ \mathbb{B} \end{pmatrix}$

The fiber over an object B of the base category

Some shorthands:

$$\begin{pmatrix} \{\, a \in A \mid \ \} \\ \downarrow \end{pmatrix}$$

and I will use this notation
I will draw

PL (1)

$$\begin{array}{rcl} 1 & : & \mathsf{Ords} \\ \Omega & : & \mathsf{Ords} \\ A & : & \mathsf{Ords} \\ B & : & \mathsf{Ords} \\ A{\times}B & : & \mathsf{Ords} \\ A{\to}\Omega & : & \mathsf{Ords} \end{array}$$

$$\begin{array}{rcccl} \top & : & \Omega & : & \mathsf{Ords} \\ \rho & : & \Omega & : & \mathsf{Ords} \\ \sigma & : & \Omega & : & \mathsf{Ords} \\ \tau & : & \Omega & : & \mathsf{Ords} \\ \sigma{\wedge}\tau & : & \Omega & : & \mathsf{Ords} \\ \sigma{\supset}\tau & : & \Omega & : & \mathsf{Ords} \\ \beta & : & B & : & \mathsf{Ords} \\ \sigma[\tau/\beta] & : & \Omega & : & \mathsf{Ords} \\ \Sigma\beta{\in}B\cdot\rho & : & \Omega & : & \mathsf{Ords} \\ \Pi\beta{\in}B\cdot\tau & : & \Omega & : & \mathsf{Ords} \end{array}$$

PL (1) in another notation

$$
\begin{array}{rcl}
1 & : & \Theta \\
\Omega & : & \Theta \\
A & : & \Theta \\
B & : & \Theta \\
A{\times}B & : & \Theta \\
A{\to}\Omega & : & \Theta
\end{array}
\qquad
\begin{array}{rcccl}
\top & : & \Omega & : & \Theta \\
P & : & \Omega & : & \Theta \\
Q & : & \Omega & : & \Theta \\
R & : & \Omega & : & \Theta \\
P{\wedge}Q & : & \Omega & : & \Theta \\
P{\supset}Q & : & \Omega & : & \Theta \\
b & : & B & : & \Theta \\
P[b := f(a)] & : & \Omega & : & \Theta \\
\exists b{\in}B.P & : & \Omega & : & \Theta \\
\forall b{\in}B.R & : & \Omega & : & \Theta
\end{array}
$$

PL (2)

$$
\begin{array}{rcrcccll}
* & : & \top & : & \Omega & : & \mathsf{Ords} & (\top I) \\
\\
a & : & \sigma & : & \Omega & : & \mathsf{Ords} & \\
b & : & \tau & : & \Omega & : & \mathsf{Ords} & \\
c & : & \sigma\wedge\tau & : & \Omega & : & \mathsf{Ords} & \\
\langle a, b\rangle & : & \sigma\wedge\tau & : & \Omega & : & \mathsf{Ords} & (\wedge I) \\
\pi_1 c & : & \sigma & : & \Omega & : & \mathsf{Ords} & (\wedge E) \\
\pi_2 c & : & \tau & : & \Omega & : & \mathsf{Ords} & (\wedge E) \\
\\
a & : & \sigma & : & \Omega & : & \mathsf{Ords} & \\
b & : & \tau & : & \Omega & : & \mathsf{Ords} & \\
f & : & \sigma\supset\tau & : & \Omega & : & \mathsf{Ords} & \\
fa & : & \tau & : & \Omega & : & \mathsf{Ords} & (\supset E) \\
\lambda x{\in}\sigma \cdot b & : & \sigma\supset\tau & : & \Omega & : & \mathsf{Ords} & (\supset I)
\end{array}
$$

PL (2) in another notation

$$
\begin{array}{rcrcccll}
\star & : & \top & : & \Omega & : & \Theta & (\top I) \\
\\
p & : & P & : & \Omega & : & \Theta & \\
q & : & Q & : & \Omega & : & \Theta & \\
s & : & P{\wedge}Q & : & \Omega & : & \Theta & \\
(p,q) & : & P{\wedge}Q & : & \Omega & : & \Theta & (\wedge I) \\
\pi_1 s & : & P & : & \Omega & : & \Theta & (\wedge E) \\
\pi_2 s & : & Q & : & \Omega & : & \Theta & (\wedge E) \\
\\
p & : & P & : & \Omega & : & \Theta & \\
q & : & Q & : & \Omega & : & \Theta & \\
f & : & P{\supset}Q & : & \Omega & : & \Theta & \\
fp & : & Q & : & \Omega & : & \Theta & (\supset E) \\
\lambda p{:}P.q & : & P{\supset}Q & : & \Omega & : & \Theta & (\supset I)
\end{array}
$$

PL (3)

$$
\begin{array}{rcrcccl}
 & & \alpha & : & A & : & \mathsf{Ords} \quad \text{(indet)} \\
 & & \sigma & : & \Omega & : & \mathsf{Ords} \\
 & & \tau & : & A & : & \mathsf{Ords} \\
b & : & \sigma[\tau/\alpha] & : & \Omega & : & \mathsf{Ords} \\
I(b) & : & \Sigma\alpha{\in}A\cdot\sigma & : & \Omega & : & \mathsf{Ords} \quad (\Sigma I) \\
I_{\Sigma\alpha\cdot\sigma,\tau}(b) & : & \Sigma\alpha{\in}A\cdot\sigma & : & \Omega & : & \mathsf{Ords} \quad (\Sigma I) \\
a & : & \sigma\supset\rho & : & \Omega & : & \mathsf{Ords} \\
 & & \alpha & : & A & : & \mathsf{Ords} \quad \text{(indet n.f.)} \\
\mathbf{V}\alpha{\in}A\cdot a & : & (\Sigma\alpha{\in}A\cdot\sigma)\supset\rho & : & \Omega & : & \mathsf{Ords} \quad (\Sigma E)
\end{array}
$$

PL (3) in another notation

$$\begin{array}{rclcccccl}
 & & b & : & B & : & \Theta & & \text{(indet)} \\
 & & P(a,b) & : & \Omega & : & \Theta & & \\
 & & f(a) & : & B & : & \Theta & & \\
p & : & P(a,f(a)) & : & \Omega & : & \Theta & & \\
I(p) & : & \exists b{:}B.P(a,b) & : & \Omega & : & \Theta & & (\Sigma I) \\
I_{\Sigma\alpha\cdot\sigma,\tau}(b) & : & \exists b{:}B.P(a,b) & : & \Omega & : & \Theta & & (\Sigma I) \\
a & : & P(a,b)\supset Q(a) & : & \Omega & : & \Theta & & \\
 & & b & : & B & : & \Theta & & \text{(indet n.f.)} \\
\mathbf{V}\alpha\in A\cdot a & : & (\exists b{:}B.P(a,b))\supset Q(a) & : & \Omega & : & \Theta & & (\Sigma E)
\end{array}$$

PL (4)

$$
\begin{array}{rcrcccl}
a & : & \sigma & : & \Omega & : & \mathsf{Ords} \\
 & & \alpha & : & A & : & \mathsf{Ords} \qquad \text{(indet n.f.)} \\
\Lambda\alpha{\in}A\cdot a & : & \Pi\alpha{\in}A\cdot\sigma & : & \Omega & : & \mathsf{Ords} \qquad (\Pi I) \\
\\
\tau & : & \alpha & : & A & : & \mathsf{Ords} \\
a & : & \Pi\alpha{\in}A\cdot\sigma & : & \Omega & : & \mathsf{Ords} \\
a\{\tau\} & : & \sigma[\tau/\alpha] & : & A & : & \mathsf{Ords} \qquad (\Pi E)
\end{array}
$$

PL (4) in another notation

$$
\begin{array}{rcrcccccc}
r & : & R(a,b) & : & \Omega & : & \Theta & & \\
 & & b & : & B & : & \Theta & & \text{(indet n.f.)} \\
\Lambda r & : & \forall b{:}B.R(a,b) & : & \Omega & : & \Theta & & (\Pi I) \\
 & & & & & & & & \\
 & & f(a) & : & B & : & \Theta & & \\
g & : & \forall b{:}B.R(a,b) & : & \Omega & : & \Theta & & \\
g(f(a)) & : & R(a,f(a)) & : & \Omega & : & \Theta & & (\Pi E)
\end{array}
$$

quants-my-1:

$$
\begin{array}{ccc}
\left\{ \begin{array}{c} P(a,b) \\ (a,b)\in A\times B \end{array} \right\} & \longmapsto & \left\{ \begin{array}{c} \exists b\in B. P(a,b) \\ a\in A \end{array} \right\} \\
\downarrow & \longleftrightarrow & \downarrow \\
\left\{ \begin{array}{c} Q(a) \\ (a,b)\in A\times B \end{array} \right\} & \longleftarrow\!\dashv & \left\{ \begin{array}{c} Q(a) \\ a\in A \end{array} \right\} \\
\downarrow & \longleftrightarrow & \downarrow \\
\left\{ \begin{array}{c} R(a,b) \\ (a,b)\in A\times B \end{array} \right\} & \longmapsto & \left\{ \begin{array}{c} \forall b\in B. R(a,b) \\ a\in A \end{array} \right\}
\end{array}
$$

$$
A\times B \xrightarrow{\pi} A \xrightarrow{\chi i Q} \Omega
$$

$$
A\times B \overset{\chi i P}{\underset{\chi i R}{\rightrightarrows}} \Omega
$$

quants-my-2:

$$\begin{array}{ccc}
P & \longmapsto & \exists_\pi P \\
\downarrow & \longleftrightarrow & \downarrow \\
\pi^* Q & \longleftarrow\!\!\!\!\dashv & Q \\
\downarrow & \longleftrightarrow & \downarrow \\
Q & \longmapsto & \forall_\pi Q
\end{array}$$

$$A \times B \xrightarrow{\ \pi\ } A \xrightarrow{\ \chi i Q\ } \Omega \qquad A \times B \underset{\chi i R}{\overset{\chi i P}{\rightrightarrows}} \Omega$$

quants-seely-1:

$$
\begin{array}{ccc}
\rho & \longmapsto & \Sigma\beta\in B\cdot\rho \\
\downarrow & \longleftrightarrow & \downarrow \\
\pi^*\sigma & \longleftarrow\!\!\dashv & \sigma \\
\downarrow & \longleftrightarrow & \downarrow \\
\tau & \longmapsto & \Pi\beta\in B\cdot\tau
\end{array}
$$

$$
A\times B \xrightarrow{\ \pi\ } A \xrightarrow{\ \sigma\ } \Omega
\qquad
A\times B \underset{\tau}{\overset{\rho}{\rightrightarrows}} \Omega
$$

The rules for '∃' in Natural Deduction

The rules are easier to understand and visualize if we use bounded quantifiers and finite sets...

Bounded: $\exists y{:}Y.P(x,y)$

Unbounded: $\exists y.P(x,y)$

Finite sets: $X = \{1,2,3,4,5,6\}$, $Y = \{0,1,2\}$

$\begin{pmatrix} A \\ \downarrow \\ B \end{pmatrix} \left(\begin{smallmatrix} A \\ \downarrow \\ B \end{smallmatrix}\right) \mathbf{Set}^{\downarrow}$

$\mathsf{Cod} : \mathbf{Set}^{\downarrow} \to \mathbf{Set}$

$\begin{Bmatrix} P(a) \\ a \in A \end{Bmatrix} := \begin{pmatrix} \{\, a \in A \mid P(a) \,\} \\ \downarrow \\ A \end{pmatrix}$

I learned hyperdoctrines