Notes on MacLane and Moerdijk's
"Sheaves in Geometry and Logic - A First Introduction to Topos Theory" (1994)
https://www.springer.com/gp/book/9780387977102

These notes are at:
http://angg.twu.net/LATEX/2020maclane-moerdijk.pdf

See:
http://angg.twu.net/LATEX/2020favorite-conventions.pdf
http://angg.twu.net/math-b.html#favorite-conventions
I wrote these notes mostly to test if the conventions above are good enough.

1. Categories of functors

(Page 25):
(viii) $\mathbf{Set}^{\mathbf{C}^{\text{op}}}$, where...

$$
\begin{array}{ccc}
C & \longmapsto & PC \\
{\scriptstyle f}\uparrow & \longmapsto & \downarrow{\scriptstyle Pf} \\
D & \longmapsto & PD \\
\mathbf{C} & & \\
\mathbf{C}^{\text{op}} & \xrightarrow{P} & \mathbf{Set}
\end{array}
\qquad
\begin{array}{ccc}
C & \longmapsto & P'C \\
{\scriptstyle f}\uparrow & \longmapsto & \downarrow{\scriptstyle P'f} \\
D & \longmapsto & P'D \\
\mathbf{C} & & \\
\mathbf{C}^{\text{op}} & \xrightarrow{P'} & \mathbf{Set}
\end{array}
\qquad
\begin{array}{cccc}
C & PC & \xrightarrow{\theta_C} & P'C \\
{\scriptstyle f}\uparrow & {\scriptstyle Pf}\downarrow & & \downarrow{\scriptstyle P'f} \\
D & PD & \xrightarrow{\theta_D} & P'D \\
 & & & \\
 & P & \xrightarrow{\theta} & P'
\end{array}
$$

$$
\begin{array}{cccccc}
C & \longmapsto & PC & x & & \\
{\scriptstyle f}\uparrow & \longmapsto & \downarrow{\scriptstyle Pf} & \downarrow & & \\
D & \longmapsto & PD & x \cdot f & & = x|f \\
{\scriptstyle g}\uparrow & \longmapsto & \downarrow{\scriptstyle Pg} & \downarrow & & \\
E & \longmapsto & PE & (x \cdot f) \cdot g & & = x \cdot (f \circ g) \\
\mathbf{C} & & & & & \\
\mathbf{C}^{\text{op}} & \xrightarrow{P} & \mathbf{Set} & & &
\end{array}
$$

(Page 26):
Each object C of $\mathbf{C}$ gives rise to a presheaf $\mathbf{y}(C)$ on $\mathbf{C}$...

$$
\begin{array}{ccccccc}
D & \longmapsto & \mathbf{y}(C)(D) & = \mathrm{Hom}_{\mathbf{C}}(D,C) & u & & D \xrightarrow{u} C \\
{\scriptstyle\alpha}\uparrow & \longmapsto & \downarrow{\scriptstyle \mathbf{y}(C)(\alpha)} & \downarrow{\scriptstyle(\circ\alpha)} & \downarrow & & {\scriptstyle\alpha}\uparrow \nearrow{\scriptstyle u\circ\alpha} \\
D' & \longmapsto & \mathbf{y}(C)(D') & = \mathrm{Hom}_{\mathbf{C}}(D',C) & u\circ\alpha & & D'
\end{array}
$$

$\mathbf{C}$
$\mathbf{C}^{\mathrm{op}} \xrightarrow{\mathbf{y}(C)} \mathbf{Set}$

$$
\begin{array}{cccccccc}
C_1 & \longmapsto & \mathbf{y}(C_1) & = \mathrm{Hom}_{\mathbf{C}}(-,C_1) & \mathrm{Hom}_{\mathbf{C}}(D,C_1) & v & & D \xrightarrow{v} C_1 \\
{\scriptstyle f}\downarrow & \longmapsto & \downarrow{\scriptstyle \mathbf{y}(f)} & \downarrow{\scriptstyle(f\circ)} & \downarrow{\scriptstyle(f\circ)} & \downarrow & & {\scriptstyle f\circ v}\searrow \quad \downarrow{\scriptstyle f} \\
C_2 & \longmapsto & \mathbf{y}(C_2) & = \mathrm{Hom}_{\mathbf{C}}(-,C_2) & \mathrm{Hom}_{\mathbf{C}}(D,C_2) & f\circ v & & C_2
\end{array}
$$

$\mathbf{C}$
$\mathbf{C} \xrightarrow{\mathbf{y}} \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$

$$
\begin{array}{ccc}
 & & 1 \\
 & & \downarrow{\scriptstyle \ulcorner\theta_\alpha\urcorner = \ulcorner\alpha_C(1_C)\urcorner} \\
C & \longmapsto & PC \\
\uparrow & \longmapsto & \downarrow \\
D & \longmapsto & PD
\end{array}
$$

$\mathbf{C}$
$\mathbf{C}^{\mathrm{op}} \xrightarrow{P} \mathbf{Set}$

$$
\begin{array}{ccc}
\mathbf{y}(C) = & \mathrm{Hom}_{\mathbf{C}}(-,C) \longrightarrow & \mathbf{Set}(1,P-) \\
 & {\scriptstyle\alpha}\searrow & \updownarrow \\
 & & P
\end{array}
$$

(Page 37):

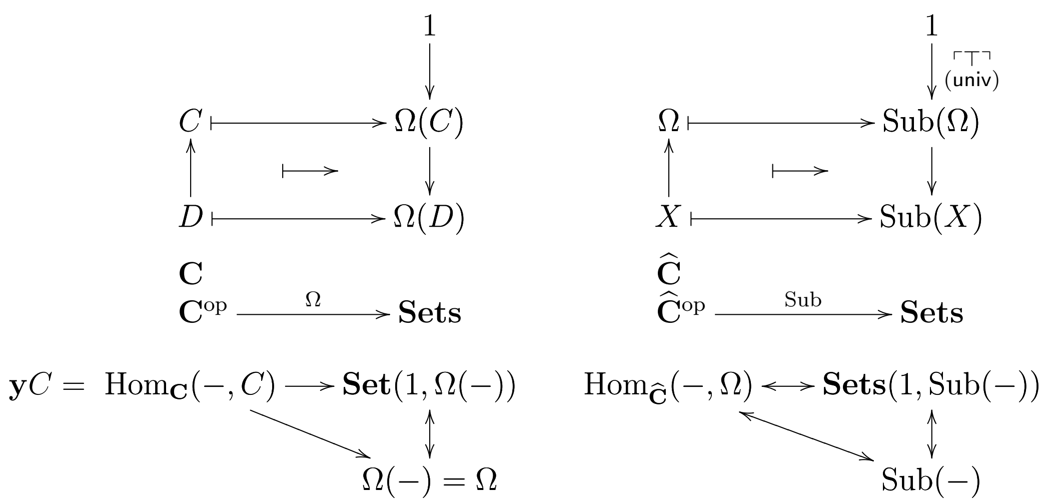
For an arbitrary presheaf category $\widehat{\mathbf{C}} = \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, if there is a subobject classifier Ω, it must, in particular, classify the subobjects of each representable presheaf $\mathbf{y}C = \mathrm{Hom}_{\mathbf{C}}(-, C) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$. Therefore,

$$\begin{aligned} \mathrm{Sub}_{\widehat{\mathbf{C}}}(\mathrm{Hom}_{\mathbf{C}}(-, C)) &\cong \mathrm{Hom}_{\widehat{\mathbf{C}}}(\mathrm{Hom}_{\mathbf{C}}(-, C), \Omega) \\ &\cong \mathrm{Nat}(\mathrm{Hom}_{\mathbf{C}}(-, C), \Omega) \end{aligned}$$

By the Yoneda Lemma [see §1(6) above], the set on the right is (up to isomorphism) $\Omega(C)$. Thus the subobject classifier Ω, if it exists, must be the functor $\Omega : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$ with object function

$$\begin{aligned} \Omega(C) &= \mathrm{Sub}_{\widehat{\mathbf{C}}}(\mathrm{Hom}_{\mathbf{C}}(-, C)) \\ &= \{\, S \mid S \text{ is a subfunctor of } \mathrm{Hom}_{\mathbf{C}}(-, C) \,\}, \end{aligned}$$

and with a suitable mapping function.



$$\begin{aligned} \Omega(C) &\cong \mathbf{Sets}(1, \Omega(C)) \\ &\cong \mathrm{Hom}_{\widehat{\mathbf{C}}}(\mathbf{y}C, \Omega) \\ &\cong \mathrm{Sub}_{\widehat{\mathbf{C}}}(\mathbf{y}C) \\ &= \mathrm{Sub}_{\widehat{\mathbf{C}}}(\mathrm{Hom}_{\mathbf{C}}(-, C)) \\ &= \{\, S \mid S \text{ is a subfunctor of } \mathrm{Hom}_{\mathbf{C}}(-, C) \,\}, \end{aligned}$$

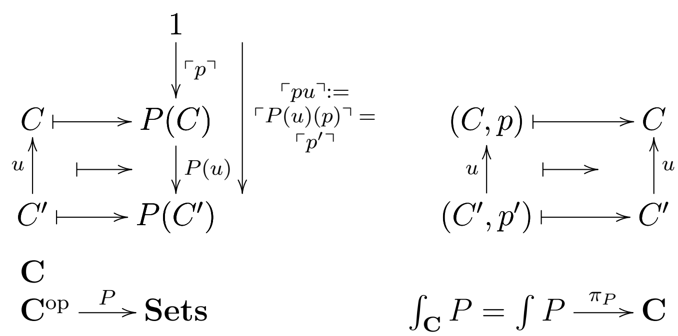
(Page 41):

Given P, the index category J which serves to prove the proposition is the so-called *category of elements* of P, denoted by $\int_{\mathbf{C}} P$ or, more briefly, $\int P$. Its objects are all pairs (C, p) where C is an object of $\mathbf{C}$ and p is an element $p \in P(C)$. Its morphisms $(C', p) \to (C, p)$ are those morphisms $u : C' \to C$ of $\mathbf{C}$ for which $pu = p'$; in other words, u must take the chosen element p in $P(C)$ "back" into p' in $P(C')$:

$$(C', p) \to (C, p) \qquad \text{by } u : C' \to C \text{ with } pu = p'.$$

These morphisms ar composed by composing the underlying arrows u of $\mathbf{C}$. This category has an evident projection functor

$$\pi_P : \int_{\mathbf{C}} P \to \mathbf{C}, \qquad (C, p) \mapsto C.$$



(Page 41):

Colimits over the category of elements can be used to construct a pair of adjoint functors which will have many uses, as follows.

Theorem 2. *If $A : \mathbf{C} \to \mathcal{E}$ is a functor from a small category $\mathbf{C}$ to a cocomplete category $\mathcal{E}$, the functor R from $\mathcal{E}$ to presheaves given by*

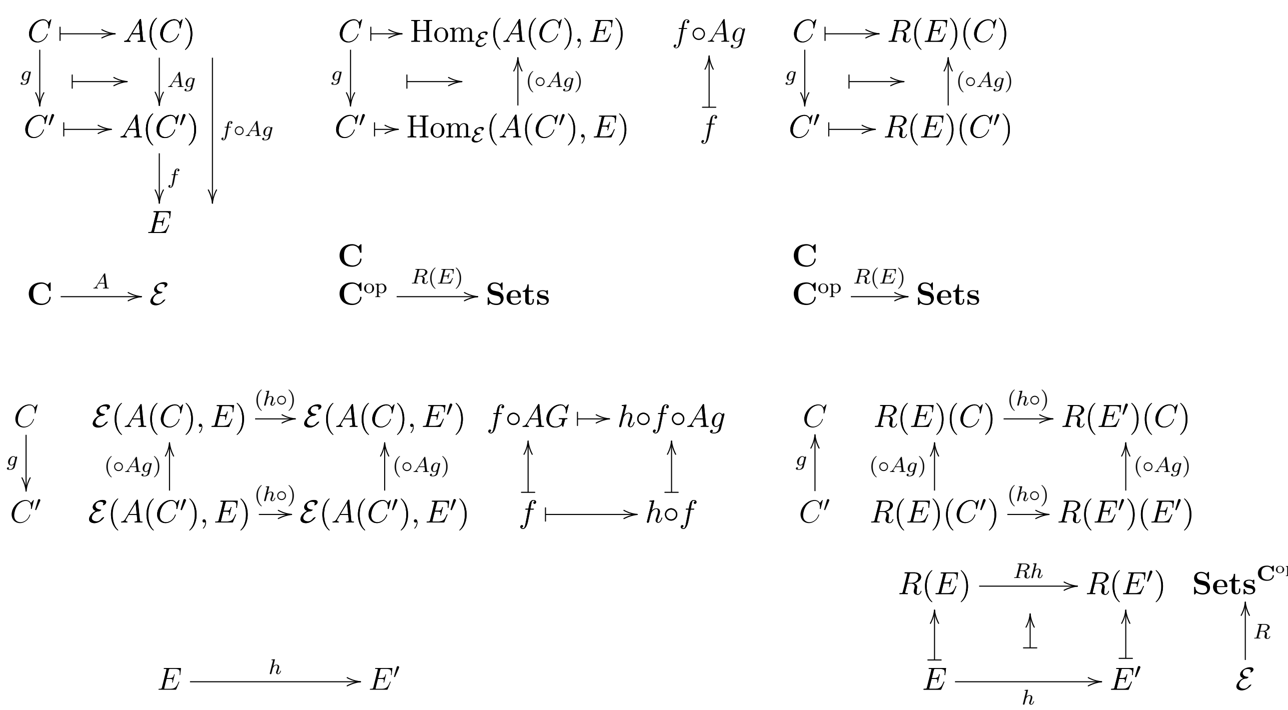
$$R(E) : C \mapsto \mathrm{Hom}_{\mathcal{E}}(A(C), E)$$

has a left adjoint $L : \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}} \to \mathcal{E}$ defined for each presheaf P in $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ as the colimit

$$L(P) = \mathrm{Colim}(\int P \overset{\pi_P}{\to} \mathbf{C} \overset{A}{\to} \mathcal{E}).$$

Here's how I found the type and a precise definition of R_1...

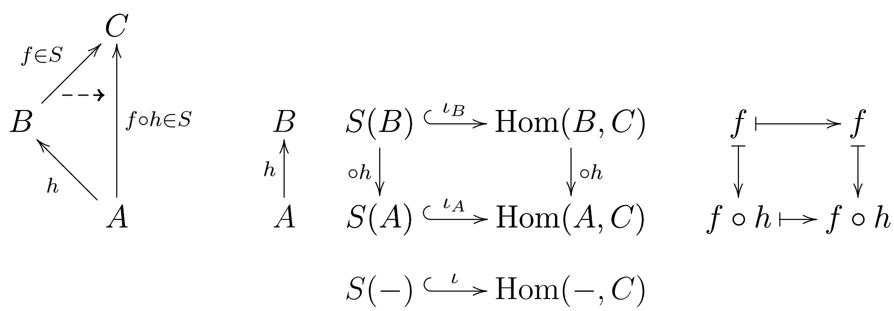
(It's too big! How do other people do this?)



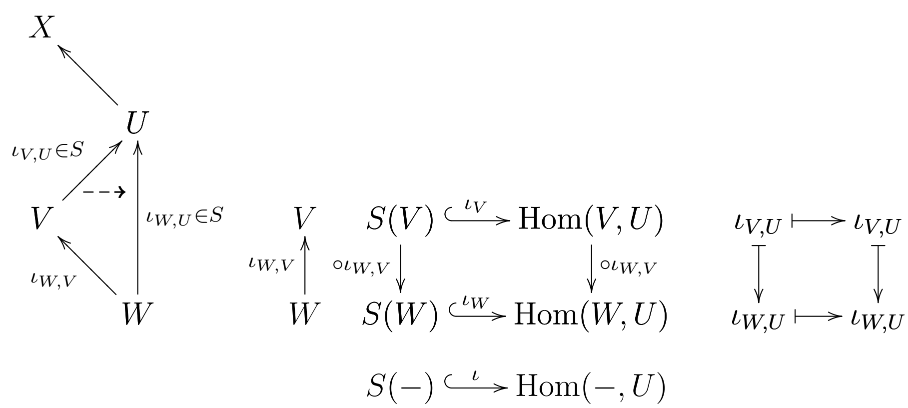
[MM92], (Page 37):

Given an object C in the category $\mathbf{C}$, a *sieve* on C is a set S of arrows with codomain C such that $f \in S$ implies $f \circ h \in S$.

A sieve S can be seen as a subfunctor $S : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ of $\mathrm{Hom}(-, C) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ — or, more explicitly, as a natural transformation $\iota : S \to \mathrm{Hom}(-, C)$ such that each ι_A is an inclusion.



(Page 70):
A sieve S on an object U of $\mathcal{O}(X)$ is a subfunctor of $\mathrm{Hom}(-, U)$:



2. Sieves and Sheaves

(From pages 69–70):

On any space X, each open set U determines a presheaf $\mathrm{Hom}(-, U)$ defined, for each open set V, by

$$\mathrm{Hom}(V, U) = \begin{cases} 1 & \text{if } V \subset U, \\ \emptyset & \text{otherwise.} \end{cases}$$

This presheaf is clearly a sheaf; it is the representable presheaf $\mathbf{y}(U) = \mathrm{Hom}(-, U)$ on the category $\mathcal{O}(X)$. Recall from section I.4 that a *sieve* S on U in this category is defined to be a subfunctor of $\mathrm{Hom}(-, U)$. Replacing the sieve S by the set (call it S again) of all those $V \subset U$ with $SV = 1$, we may also describe a sieve on U as a subset $S \subset \mathcal{O}(U)$ of objects such that $V_0 \subset V \in S$ implies $V_0 \in S$. Each indexed family $\{ V_i \subset U \mid i \in I \}$ of subsets of U generates (= "spans") a sieve S on U; namely, the set S consisting of all those open V with $V \subseteq V_i$ for some i; in particular, each $V_0 \subset U$ determines a *principal sieve* (V_0) on U, consisting of all V with $V \subseteq V_0$. It is not difficult to see that a sieve S on U is principal iff the subfunctor S of $\mathbf{y}(U)$ is a subsheaf (Exercise 1). A sieve S on U is said to be a *covering sieve* for U when U is the union of all the open sets V in S.

Let's see how to visualize this.
Definitions:
if $V \in \mathcal{O}(X)$ then $\downarrow V = \{\, W \in \mathcal{O}(X) \mid W \subseteq V \,\}$;
if $\mathcal{V} \subseteq \mathcal{O}(X)$ then $\downarrow \mathcal{V} = \bigcup_{V \in \mathcal{V}} (\downarrow V)$.
Let's use this topology from [PH1, sections 12 and 13]:
$X = H =$  (the "house" DAG), and

$$\mathcal{O}(X) \quad = \quad \begin{array}{ccccc} & 32 & & & \\ & & 22 & & \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{array} .$$

Writing 0 for $\emptyset$,

$$\mathbf{y}(22) = \mathrm{Hom}(-, 22) = \begin{array}{ccccc} & 0 & & & \\ & & 1 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{array} ,$$

This is also a sieve on 22: $S = \begin{array}{ccccc} & 0 & & & \\ & & 0 & & \\ & 0 & & 0 & \\ 1 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array}$.

Let $\mathcal{V} = \{ V_i \subset U \mid i \in I \} = \begin{array}{ccccc} & 0 & & & \\ & & 0 & & \\ & 0 & & 0 & \\ 1 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array}$;

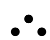
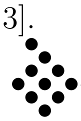
then $\mathcal{V}$ spans $\downarrow\mathcal{V} = \begin{array}{ccccc} & 0 & & & \\ & & 0 & & \\ & 0 & & 0 & \\ 1 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array}$.

Note that this $\downarrow\mathcal{V}$ is not a principal sieve.
We have $\bigcup(\downarrow\mathcal{V}) = 21 \neq 22$, so $\downarrow\mathcal{V}$ is not a covering sieve on U.

A subset $\mathcal{V} \subseteq \mathcal{O}(X)$ is a sieve on X if and only if $\mathcal{V} = \downarrow\mathcal{V}$.
Let's use the letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ to denote sieves on X.
For every sieve $\mathcal{A}$ on X we have:
$\mathcal{A}$ is a covering sieve on $\bigcup \mathcal{A}$,
and $\downarrow\bigcup\mathcal{A}$ is a principal sieve (generated by $\bigcup\mathcal{A}$).

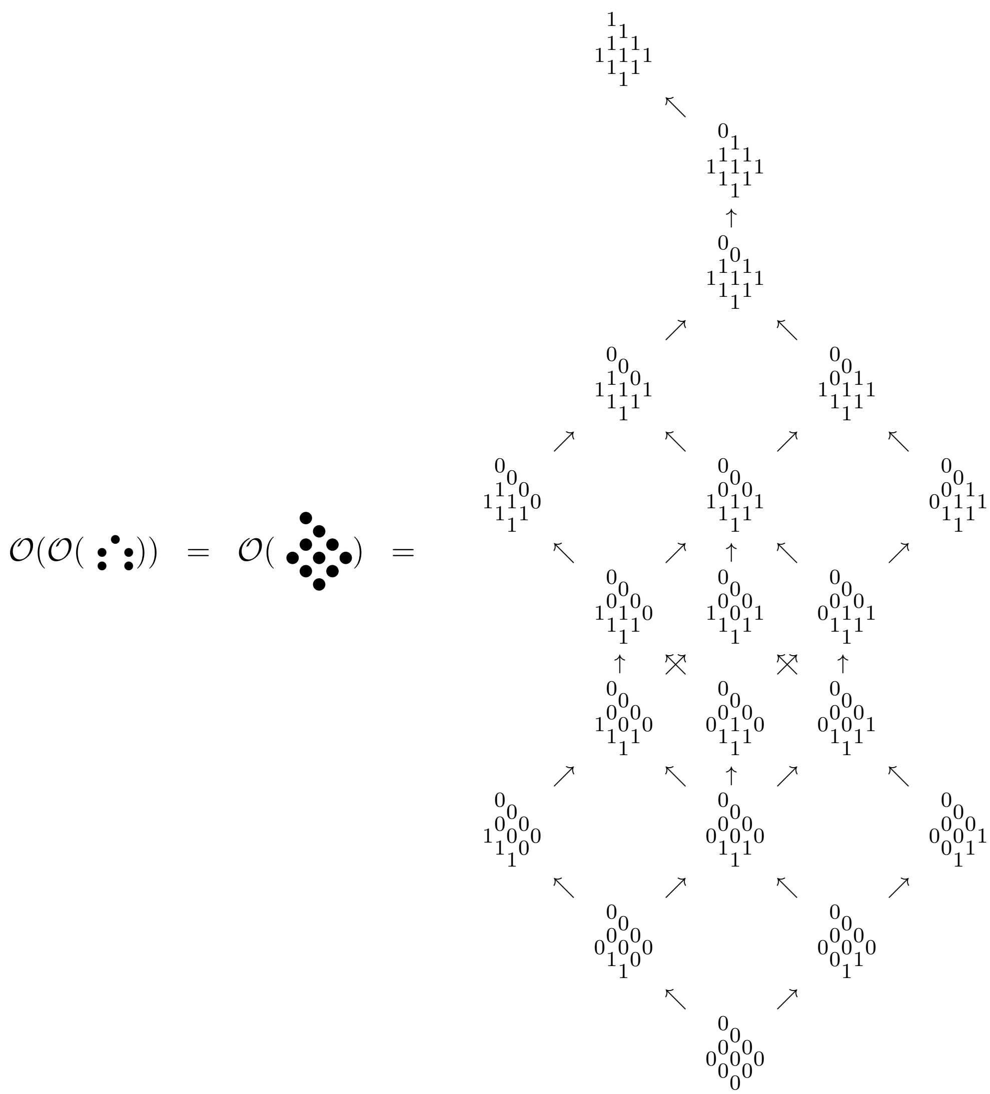
The operation $\mathcal{A} \mapsto \mathcal{A}^* := \downarrow\bigcup\mathcal{A}$ takes sieves to principal sieves.
This operation obeys $\mathcal{A} \subseteq \mathcal{A}^* = \mathcal{A}^{**}$.
Fact (true but not obvious): $\mathcal{A}^* \cap \mathcal{B}^* = (\mathcal{A} \cap \mathcal{B})^*$.

Now reread [PH1, sections 12 and 13].

Remember that $\mathcal{O}(H) = \mathcal{O}($  $) =$ .

In [DP02, p.20] the operation '$\mathcal{O}$' is defined in a different, but equivalent, way: if X is an ordered set then $\mathcal{O}(X)$ is the set of the down-sets of X, ordered by inclusion.

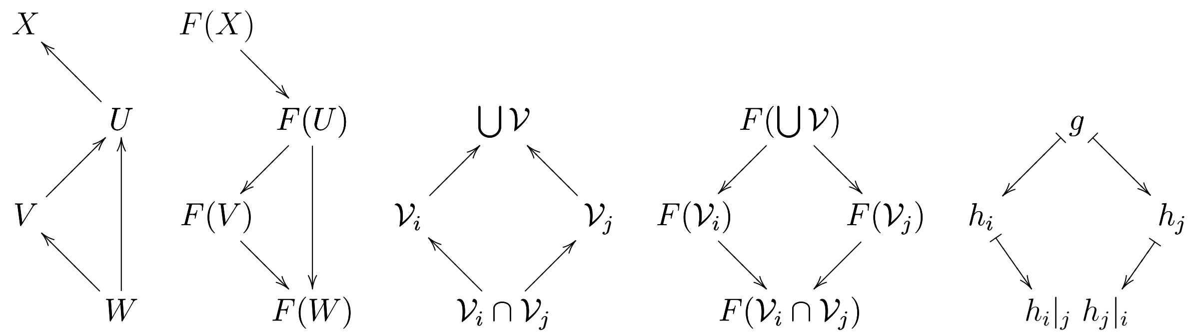
With the definition in [DP02] it is easy to calculate $\mathcal{O}(\mathcal{O}(H))$ as a set of down-sets, and then interpret it as a topology. We have:



The elements of $\mathcal{O}(\mathcal{O}(H))$ are exactly the sieves on H.

The operation $\mathcal{A} \mapsto \mathcal{A}^*$ that takes sieves on H to principal sieves is a J-operator on $\mathcal{O}(\mathcal{O}(H))$ (see [PH2]).

Topological sheaves in my notation



$\mathcal{O}(X)$

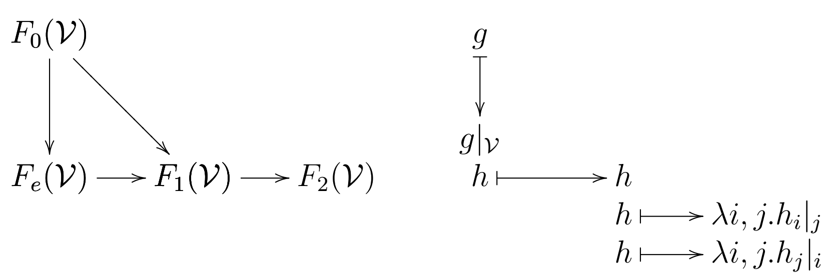
$\mathcal{O}(X)^{\mathrm{op}} \longrightarrow \mathbf{Set}$

$\mathcal{O}(X)$

$\mathcal{O}(X)^{\mathrm{op}} \longrightarrow \mathbf{Set}$

I : a set
$\mathcal{V} : I \to \mathcal{O}(X)$
$\bigcup \mathcal{V} := \bigcup_{i \in I} \mathcal{V}_i$
$g : F(\bigcup \mathcal{V})$
$h : (i : I) \to F(\mathcal{V}_i)$
$i, j : I$
$g|_{\mathcal{V}} : F_e(\mathcal{V})$
$g|_{\mathcal{V}} := \lambda i. F(\iota : \mathcal{V}_i \to \bigcup \mathcal{V})(g)$

$h_i|_j := F(\iota : \mathcal{V}_i \cap \mathcal{V}_j \to \mathcal{V}_i)(h_i)$
$h_j|_i := F(\iota : \mathcal{V}_j \cap \mathcal{V}_j \to \mathcal{V}_j)(h_j)$
$F_0(\mathcal{V}) := F(\bigcup \mathcal{V})$
$F_1(\mathcal{V}) := (i : I) \to F(\mathcal{V}_i)$
$F_e(\mathcal{V}) := \{\, h : (i : I) \to F(\mathcal{V}_i) \mid \forall (i, j : I). h_i|_j = h_j|_i \,\}$
$F_2(\mathcal{V}) := (i, j : I) \to F(\mathcal{V}_i \cap \mathcal{V}_j)$



II.5. Sheaves and cross-sections
(Page 87):

$$\begin{array}{ccccc} \Gamma\Lambda_P & \longleftarrow\!\!\!\shortmid & P & \qquad & P \\ {\scriptstyle\sigma}\downarrow & \longleftrightarrow & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\eta} \\ F & \longmapsto & F & & \Gamma\Lambda_P \end{array}$$

$$\mathrm{Sh}(X) \underset{\mathrm{inc}}{\overset{\Gamma\Lambda}{\leftrightarrows}} \mathbf{Sets}^{\mathcal{O}(X)^{\mathrm{op}}} = \widehat{\mathcal{O}(X)}$$

(Page 88):
6. Sheaves as Étale Spaces

$$\begin{array}{ccccccc} \Lambda\Gamma Y & \qquad & \Lambda P & \longleftarrow\!\!\!\shortmid & P & \qquad & P \\ {\scriptstyle\epsilon_Y}\downarrow & & \downarrow & \longleftrightarrow & \downarrow & & \downarrow{\scriptstyle\eta_P} \\ Y & & Y & \longmapsto & \Gamma Y & & \Gamma\Lambda P \end{array}$$

$$\mathbf{Top}/X = \mathbf{Bund}(X) \underset{\Gamma}{\overset{\Lambda}{\leftrightarrows}} \mathbf{Sets}^{\mathcal{O}(X)^{\mathrm{op}}}$$

III.4. Sheaves on a Site

(Page 121)
Definition in page 122:

$$
\begin{array}{cccc}
C & J(C) & \mathbf{y}C \xrightarrow{\exists ! g} P & \mathrm{Hom}(\mathbf{y}C, P) \quad g \\
 & \uparrow{\scriptstyle \in} & {\scriptstyle i}\uparrow \nearrow {\scriptstyle \forall f} & \downarrow{\scriptstyle (\circ i) \atop \text{iso}} \qquad \downarrow \\
 & \forall S & S & \mathrm{Hom}(S, P) \quad g \circ i
\end{array}
$$

Archetypal case:

$$
\begin{array}{cccc}
U & J(U) & {\downarrow} U \xrightarrow{\exists ! g} F & \mathrm{Hom}({\downarrow} U, F) \quad g \\
 & \uparrow{\scriptstyle \in} & {\scriptstyle i}\uparrow \nearrow {\scriptstyle \forall f} & \downarrow{\scriptstyle (\circ i) \atop \text{iso}} \qquad \downarrow \\
 & \forall \mathcal{U} & \mathcal{U} & \mathrm{Hom}(\mathcal{U}, F) \quad g \circ i
\end{array}
$$

Only the 'j's:

$$
\begin{array}{ccc}
P^* & P^* \xrightarrow{\exists ! g} F & \mathrm{Hom}(P^*, F) \quad g \\
{\scriptstyle d \atop \text{dense}}\uparrow & {\scriptstyle d}\uparrow \nearrow {\scriptstyle \forall f} & \downarrow{\scriptstyle (\circ d) \atop \text{iso}} \qquad \downarrow \\
\forall P & P & \mathrm{Hom}(P, F) \quad g \circ d
\end{array}
$$

All dense truth values:

$$
\begin{array}{ccc}
\forall Q & Q \xrightarrow{\exists ! g} F & \mathrm{Hom}(Q, F) \quad g \\
{\scriptstyle d \atop \text{dense}}\uparrow & {\scriptstyle d}\uparrow \nearrow {\scriptstyle \forall f} & \downarrow{\scriptstyle (\circ d) \atop \text{iso}} \qquad \downarrow \\
\forall P & P & \mathrm{Hom}(P, F) \quad g \circ d
\end{array}
$$

All dense maps (see Bell p.174):

$$
\begin{array}{ccc}
\forall B & B \xrightarrow{\exists ! g} F & \mathrm{Hom}(B, F) \quad g \\
{\scriptstyle d \atop \text{dense}}\uparrow & {\scriptstyle d}\uparrow \nearrow {\scriptstyle \forall f} & \downarrow{\scriptstyle (\circ d) \atop \text{iso}} \qquad \downarrow \\
\forall A & A & \mathrm{Hom}(A, F) \quad g \circ d
\end{array}
$$

III.5. The associated sheaf functor

(Page 128):

$$(P^+)^+ = \quad \begin{array}{ccc} \mathbf{a}P & \longleftarrow\!\shortmid & P \\ {\scriptstyle ?}\big\downarrow & \longleftrightarrow & \big\downarrow{\scriptstyle ?} \\ F & \longmapsto & F \end{array} \qquad \begin{array}{c} P \\ \big\downarrow{\scriptstyle \eta} \\ i\mathbf{a}P \end{array}$$

$$\mathrm{Sh}(\mathbf{C}, J) \underset{i}{\overset{\mathbf{a}}{\leftrightarrows}} \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}} \quad = \widehat{\mathcal{O}(X)}$$

V.4 Lawvere-Tierney Subsumes Grothendieck

IX. Localic Topoi

(Page 471)
Lemma IX.1.1:

$$\begin{array}{ccc} \Phi(V) & \longleftarrow\!\!\shortmid & V \\ \downarrow & \longleftrightarrow & \downarrow \\ U & \longmapsto & \Psi(U) \end{array} \quad = \bigvee\{ V \in B \mid \Phi(V) \le U \}$$

$$A \underset{\Psi}{\overset{\Phi}{\leftrightarrows}} B$$

$$\begin{array}{ccc} f^{-1}(V) & \longleftarrow\!\!\shortmid & V \\ \downarrow & \longleftrightarrow & \downarrow \\ U & \longmapsto & f_*(U) \end{array} \quad = \bigcup\{ V \in \mathcal{O}(T) \mid f^{-1}V \subseteq U \}$$

$$\mathcal{O}(S) \underset{f_*}{\overset{f^{-1}}{\leftrightarrows}} \mathcal{O}(T)$$

$$S \xrightarrow{f} T$$

(Page 472):

$$\begin{array}{c} (\textbf{Locales}) \\ \updownarrow \\ (\textbf{Frames})^{\mathrm{op}} \quad (\textbf{Frames}) \\ \uparrow \\ (\textbf{Spaces}) \end{array} \qquad \begin{array}{ccc} S & \xrightarrow{f} & T \\ \updownarrow & \updownarrow & \updownarrow \\ \mathcal{O}(S) & \xleftarrow{f^{-1}} & \mathcal{O}(T) \\ \updownarrow & \uparrow & \updownarrow \\ S & \xrightarrow{f} & T \end{array} \qquad \begin{array}{ccc} (\mathcal{O}(S), \subseteq) & \longleftarrow & (\mathcal{O}(T), \subseteq) \\ \updownarrow & \updownarrow & \updownarrow \\ (\mathcal{O}(S), \subseteq) & \longleftarrow & (\mathcal{O}(T), \subseteq) \\ \updownarrow & \uparrow & \updownarrow \\ (S, \mathcal{O}(S)) & \longrightarrow & (T, \mathcal{O}(T)) \end{array}$$

(Page 474): Lemma 1, archetypal case:

$$\begin{array}{cccrcl}
 & & & \{\,* \in 1 \mid p(*) \notin U\,\} = & p^{-1}U & \longleftarrow\!\!\shortmid\; U \\
(\mathbf{Frames}) & & & \{\emptyset, \{*\}\} = & \{0,1\} & \xleftarrow{p^{-1}} \mathcal{O}(X) \\
\uparrow & & & & & \\
(\mathbf{Spaces})^{\mathrm{op}} & (\mathbf{Spaces}) & & \{*\} = & 1 & \xrightarrow{\;p\;} X
\end{array}$$

For each $p : 1 \to X$ we define K and P as:

$$\begin{array}{rclcl}
\mathcal{O}(X) & \supseteq & K & := & \operatorname{Ker} p^{-1} \\
 & & & = & \{\, U \in \mathcal{O}(X) \mid p^{-1}U = 0 \,\} \\
 & & & = & \{\, U \in \mathcal{O}(X) \mid p^{-1}U = \emptyset \,\} \\
 & & & = & \{\, U \in \mathcal{O}(X) \mid * \notin p^{-1}U \,\} \\
 & & & = & \{\, U \in \mathcal{O}(X) \mid p(*) \notin U \,\} \\
\mathcal{O}(X) & \ni & P & := & \bigcup K \\
 & & & = & \bigcup \{\, U \in \mathcal{O}(X) \mid p(*) \notin U \,\} \\
 & & & = & \mathsf{int}(X - \{p(*)\})
\end{array}$$

This K is a *kernel* on $\mathcal{O}(X)$. The definition is: a kernel on $\mathcal{O}(X)$ is a subset $K \subseteq \mathcal{O}(X)$ that is closed downwards, closed by taking arbitrary unions, and it obeys $1 \notin K$ and, for all $U, V \in \mathcal{O}(X)$:

$$U \wedge V \in K \text{ implies } U \in K \text{ or } V \in K.$$

This P is a *proper prime element* of $\mathcal{O}(X)$. The definition is: a $P \in \mathcal{O}(X)$ is a proper prime element iff $1 \neq P$, and, for all $U, V \in \mathcal{O}(X)$:

$$U \cap V \subseteq P \text{ implies } U \subseteq P \text{ or } V \subseteq P.$$

(Page 473):
There is an obvious functor Loc from spaces to locales:

$$
\begin{array}{ccc}
S & \longmapsto & \mathrm{Loc}(S) \\
{\scriptstyle f}\downarrow & \longmapsto & \downarrow{\scriptstyle \mathrm{Loc}(f)} \\
T & \longmapsto & \mathrm{Loc}(T)
\end{array}
\qquad
\begin{array}{ccc}
(S, \mathcal{O}(S)) & \longmapsto & (\mathcal{O}(S), \subseteq) \\
{\scriptstyle f}\downarrow & \longmapsto & \uparrow{\scriptstyle f^{-1}} \\
(T, \mathcal{O}(S)) & \longmapsto & (\mathcal{O}(T), \subseteq)
\end{array}
$$

$$(\mathbf{Spaces}) \xrightarrow{\mathrm{Loc}} (\mathbf{Locales})$$

(Page 475):
IX.3: Spaces from locales

On locales that come from a topological spaces we define the functor $\mathrm{pt} : (\mathbf{Locales}) \to (\mathbf{Spaces})$ as this. The functor takes each locale $X \equiv (\mathcal{O}(X), \subseteq)$ to a topological space $\mathrm{pt}(X) \equiv (\mathrm{pt}(X), \mathcal{O}(\mathrm{pt}(X))$, where:

$$
\begin{array}{rrcl}
 & \mathrm{pt}(X) & := & \{\, p \mid p : 1 \to X \,\} \\
\text{for } U \in \mathcal{O}(X), & \mathrm{pt}(U) & := & \{\, p : 1 \to X \mid p(*) \in U \,\} \\
 & & = & \{\, p : 1 \to X \mid * \in p^{-1}(U) \,\} \\
 & & = & \{\, p : 1 \to X \mid p^{-1}U = 1 \,\} \\
 & \mathcal{O}(\mathrm{pt}(X)) & := & \{\, \mathrm{pt}(U) \mid U \in \mathcal{O}(X) \,\}
\end{array}
$$

On locales $X \equiv (X, \leq)$ we define the functor $\mathrm{pt} : (\mathbf{Locales}) \to (\mathbf{Spaces})$ as this generalization of the idea above:

$$
\begin{array}{rrcl}
 & \mathrm{pt}(X) & := & \{\, p \mid p : 1 \to X \,\} \\
\text{for } U \in X, & \mathrm{pt}(U) & := & \{\, p : 1 \to X \mid p^{-1}U = 1 \,\} \\
 & \mathcal{O}(\mathrm{pt}(X)) & := & \{\, \mathrm{pt}(U) \mid U \in X \,\}
\end{array}
$$

We draw that functor as:

$$
\begin{array}{ccc}
X & \longmapsto & \mathrm{pt}(X) \\
{\scriptstyle f}\downarrow & \longmapsto & \downarrow{\scriptstyle \mathrm{pt}(f)} \\
Y & \longmapsto & \mathrm{pt}(Y)
\end{array}
\qquad
\begin{array}{ccc}
(X, \leq) & \longmapsto & (\mathrm{pt}(X), \mathcal{O}(\mathrm{pt}(X)) \\
\uparrow & \longmapsto & \downarrow \\
(Y, \leq) & \longmapsto & (\mathrm{pt}(Y), \mathcal{O}(\mathrm{pt}(Y))
\end{array}
\qquad
\begin{array}{c}
p \\
\downarrow \\
f \circ p
\end{array}
$$

$$(\mathbf{Locales}) \xrightarrow{\mathrm{pt}} (\mathbf{Spaces})$$

(Page 476):

Theorem 1: The functor pt : (**Locales**) → (**Spaces**) is right adjoint to the functor Loc : (**Spaces**) → (**Locales**).

$$\begin{array}{ccccccc} \mathrm{Loc}(S) & \longleftarrow\!\shortmid & S & & (\mathcal{O}(S), \subseteq) & \longleftarrow\!\shortmid & (S, \mathcal{O}(S)) \\ {\scriptstyle f}\downarrow & \longleftrightarrow & \downarrow{\scriptstyle g} & & {\scriptstyle f^{-1}}\uparrow & \longleftrightarrow & \downarrow{\scriptstyle g} \\ X & \longmapsto & \mathrm{pt}(X) & & (X, \leq) & \longmapsto & (\mathrm{pt}(X), \mathcal{O}(\mathrm{pt}(X))) \end{array}$$

$$(\mathbf{Locales}) \underset{\mathrm{pt}}{\overset{\mathrm{Loc}}{\rightleftarrows}} (\mathbf{Spaces})$$

Proof:

$$f^{-1} \quad :=$$

IX.5. Localic topoi

(Page 487):

(Page 488):

Theorem 1: For a Grothendieck topos the following are equivalent:

(i) $\mathcal{E}$ is localic,

(ii) there exists a site for $\mathcal{E}$ with a poset as underlying category,

(iii) $\mathcal{E}$ is generated by the subobjects of its terminal object 1.

Proof. Since a frame is a poset, (i) trivially implies (ii).

(ii) ⇒ (iii) Suppose that $\mathcal{E} = \mathrm{Sh}(\mathbf{P}, J)$, where J is a Grothendieck topology on a poset $\mathbf{P}$, and write $\mathbf{ay} : \mathbf{P} \to \mathcal{E}$ for the process of sheafification $\mathbf{a}$ followed by the Yoneda embedding. Now for each $p \in \mathbf{P}$ the map is necessarily monic in presheaves, while sheafification $\mathbf{a}$ is left exact, hence preserves monics. Thus every map $\mathbf{ay}(p) \to 1$ is monic, hence gives a subobject of 1. But III.6(17) showed that the images of the $\mathbf{ay}$ generate the topos $\mathcal{E}$.

$$\begin{array}{ccccc} p & \longmapsto & \mathbf{y}p & \longmapsto & \mathbf{ay}p \\ & & \downarrow & & \downarrow \\ & & 1 & \longmapsto & 1 \end{array}$$

$$\mathbf{P} \xrightarrow{\mathbf{y}} \mathbf{Sets}^{\mathbf{P}^{\mathrm{op}}} \xrightarrow{\mathbf{a}} \mathrm{Sh}(\mathbf{P}, J) = \mathcal{E}, \qquad \mathbf{P} \xrightarrow{\mathbf{ay}} \mathcal{E}$$

$$\left(\begin{array}{ccccc} & \cdot & & & \\ & & \cdot & & \\ & \cdot & & \cdot & \\ 20 & & \cdot & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{array}\right) \text{ is a covering sieve for } 22$$

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