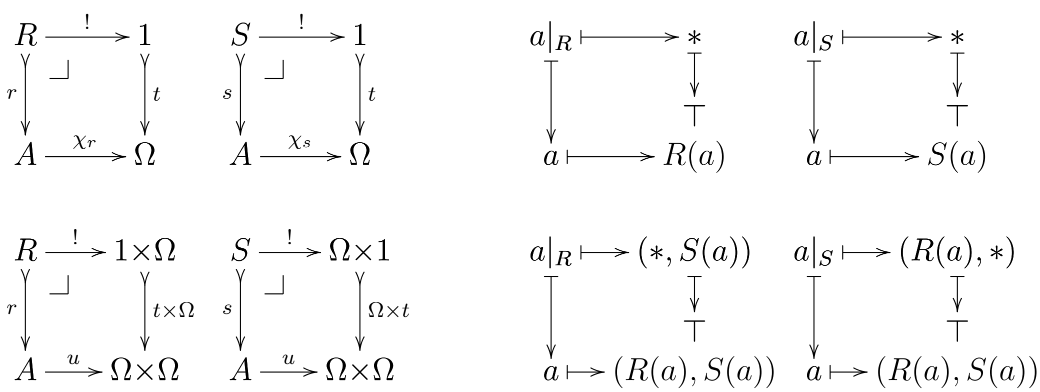
Notes on Colin McLarty's "Elementary Categories, Elementary Toposes" (1992).

These notes are at:
http://angg.twu.net/LATEX/2020mclarty.pdf
See:
http://angg.twu.net/LATEX/2020favorite-conventions.pdf
http://angg.twu.net/math-b.html#favorite-conventions

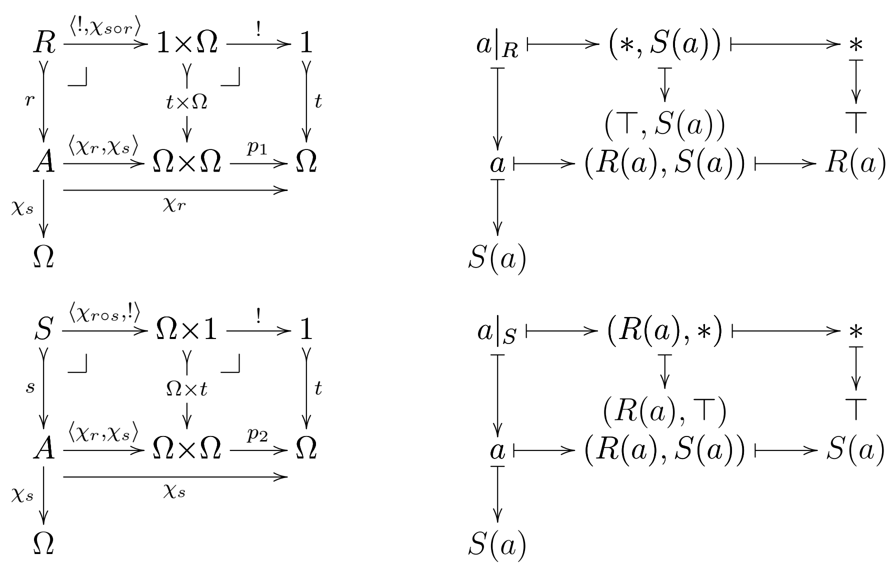
13.3. Conjunction and intersection

The arrow t is sometimes called the *generic sub-object* because it is a sub-object of Ω itself, and every sub-object is a pullback of it along exactly one arrow. There is also a generic pair of sub-objects, namely $t\times\Omega : 1\times\Omega \to \Omega\times\Omega$ and $\Omega \times t : \Omega \times 1 \to \Omega \times \Omega$.

Theorem 13.2. Given any pair of sub-objects of an object A, $r : R \rightarrowtail A$ and $s : S \rightarrowtail A$, there is a unique arrow $u : A \to \Omega \times \Omega$ that makes both the (lower) squares below pullbacks, and that arrow is $u := \langle \chi_r, \chi_s \rangle$:



Proof. Consider the following diagram:



The left-hand square is a pullback iff the outer rectangle is; that is, iff $p_1 \circ u = \chi_s$. Similarly, $p_2 \circ u = \chi_r$.

21. Topologies

Theorem 21.1. For any topology j, subobjects s and w of A, and arrow $f : B \to A$:

(1) $s \subseteq \overline{s}$,

(2) $\overline{s} \equiv \overline{\overline{s}}$,

(3) $\overline{(s \cap w)} \equiv \overline{s} \cap \overline{w}$,

(4) if $s \subseteq w$ then $\overline{s} \subseteq \overline{w}$,

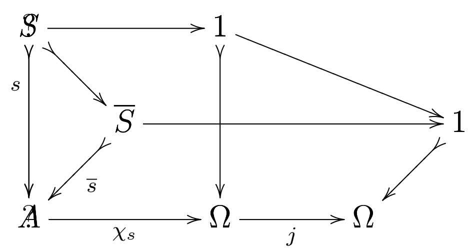
(5) the j-closure of $f^{-1}(s)$ is $f^{-1}(\overline{s})$.

We say the the j-closure operator is inflationary, idempotent, it preserves intersections, it preserves order, and it is stable under pullback.

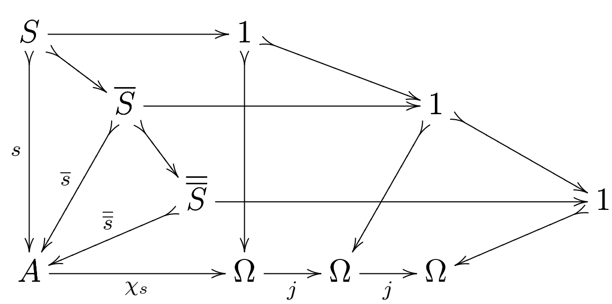
Here are the constructions.

(Thx to David Michael Roberts for helping me with item (1)!)

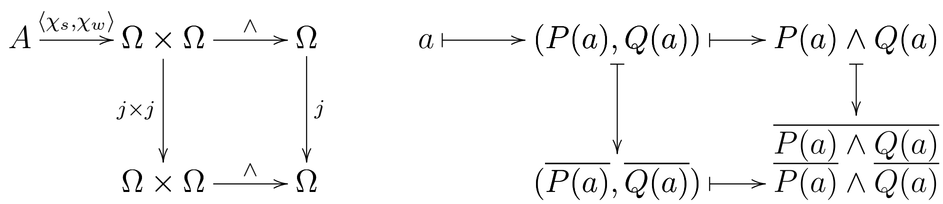
(1) The arrow $S \rightarrowtail \overline{S}$ is a factorization through a pullback:



(2) Idempotent: we have $\overline{\overline{s}} = (j \circ j \circ \chi_s)^{-1}(t) = (j \circ \chi_s)^{-1}(t) = \overline{s}$, so $\overline{\overline{s}}$ and $\overline{s}$ are the same subobject and $\overline{S} \rightarrowtail \overline{\overline{S}}$ is the identity map.



(3) Preserves intersections:

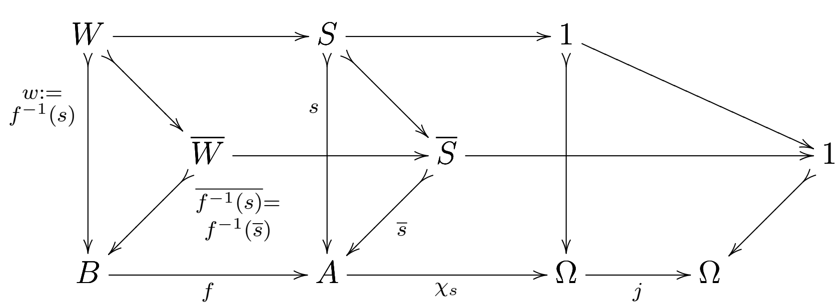


$$\begin{aligned}
j \circ \wedge \circ \langle \chi_s, \chi_w \rangle &= j \circ \chi_{s \cap w} \\
&= \chi_{\overline{s \cap w}} \\
j \circ \wedge \circ \langle \chi_s, \chi_w \rangle &= \wedge \circ (j \times j) \circ \langle \chi_s, \chi_w \rangle \\
&= \wedge \circ \langle j \circ \chi_s, j \circ \chi_w \rangle \\
&= \wedge \circ \langle \chi_{\overline{s}}, \chi_{\overline{w}} \rangle \\
&= \chi_{\overline{s} \cap \overline{w}}
\end{aligned}$$

(4) Preserves order:

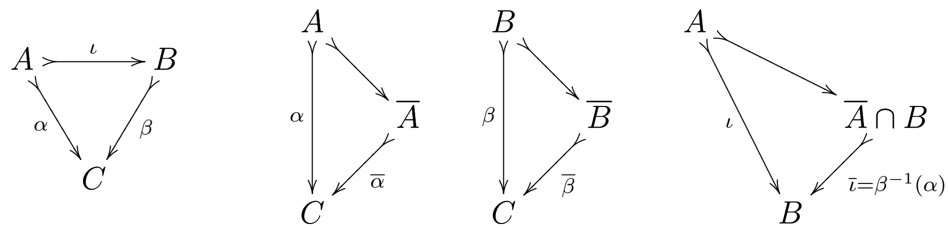
$$\begin{array}{c} s \subseteq w \\ \hline s = s \cap w \\ \hline \overline{s} = \overline{s \cap w} \\ \hline \overline{s} = \overline{s} \cap \overline{w} \\ \hline \overline{s} \subseteq \overline{w} \end{array} \qquad \begin{array}{c} P \leq Q \\ \hline P = P \wedge Q \\ \hline \overline{P} = \overline{P \wedge Q} \\ \hline \overline{P} = \overline{P} \wedge \overline{Q} \\ \hline \overline{P} \leq \overline{Q} \end{array}$$

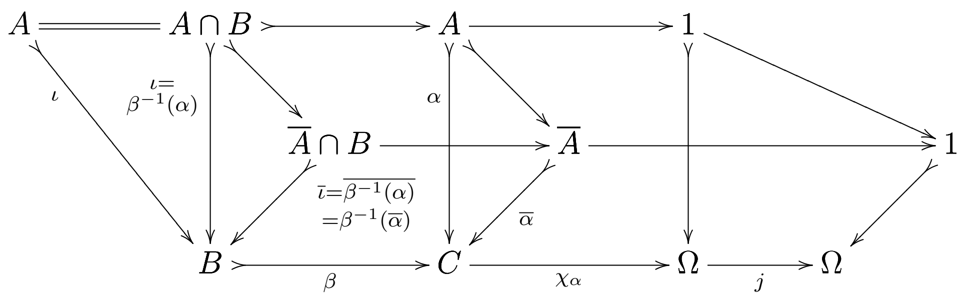
(5) Stable under pullback. Here W is a subobject of B, not of A:



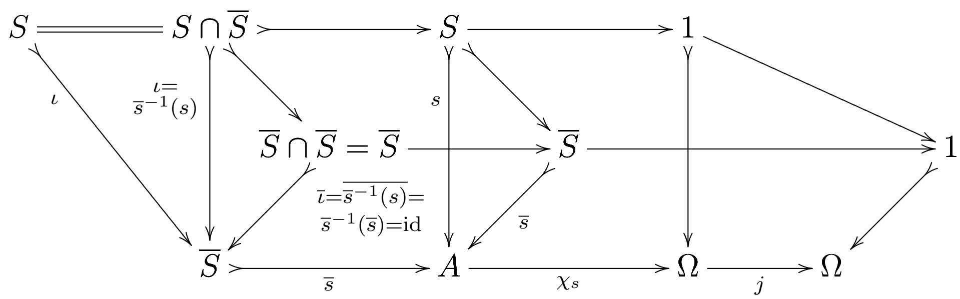
Some consequences of stability by pullbacks

Theorem. If $\alpha \subseteq \beta$ are subobjects of C with mediating map $\iota : A \rightarrowtail B$, as in the first triangle below, then we have $\overline{\iota} = \overline{\beta^{-1}(\alpha)} = \beta^{-1}(\overline{\alpha})$. If $\alpha : A \rightarrowtail C$ and $\beta : B \rightarrowtail C$ then the domain of $\overline{\iota} = \beta^{-1}(\overline{\alpha})$ is $\overline{A} \cap B$. Proof:





First corollary. Take any monic $s : S \rightarrowtail A$; its closure is $\overline{s} : \overline{S} \rightarrowtail A$. Call its mediating map $\iota : S \rightarrowtail \overline{S}$. Then $\overline{\iota} = \mathrm{id}_{\overline{S}} : \overline{S} \rightarrowtail \overline{S}$. Proof:



Two definitions: dense and closed. Take a monic $s : S \rightarrowtail A$. We say that s is *dense* when $\overline{s} = \mathrm{id}_A$, i.e., when $\overline{S} = A$. And we say that s is *closed* when $\overline{s} = s$, i.e., when $\overline{S} = S$.

Note that for any monic $s : S \rightarrowtail A$ with mediating map $\iota : S \rightarrowtail \overline{S}$ this mediating map is dense (by the First Corollary) and $\overline{s}$ is closed (because $\overline{\overline{s}} = \overline{s}$).

Second corollary. Let $p : P \rightarrowtail 1$, $q : Q \rightarrowtail 1$, and $p \subseteq q$, with mediating map $\iota : P \rightarrowtail Q$. Then $\bar{i} = q^{-1}(\bar{p}) : \overline{P} \cap Q \rightarrowtail Q$. Proof: