

Adjoints to an arbitrary f^* :

$$\begin{array}{ccc}
 A_1 & \dashrightarrow & \Sigma_f A_1 \\
 \alpha \downarrow & \dashrightarrow & \downarrow \Sigma_f \alpha \\
 A_2 & \dashrightarrow & \Sigma_f A_2 \\
 \begin{array}{c} f \\ (\Sigma_f^\sharp)g \end{array} \downarrow & \rightleftarrows & \downarrow \begin{array}{c} (\Sigma_f^\flat) f \\ g \end{array} \\
 f^* B_1 & \dashleftarrow & B_1 \\
 f^* \beta \downarrow & \dashleftarrow & \downarrow \beta \\
 f^* B_2 & \dashleftarrow & B_2 \\
 \begin{array}{c} (\Pi_f^\flat)h \\ k \end{array} \downarrow & \rightleftarrows & \downarrow \begin{array}{c} h \\ (\Pi_f^\sharp)k \end{array} \\
 C_1 & \dashrightarrow & \Pi_f C_1 \\
 \gamma \downarrow & \dashrightarrow & \downarrow \Pi_f \gamma \\
 C_2 & \dashrightarrow & \Pi_f C_2 \\
 \\
 \mathbf{P}(X) & \begin{array}{c} \xrightarrow{\Sigma_f} \\ \xleftarrow{f^*} \\ \xrightarrow{\Pi_f} \end{array} & \mathbf{P}(Y) \\
 \\
 X & \xrightarrow{f} & Y
 \end{array}$$

Equality as an adjoint to Δ^* :

$$\begin{array}{c}
A_1(x) \vdash \rightarrow x=x' \wedge A_1(x) \\
\alpha \downarrow \quad \vdash \rightarrow \quad \downarrow \Sigma_\Delta \alpha \\
A_2(x) \vdash \rightarrow x=x' \wedge A_2(x) \\
f \downarrow \quad \rightleftarrows \quad \downarrow (\Sigma_\Delta^b) f \\
(\Sigma_\Delta^\sharp)g \downarrow \quad \rightleftarrows \quad \downarrow (\Sigma_\Delta^b)g \\
B_1(x, x) \leftarrow \vdash B_1(x, x') \\
\Delta^* \beta \downarrow \quad \leftarrow \vdash \quad \downarrow \beta \\
B_2(x, x) \leftarrow \vdash B_2(x, x') \\
(\Pi_\Delta^b)h \downarrow \quad \rightleftarrows \quad \downarrow (\Pi_\Delta^\sharp)h \\
C_1(x) \vdash \rightarrow x=x' \supset C_1(x) \\
\gamma \downarrow \quad \vdash \rightarrow \quad \downarrow \Pi_\Delta \gamma \\
C_2(x) \vdash \rightarrow x=x' \supset C_2(x)
\end{array}
\quad
\begin{array}{c}
\Sigma_\Delta \alpha := \left(\frac{\frac{x=x' \wedge A_1(x)}{A_1(x)} \quad \vdots \quad \alpha}{\frac{x=x' \wedge A_2(x)}{A_2(x)}} \right) \\
(\Sigma_\Delta^b) f := \left(\frac{\frac{x=x' \wedge A_2(x)}{A_2(x)} \quad \vdots \quad f}{\frac{x=x'}{B_1(x, x')}} \right) \\
(\Pi_\Delta^\sharp) k := \left(\frac{\frac{[B_2(x, x')]^1 \quad \vdots \quad k}{C_1(x)} \quad \vdots \quad 1}{\frac{[x=x']^1 \quad \frac{B_2(x, x) \supset C_1(x)}{B_2(x, x') \supset C_1(x)}}{C_1(x)}} \right) \\
\Pi_\Delta \gamma := \left(\frac{\frac{[x=x']^1 \quad x=x' \supset C_1(x)}{C_1(x)} \quad \vdots \quad \gamma}{\frac{C_2(x)}{x=x' \supset C_2(x)} \quad 1} \right)
\end{array}$$

$$\mathbf{P}(X) \xrightleftharpoons[\Pi_\Delta]{\Sigma_\Delta} \mathbf{P}(X \times X)$$

$$\begin{array}{ccc}
x & \vdash \rightarrow & (x, x') \\
X & \xrightarrow{\Delta} & X \times X
\end{array}$$

Quantifiers as adjoints to π^* :

$$\begin{array}{c}
 Pxy \vdash \longrightarrow \exists y.Pxy \\
 \alpha \downarrow \quad \vdash \quad \downarrow \Sigma_\pi \alpha \\
 Qxy \vdash \longrightarrow \exists y.Qxy \\
 (\Sigma_\pi^\sharp)g := \left(\frac{Qxy}{\exists y.Qxy} \right) \quad \downarrow f \quad \vdash \quad \downarrow (\Sigma_\pi^b)f \\
 (\Sigma_\pi^\sharp)g := \left(\frac{Qxy}{\exists y.Qxy} \right) \quad \downarrow f \quad \vdash \quad \downarrow (\Sigma_\pi^b)f \\
 Rxx \longleftarrow \vdash Rxx \\
 \pi^*\beta := \left(\frac{Rxx}{Sxx} \right) \quad \downarrow \pi^*\beta \quad \longleftarrow \vdash \quad \downarrow \beta \\
 Sxx \longleftarrow \vdash Sxx \\
 (\Pi_\pi^b)h := \left(\frac{Sxx}{\forall y.Txy} \right) \quad \downarrow (\Pi_\pi^b)h \quad \vdash \quad \downarrow (\Pi_\pi^\sharp)k \\
 (\Pi_\pi^b)h := \left(\frac{Sxx}{\forall y.Txy} \right) \quad \downarrow (\Pi_\pi^b)h \quad \vdash \quad \downarrow (\Pi_\pi^\sharp)k \\
 Txy \vdash \longrightarrow \forall y.Txy \\
 \gamma \downarrow \quad \vdash \quad \downarrow \Pi_\pi \gamma \\
 Uxy \vdash \longrightarrow \forall y.Uxy \\
 \Pi_\pi \gamma := \left(\frac{\forall y.Txy}{Txy} \right) \quad \downarrow \gamma \quad \vdash \quad \downarrow \Pi_\pi \gamma \\
 \Pi_\pi \gamma := \left(\frac{\forall y.Txy}{Txy} \right) \quad \downarrow \gamma \quad \vdash \quad \downarrow \Pi_\pi \gamma
 \end{array}$$

$$\mathbf{P}(X \times Y) \begin{array}{c} \xrightarrow{\Sigma_\pi} \\ \xleftarrow{\pi^*} \\ \xrightarrow{\Pi_\pi} \end{array} \mathbf{P}(X)$$

$$\begin{array}{c}
 (x, y) \vdash \longrightarrow x \\
 X \times Y \xrightarrow{\pi} X
 \end{array}$$

Adjoints to $(y := f(x))^*$ can be built using quantifiers and equality:

$$\begin{array}{c}
 \begin{array}{c}
 (\Sigma_{\pi}^{\sharp})g := \left(\frac{\frac{fx=fx \quad Qx}{fx=fx \wedge Qx} \quad \frac{\frac{[fx=y \wedge Qx]^1}{\exists x.fx=y \wedge Qx} \quad R_y}{fx=y}}{Rfx} \quad Rfx \quad 1 \right) \\
 \\
 \pi^* \beta := \left(\frac{[Ry]^1}{Rfx} \quad \beta}{Sfx} \right) \\
 \\
 (\Pi_{\pi}^{\flat})h := \left(\frac{[Sy]^1}{Sfx} \quad \frac{\forall y.fx=y \supset Tx}{fx=y \supset Tx}}{fx=fx} \quad Tx}{Tx} \right)
 \end{array}
 \quad
 \begin{array}{c}
 Px \vdash \exists x.fx=y \wedge Px \\
 \alpha \downarrow \quad \vdash \quad \downarrow \Sigma_{\pi} \alpha \\
 Qx \vdash \exists x.fx=y \wedge Qx \\
 (\Sigma_{\pi}^{\sharp})g \downarrow \quad \rightleftharpoons \quad \downarrow (\Sigma_{\pi}^{\flat})f \\
 Rfx \longleftarrow \vdash Ry \\
 \pi^* \beta \downarrow \quad \longleftarrow \vdash \quad \downarrow \beta \\
 Sfx \longleftarrow \vdash Sy \\
 (\Pi_{\pi}^{\flat})h \downarrow \quad \rightleftharpoons \quad \downarrow (\Pi_{\pi}^{\sharp})k \\
 Tx \vdash \exists x.fx=y \supset Tx \\
 \gamma \downarrow \quad \vdash \quad \downarrow \Pi_{\pi} \gamma \\
 Ux \vdash \exists x.fx=y \supset Ux
 \end{array}
 \quad
 \begin{array}{c}
 \Sigma_{\pi} \alpha := \left(\frac{\frac{[fx=y \wedge Px]^1}{Px} \quad \alpha}{fx=y} \quad Qx}{\exists y.fx=y \wedge Px} \quad \frac{fx=y \wedge Qx}{\exists x.fx=y \wedge Qx} \quad 1 \right) \\
 \\
 (\Sigma_{\pi}^{\flat})f := \left(\frac{\frac{[fx=y \wedge Qx]^1}{Qx} \quad f}{fx=y} \quad Rfx}{\exists x.fx=y \wedge Qx} \quad Ry \quad 1 \right) \\
 \\
 (\Pi_{\pi}^{\sharp})k := \left(\frac{[fx=y]^1 \quad [Sy]^2}{Sfx} \quad \frac{Tx}{fx=y \supset Tx} \quad 1}{\exists x.fx=y \supset Tx} \quad 2 \right) \\
 \\
 \Pi_{\pi} \gamma := \left(\frac{[fx=y]^1 \quad \frac{[fx=y \supset Tx]^2}{Tx}}{fx=y \supset Tx} \quad \frac{Ux}{fx=y \supset Ux} \quad 1}{\exists x.fx=y \supset Ux} \quad 2 \right)
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \mathbf{P}(X) \begin{array}{c} \xrightarrow{\Sigma_f} \\ \xleftarrow{f^*} \\ \xrightarrow{\Pi_f} \end{array} \mathbf{P}(Y) \\
 \\
 x \vdash \longrightarrow fx \\
 X \xrightarrow{\pi} Y
 \end{array}$$