Notes on [Seely83], a.k.a.:
"Hyperdoctrines, Natural Deduction and The Beck Condition", available at:
http://www.math.mcgill.ca/rags/ZML/ZML.PDF

These notes are at:
http://angg.twu.net/LATEX/2020seelyhyp.pdf

See:
http://angg.twu.net/LATEX/2020favorite-conventions.pdf
http://angg.twu.net/math-b.html#favorite-conventions
I wrote these notes mostly to test if the conventions above are good enough.

[See83] was one of the first papers that made me think: "wow, this is really beautiful but it's written in the *wrong language!* It shows how to generalize *something*, but it doesn't show in a way that is clear enough what this 'something' is, and I find the translation between the original and the generalization *very hard* to follow... *I need a version 'for children' of this!!!*"...

My current favorite way of creating a version "for children" of the constructions in [See83] takes two steps: in the first we draw the "missing diagrams" following the conventions in [FavC], and in the second we draw diagrams with the same shapes as these, but in the "archetypal case" ([IDARCT, sec.16]). The *archetypal hyperdoctrine* is the fibration of subsets:

$$\begin{array}{ccc} \mathbf{Set}^{\hookleftarrow} & & \begin{pmatrix} A \\ \downarrow \\ B \end{pmatrix} \\ {\scriptstyle \mathrm{Cod}}\downarrow & & \downarrow \\ \mathbf{Set} & & B \end{array}$$

We usually don't draw the vertical '$\mapsto$'s — we just draw $\begin{pmatrix} A \\ \downarrow \\ B \end{pmatrix}$ above B.

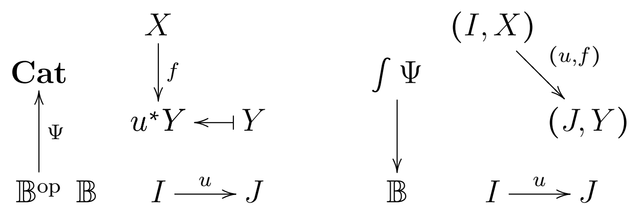
We abbreviate subsets defined by propositions as this:

$$\begin{pmatrix} \{\, b \in B \mid P(b) \,\} \\ \downarrow \\ B \end{pmatrix} \quad \equiv \quad \begin{Bmatrix} P(b) \\ b \in B \end{Bmatrix}$$

Most of the time we will just write the '$P(b)$' above the 'B', omitting the '$b \in$' and pretending that the '$b \in$' part is obvious to reconstruct.

In this paper Seely uses indexed categories instead of fibrations. The translation between indexed categories and fibrations is nicely explained in

[Jac99, p.107]; one of the directions of the translation is the Grothendieck Construction, that takes an index category $\Psi : \mathbb{B}^{\text{op}} \to \mathbf{Cat}$ and produces an fibration $\int \Psi \to \mathbf{B}$, and in the notation for defining new categories diagrammatically in [FavC, sec.7.1] the Grothendieck Construction is this:

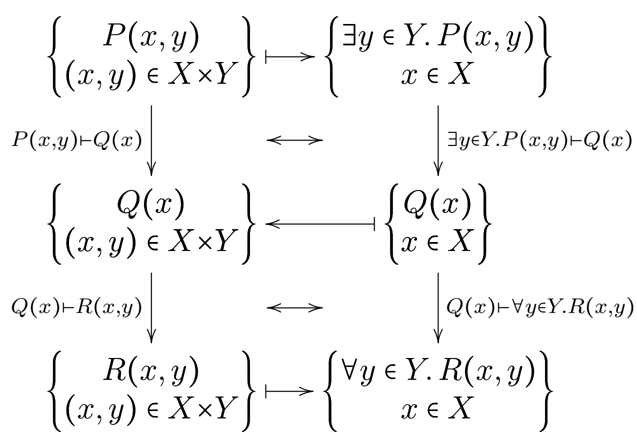


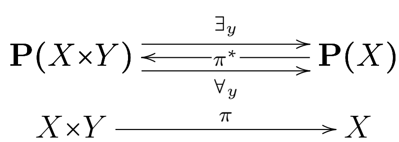
Motivation: some adjunctions

It is easy to see — in the sense of *visualize* — that the functor that adds a dummy variable to the context,

$$\begin{Bmatrix} Q(x) \\ (x,y) \in X{\times}Y \end{Bmatrix} \longleftarrow\!\!\shortmid \begin{Bmatrix} Q(x) \\ x \in X \end{Bmatrix}$$

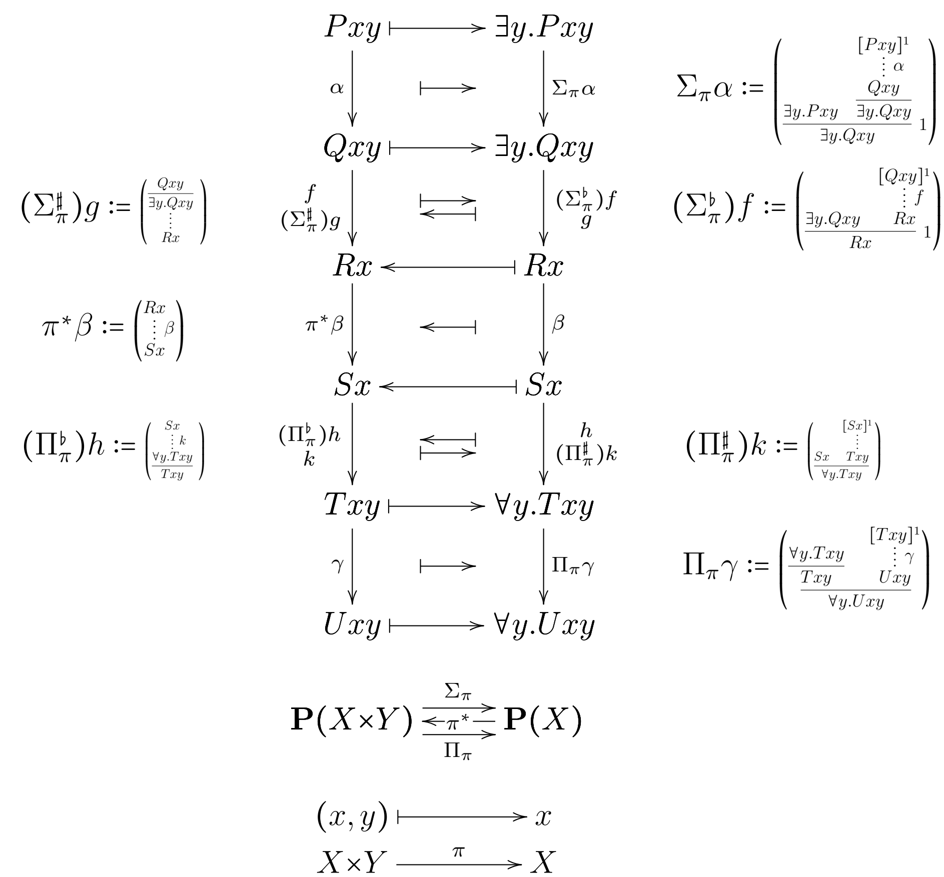
is "adjoint to the quantifiers":





See the big figure in section 13 of [IDARCT].

The '$\longleftrightarrow$' in the middle of the top square in the diagram above can also be interpreted as: *if we have an intuitionistic proof of* $P(x, y) \vdash Q(x)$ *then we can produce from it an intuitionistic proof of* $\exists y \in Y.P(x, y) \vdash Q(x)$, *and vice-versa* — and the same for the '$\longleftrightarrow$' in the middle of the lower square. Here is how, in details, and showing also the actions of the three functors on morphisms:

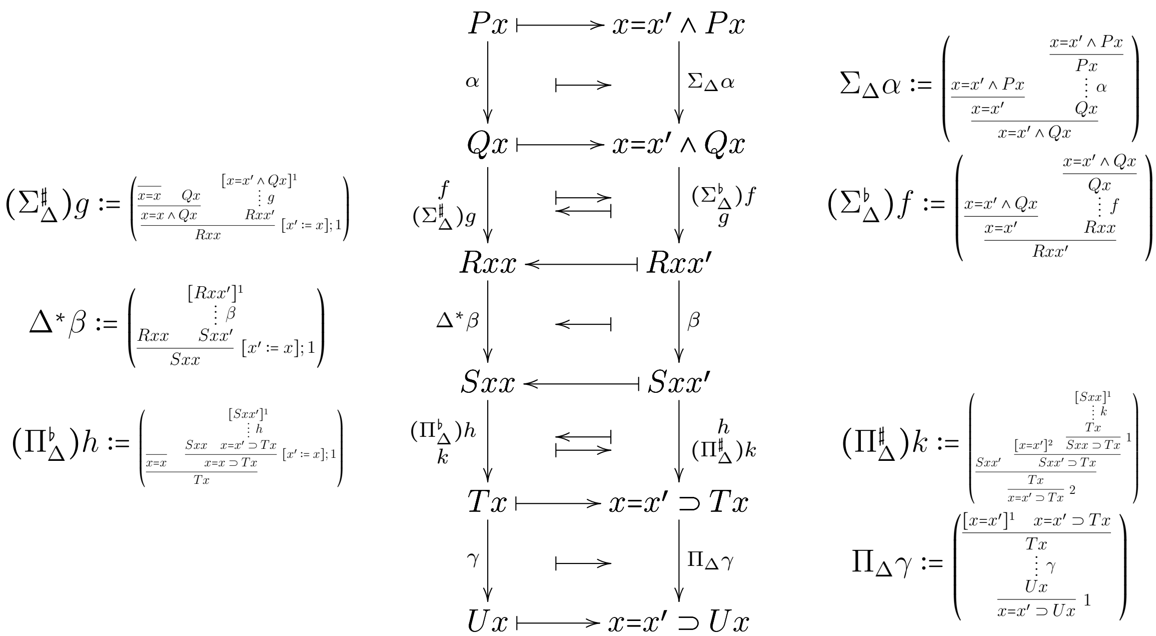


Not everybody who knows Natural Deduction for Propositional Calculus knows how to use quantifiers in ND... some of the rules for introduction and elimination of quantifiers have restrictions that are (or: that I found) hard to understand. Seely uses a system of ND that is equivalent to the one in [Pra65], but that only allows discarding hypotheses in subtrees "with a single hypothesis"; I find that system much easier to understand than the one in [Pra65].

...so, the diagram in the previous page can be used to learn and to teach Natural Deduction with quantifiers — and we can use the adjoints to the functor

$$\left\{\begin{matrix} R(x,x) \\ x \in X \end{matrix}\right\} \longleftarrow\!\shortmid \left\{\begin{matrix} R(x,x') \\ (x,x') \in X \times X \end{matrix}\right\}$$

to learn how to use the rules for equality in ND:



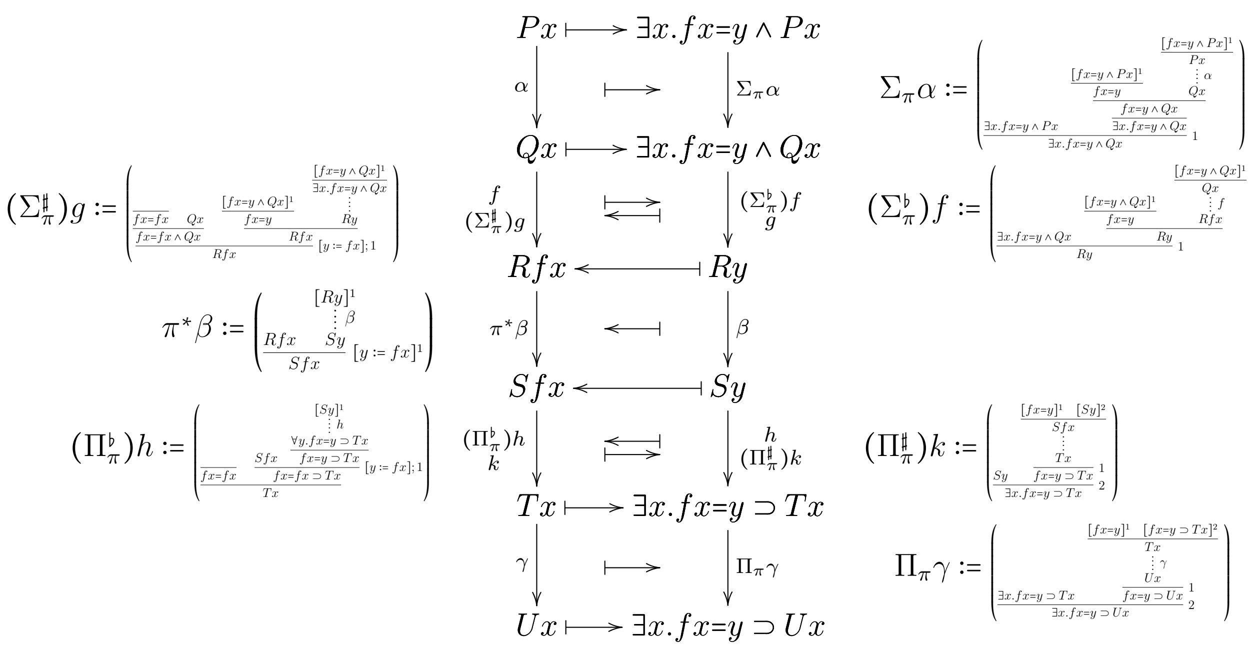
$$\mathbf{P}(X) \underset{\Pi_\Delta}{\overset{\Sigma_\Delta}{\underset{\longleftarrow \Delta^* \longrightarrow}{\longrightarrow}}} \mathbf{P}(X \times X)$$

$$\begin{array}{ccc} x & \longmapsto & (x,x') \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

We also have adjoints to this functor,

$$\left\{\begin{matrix} R(f(x)) \\ x \in X \end{matrix}\right\} \longleftarrow\!\shortmid \left\{\begin{matrix} R(y) \\ y \in Y \end{matrix}\right\}$$

and they are a bit harder to build than the ones in the previous page — and they force us to learn how to handle functions in ND.



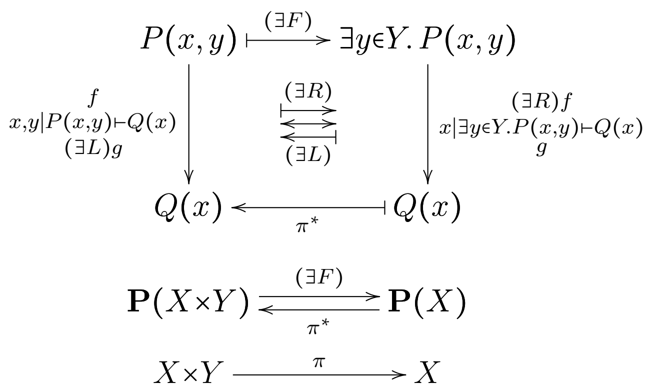
$$\mathbf{P}(X) \underset{\Pi_f}{\overset{\Sigma_f}{\underset{\longleftarrow f^* \longrightarrow}{\rightrightarrows}}} \mathbf{P}(Y)$$

$$\begin{matrix} x & \longmapsto & fx \\ X & \xrightarrow{\pi} & Y \end{matrix}$$

All this *suggests* that it should be possible to interpret first-order logic in categories in which all functors of the form f^* have both adjoints... but it turns out that we need Frobenius, Beck-Chevalley, and lots of other technicalities.

The adjunction $\exists \dashv \pi^*$ for children

The main diagram:



The '↔' in the middle of the square says that $x,y|P(x,y) \vdash Q(x)$ is true if and only if $x|\exists y \in Y.P(x,y) \vdash Q(x)$ is true. The operation $(\exists R)$ receives an inclusion morphism

$$f: \left\{\begin{matrix} P(x,y) \\ (x,y) \in X \times Y \end{matrix}\right\} \to \left\{\begin{matrix} \exists y \in Y.P(x,y) \\ x \in X \end{matrix}\right\}$$

and returns an inclusion morphism

$$(\exists R)f: \left\{\begin{matrix} \exists y \in Y.P(x,y) \\ x \in X \end{matrix}\right\} \to \left\{\begin{matrix} P(x,y) \\ (x,y) \in X \times Y \end{matrix}\right\};$$

The operation $(\exists L)$ does the reverse.

The '↔' is usually represented as bidirectional rule, and there's a similar bidirectional rule for the universal. Here they are (see [Jac99, p.230]):

$$\frac{x \in Y, y \in Y|P(x,y) \vdash Q(x)}{x \in Y|\exists y \in Y.P(x,y) \vdash Q(x)}\ (\exists\text{-mate}) \qquad \frac{x \in Y, y \in Y|Q(x) \vdash R(x,y)}{x \in Y|Q(x) \vdash \forall y \in Y.R(x,y)}\ (\forall\text{-mate})$$

The adjunction $\exists \dashv \pi^*$ for children (2)

Let's write $A \overset{=(}{\longrightarrow} B$ for "there are no morphisms from A to B", i.e., $\mathrm{Hom}(A, B) = \varnothing$. Let $X = \{0, 1, 2, 3, 4\}$ and $Y = \{0, 1\}$. Let $Q(x) = (x \geq 3)$, and let's compare what happens for two different choices of P; in the left we have $P(x, y) = (x - y \geq 3)$, and in the right we have $P(x, y) = (x - y \geq 1)$.

$$\begin{array}{ccc}
\begin{smallmatrix}00001\\00011\end{smallmatrix} & \overset{(\exists F)}{\longmapsto} & 00011 \\
{\scriptstyle !}\downarrow & \longleftrightarrow & \downarrow{\scriptstyle !} \\
\begin{smallmatrix}00111\\00111\end{smallmatrix} & \underset{\pi^*}{\longleftarrow\!\shortmid} & 00111 \\
\begin{smallmatrix}\bullet\bullet\bullet\bullet\bullet\\\bullet\bullet\bullet\bullet\bullet\end{smallmatrix} & \overset{\pi}{\longrightarrow} & \bullet\bullet\bullet\bullet\bullet
\end{array}
\qquad
\begin{array}{ccc}
\begin{smallmatrix}00111\\01111\end{smallmatrix} & \overset{(\exists F)}{\longmapsto} & 01111 \\
{\scriptstyle =(}\downarrow & \longleftrightarrow & \downarrow{\scriptstyle =(} \\
\begin{smallmatrix}00111\\00111\end{smallmatrix} & \underset{\pi^*}{\longleftarrow\!\shortmid} & 00111 \\
\begin{smallmatrix}\bullet\bullet\bullet\bullet\bullet\\\bullet\bullet\bullet\bullet\bullet\end{smallmatrix} & \overset{\pi}{\longrightarrow} & \bullet\bullet\bullet\bullet\bullet
\end{array}$$

$$\begin{array}{ccc}
\left\{\begin{matrix} x - y \geq 3 \\ (x,y) \in X \times Y \end{matrix}\right\} & \overset{(\exists F)}{\longmapsto} & \left\{\begin{matrix} \exists y \in Y.\ x - y \geq 3 \\ x \in X \end{matrix}\right\} \\
{\scriptstyle !}\downarrow & \longleftrightarrow & \downarrow{\scriptstyle !} \\
\left\{\begin{matrix} x \geq 2 \\ (x,y) \in X \times Y \end{matrix}\right\} & \underset{\pi^*}{\longleftarrow\!\shortmid} & \left\{\begin{matrix} x \geq 2 \\ x \in X \end{matrix}\right\} \\
X \times Y & \overset{\pi}{\longrightarrow} & X
\end{array}
\qquad
\begin{array}{ccc}
\left\{\begin{matrix} x - y \geq 1 \\ (x,y) \in X \times Y \end{matrix}\right\} & \overset{(\exists F)}{\longmapsto} & \left\{\begin{matrix} \exists y \in Y.\ x - y \geq 1 \\ x \in X \end{matrix}\right\} \\
{\scriptstyle =(}\downarrow & \longleftrightarrow & \downarrow{\scriptstyle =(} \\
\left\{\begin{matrix} x \geq 2 \\ (x,y) \in X \times Y \end{matrix}\right\} & \underset{\pi^*}{\longleftarrow\!\shortmid} & \left\{\begin{matrix} x \geq 2 \\ x \in X \end{matrix}\right\} \\
X \times Y & \overset{\pi}{\longrightarrow} & X
\end{array}$$

Translating the categorical rules for 'exists'

The main diagram, again:

$$\begin{array}{ccc}
P(x,y) & \xmapsto{(\exists F)} & \exists y{\in}Y.\,P(x,y) \\
{\scriptstyle f \atop {x,y|P(x,y)\vdash Q(x) \atop (\exists L)g}}\Big\downarrow & {\scriptstyle (\exists R)} \atop {\rightleftarrows \atop {\scriptstyle (\exists L)}} & \Big\downarrow{\scriptstyle (\exists R)f \atop {x|\exists y\in Y.P(x,y)\vdash Q(x) \atop g}} \\
Q(x) & \xleftarrow[\pi^*]{} & Q(x)
\end{array}$$

$$\mathbf{P}(X{\times}Y) \underset{\pi^*}{\overset{(\exists F)}{\rightleftarrows}} \mathbf{P}(X)$$

$$X{\times}Y \xrightarrow{\pi} X$$

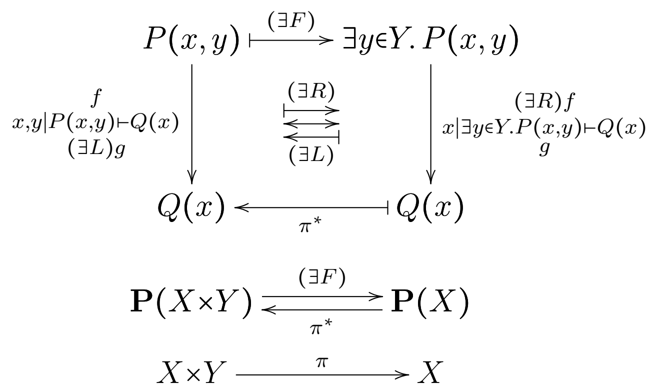
Here's how to translate the categorical rules $(\exists R)$ and $(\exists L)$ (the "transpositions") to Natural Deduction:

$$\frac{x,y|Pxy \vdash Qx}{x|\exists y.Pxy \vdash Qx}\,(\exists R) \qquad \frac{f}{(\exists R)f}\,(\exists R) \qquad (\exists R)f := \left(\frac{\exists x.Pxy \qquad \begin{array}{c}[Pxy]^1 \\ \vdots\, f \\ Qx\end{array}}{Qx}\,(\exists E);1\right)$$

$$\frac{x|\exists y.Pxy \vdash Qx}{x,y|Pxy \vdash Qx}\,(\exists L) \qquad \frac{g}{(\exists L)g}\,(\exists L) \qquad (\exists L)g := \left(\begin{array}{c}\dfrac{Pxy}{\exists y.Pxy}\,(\exists I) \\ \vdots\, g \\ Qx\end{array}\right)$$

Translating the rule $(\exists E)$ from ND to categories

The main diagram, again:



Here's how to translate an application of the rule $(\exists E)$ from Natural Deduction to a series of steps that are easy to interpret categorically. The rules $(\supset I)$ and $(\supset E)$ are transpositions of another adjunction. TODO: the categorical drawings (the "missing diagrams"), and the translation of $(\exists I)$.

$$\dfrac{\exists y.Pxy \qquad \begin{array}{c}[Pxy]^1 \quad Qx \quad Rx \\ \vdots \\ Sx\end{array}}{Sx}\,(\exists E);1 \quad \Rightarrow \quad \dfrac{Rx \qquad \dfrac{Qx \qquad \dfrac{\exists y.Pxy \qquad \dfrac{\dfrac{\begin{array}{c}[Pxy]^3 \quad [Qx]^2 \quad [Rx]^1 \\ \vdots \\ Sx\end{array}}{Rx \supset Sx}\,(\supset I);1}{Qx \supset (Rx \supset Sx)}\,(\supset I);2}{Qx \supset (Rx \supset Sx)}\,(\exists R);3}{Rx \supset Sx}\,(\supset E)}{Sx}\,(\supset E)$$

$$\dfrac{x,y|Pxy,Qx,Rx \vdash Sx}{x|\exists y.Pxy,Qx,Rx \vdash Sx}\,(\exists E) \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x,y|Pxy,Qx,Rx \vdash Sx}{x,y|Pxy,Qx \vdash Rx \supset Sx}\,(\supset I)}{x,y|Pxy \vdash Qx \supset (Rx \supset Sx)}\,(\supset I)}{x|\exists y.Pxy \vdash Qx \supset (Rx \supset Sx)}\,(\exists R)}{x|\exists y.Pxy, Qx \vdash Rx \supset Sx}\,(\supset E)}{x|\exists y.Pxy,Qx,Rx \vdash Sx}\,(\supset E)$$

Translating the rule $(\exists I)$ from ND to categories

Let's divide $(\exists I)$ in two cases...

$$\frac{P(x,y)}{\exists y.P(x,y)}\ (\exists I) \qquad \dfrac{\dfrac{}{\exists y.P(x,y) \vdash \exists y.P(x,y)}\ \text{id}}{P(x,y) \vdash \exists y.P(x,y)}\ (\exists L)$$

$$\frac{P(x,b(x))}{\exists y.P(x,y)}\ (\exists I) \qquad \dfrac{\dfrac{\dfrac{}{\exists y.P(x,y) \vdash \exists y.P(x,y)}\ \text{id}}{P(x,y) \vdash \exists y.P(x,y)}\ (\exists L)}{P(x,b(x)) \vdash \exists y.P(x,y)}\ y := b(x)$$

A categorical diagram (for the two cases):

$$\begin{array}{ccccc}
P(x,b(x)) & \longleftarrow\!\!\dashv & P(x,y) & \vdash\!\!\longrightarrow & \exists y{\in}Y.P(x,y) \\
\downarrow{\scriptstyle \langle \text{id},b\rangle^*(\exists L)\text{id}} & \overset{\langle \text{id},b\rangle^*}{\longleftarrow\!\!\dashv} & \downarrow{\scriptstyle (\exists L)\text{id}} & \overset{(\exists L)}{\longleftarrow\!\!\dashv} & \downarrow{\scriptstyle \text{id}} \\
\exists y{\in}Y.P(x,y) & \longleftarrow\!\!\dashv & \exists y{\in}Y.P(x,y) & \longleftarrow\!\!\dashv & \exists y{\in}Y.P(x,y)
\end{array}$$

$$\begin{array}{ccccc}
x & \longmapsto & (x,b(x)) & \longmapsto & x \\
X & \xrightarrow{\langle \text{id},b\rangle} & X{\times}Y & \xrightarrow{\pi} & Y
\end{array}$$

Bounded quantifiers

$$
\begin{array}{rcl}
\forall y \in Y.Pxy &:=& \forall y.y \in Y \to Pxy \\
\exists y \in Y.Pxy &:=& \exists y.y \in Y \wedge Pxy \\
(!)y.Pxy &:=& Pxy \wedge Pxy' \to y = y' \\
(!)y \in Y.Pxy &:=& y \in Y \wedge y' \in Y \wedge Pxy \wedge Pxy' \to y = y' \\
\exists! y.Pxy &:=& (\exists y.Pxy) \wedge ((!)y.Pxy) \\
\exists! y \in Y.Pxy &:=& (\exists y \in Y.Pxy) \wedge ((!)y \in Y.Pxy)
\end{array}
$$

Interpreting ND in hyperdoctrines

See pages 223–225 of [Jac99]. We have:

$$\begin{array}{ccl}
\left\{\begin{matrix} P(x,y) \\ (x,y)\in X{\times}Y \end{matrix}\right\} & = & x:X, y:Y \vdash P(x,y) : \mathsf{Prop} \\
{\scriptstyle f}\big\downarrow & = & x:X, y:Y \mid P(x,y) \vdash Qy \\
\left\{\begin{matrix} Q(y) \\ (x,y)\in X{\times}Y \end{matrix}\right\} & = & x:X, y:Y \vdash Q(y) : \mathsf{Prop}
\end{array}$$

Two examples of translations (to sequents à là Jacobs):

$$\dfrac{\exists y.Pxy \qquad \dfrac{\begin{matrix}[Pxy]^1 \\ \vdots\,\alpha \\ Qxy\end{matrix}}{\exists y.Qxy}}{\exists y.Qxy}\;1 \qquad\Rightarrow\qquad \dfrac{\dfrac{x,y;\ Pxy \vdash Qxy}{x,y;\ Pxy \vdash \exists y \in Y.Qxy}}{x;\ \exists y \in Y.Pxy \vdash \exists y \in Y.Qxy}$$

$$\dfrac{\exists x.fx{=}y \wedge Px \qquad \dfrac{\dfrac{\dfrac{[fx{=}y \wedge Px]^1}{fx{=}y} \qquad \begin{matrix}\dfrac{[fx{=}y \wedge Px]^1}{Px} \\ \vdots\,\alpha \\ Qx\end{matrix}}{fx{=}y \wedge Qx}}{\exists x.fx{=}y \wedge Qx}}{\exists x.fx{=}y \wedge Qx}\;1$$

$$\Rightarrow\qquad \dfrac{\dfrac{\dfrac{\dfrac{}{x,y;\ fx{=}y \wedge Px \vdash fx{=}y} \qquad \dfrac{\dfrac{\dfrac{}{x,y;\ fx{=}y \wedge Px \vdash Px} \qquad \dfrac{x;\ Px \vdash Qx}{x,y;\ Px \vdash Qx}}{x,y;\ fx{=}y \wedge Px \vdash Qx}}{x,y;\ fx{=}y \wedge Px \vdash Qx}}{x,y;\ fx{=}y \wedge Px \vdash fx{=}y \wedge Qx}}{x,y;\ fx{=}y \wedge Px \vdash \exists y.fx{=}y \wedge Qx}}{x;\ \exists y.fx{=}y \wedge Px \vdash \exists y.fx{=}y \wedge Qx}$$

Page 513 (5') (i):

$$\begin{array}{ccc} \gamma & \longmapsto & \Sigma_t\gamma \\ {\scriptstyle \overline{P}}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle \Sigma_t\overline{P}} \\ \varphi & \longmapsto & \Sigma_t\varphi \end{array} \qquad \begin{array}{ccc} \gamma(x) & \longmapsto & \exists\xi(t\xi{=}y\wedge\gamma(\xi)) \\ {\scriptstyle \overline{P}}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle \Sigma_t\overline{P}} \\ \varphi(x) & \longmapsto & \exists\xi(t\xi{=}y\wedge\varphi(\xi)) \end{array}$$

$$X \xrightarrow{\;t\;} Y$$

If $\overline{P}$ is represented by a derivation $\begin{pmatrix}\gamma(x)\\ P\\ \varphi(x)\end{pmatrix}$

then $\Sigma_t\overline{P}$ i.r. by

$$\left(\dfrac{\exists\xi(t\xi{=}y\wedge\gamma(\xi)) \qquad \dfrac{\dfrac{\begin{array}{c}\dfrac{tx{=}y\wedge\gamma(x)}{[\gamma(x)]}\,(\wedge E)\\ P\\ \varphi(x)\end{array} \qquad \dfrac{tx{=}y\wedge\gamma(x)}{tx{=}y}\,(\wedge E)}{tx{=}y\wedge\varphi(x)}\,(\wedge I)}{\exists\xi(t\xi{=}y\wedge\varphi(\xi))}\,(\exists I)}{\exists\xi(t\xi{=}y\wedge\varphi(\xi))}\,(\exists E)\right)$$

In my notation:

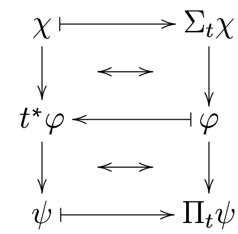
$$\begin{array}{ccc} P & \longmapsto & \Sigma_f P \\ {\scriptstyle g}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle \Sigma_f g} \\ Q & \longmapsto & \Sigma_f Q \end{array} \qquad \begin{array}{ccc} P(x) & \longmapsto & \exists x{:}X.(f(x){=}y\wedge P(x)) \\ {\scriptstyle g}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle \Sigma_f g} \\ Q(x) & \longmapsto & \exists x{:}X.(f(x){=}y\wedge Q(x)) \end{array}$$

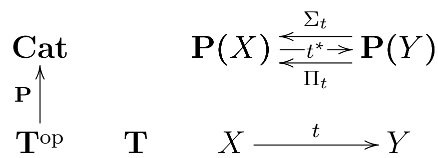
$$X \xrightarrow{\;f\;} Y$$

$$\Sigma_f g = \left(\dfrac{\exists x{:}X.(f(x){=}y\wedge P(x)) \qquad \dfrac{\dfrac{\begin{array}{c}\dfrac{[f(x){=}y\wedge P(x)]^1}{P(x)}\,(\wedge E)\\ \vdots\; g\\ Q(x)\end{array} \qquad \dfrac{[f(x){=}y\wedge P(x)]^1}{f(x){=}y}\,(\wedge E)}{f(x){=}y\wedge Q(x)}\,(\wedge I)}{\exists x{:}X.(f(x){=}y\wedge Q(x))}\,(\exists I)}{\exists x{:}X.(f(x){=}y\wedge Q(x))}\,(\exists E);1\right)$$

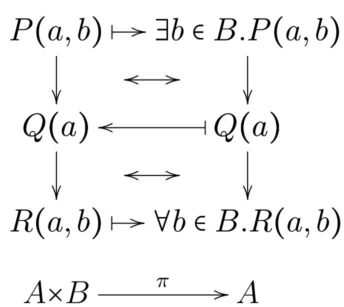
§3. Hyperdoctrines

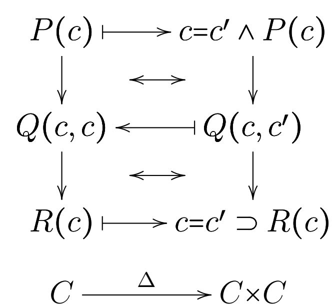
(P.510):

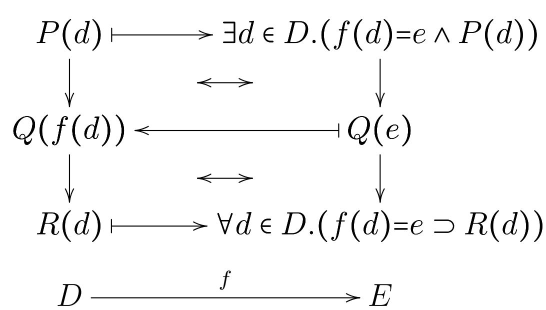




Main cases, in the archetypal model:







The inverse image functor preserves products, coproducts, top, and bottom:

$$\dfrac{\dfrac{\dfrac{\mathrm{Hom}_X(\alpha, t^*(\beta \wedge \gamma))}{\mathrm{Hom}_Y(\Sigma_t\alpha, \beta \wedge \gamma)}\cong}{\mathrm{Hom}_Y(\Sigma_t\alpha, \beta) \times \mathrm{Hom}_Y(\Sigma_t\alpha, \gamma)}\cong}{\dfrac{\mathrm{Hom}_X(\alpha, t^*\beta) \times \mathrm{Hom}_Y(\alpha, t^*\gamma)}{\mathrm{Hom}_X(\alpha, t^*\beta \wedge t^*\gamma)}\cong}\cong \qquad \dfrac{\dfrac{\dfrac{\mathrm{Hom}_X(t^*(\beta \vee \gamma), \delta)}{\mathrm{Hom}_Y(\beta \vee \gamma, \Pi_t\delta)}\cong}{\mathrm{Hom}_Y(\beta, \Pi_t\delta) \times \mathrm{Hom}_Y(\gamma, \Pi_t\delta)}\cong}{\dfrac{\mathrm{Hom}_X(t^*\beta, \delta) \times \mathrm{Hom}_Y(t^*\gamma, \delta)}{\mathrm{Hom}_X(t^*\beta \vee t^*\gamma, \delta)}\cong}\cong$$

$$t^*(\beta \wedge \gamma) \cong t^*\beta \wedge t^*\gamma \qquad t^*(\beta \vee \gamma) \cong t^*\beta \vee t^*\gamma$$

$$\dfrac{\dfrac{\dfrac{\mathrm{Hom}_X(\alpha, t^*\top_Y)}{\mathrm{Hom}_Y(\Sigma_t\alpha, \top_Y)}\cong}{1}\cong}{\mathrm{Hom}_X(\alpha, \top_X)}\cong \qquad \dfrac{\dfrac{\dfrac{\mathrm{Hom}_X(t^*\bot_Y, \delta)}{\mathrm{Hom}_Y(\bot_Y, \Pi_t\delta)}\cong}{1}\cong}{\mathrm{Hom}_X(\bot_X, \delta)}\cong$$

$$t^*\top_Y \cong \top_X \qquad t^*\bot_Y \cong \bot_X$$

3. (5') (ii): Beck-Chevalley

(P.511):

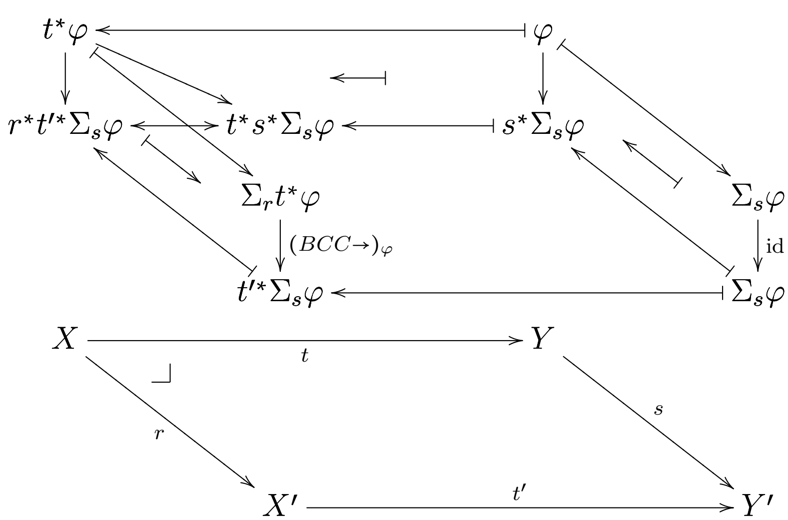
Choose any commuting square in the base category. Name its objects and morphisms like this:

$$\begin{array}{ccc} X & \xrightarrow{t} & Y \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ X' & \xrightarrow{t'} & Y' \end{array}$$

Then for any object $\varphi \in |\mathbf{P}(Y)|$ we can construct a map

$$(BCC \rightarrow)_\varphi : \Sigma_r t^* \varphi \rightarrow t'^* \Sigma_s \varphi.$$

Here's how:



The Beck-Chevalley Condition says that when the base square is a pullback then for any $\varphi \in |\mathbf{P}(Y)|$ the map $(BCC \rightarrow)_\varphi$ is an iso.

3. (4') and (4"): Frobenius

(P.511):

Note that we can replace (4) by:

(4') The "inverse image" functors t^* preserve exponentiation.

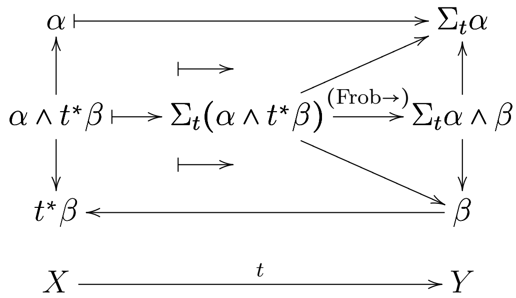
(4") for each $t : X \to Y$ in $\mathbf{T}$, $\varphi \in |\mathbf{P}(X)|$, $\psi \in |\mathbf{P}(Y)|$, the morphism

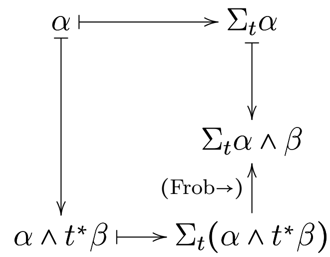
$$(\text{Frob}\to) : \Sigma_t(t^*\psi \wedge \varphi) \to \psi \wedge \Sigma_t\varphi$$

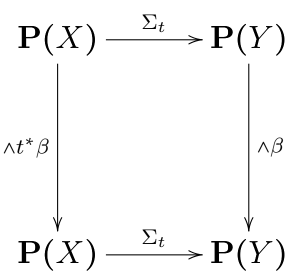
is an isomorphism.

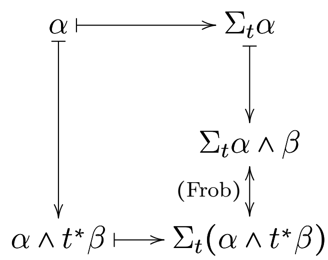
Condition (4') (or (4")) is called *Frobenius Reciprocity*.

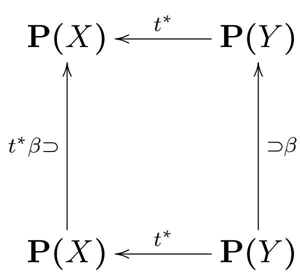
It is possible to prove that (4') implies (4") and vice-versa. The middle diagrams in the next page have the usual high-level proof. The lower diagrams have a lower-level proof.

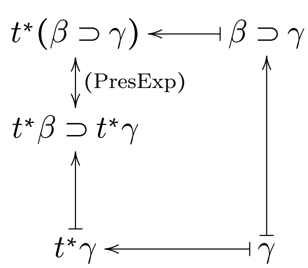

α $\Sigma_t\alpha$
$\alpha \wedge t^*\beta$ $\Sigma_t(\alpha \wedge t^*\beta)$ (Frob→) $\Sigma_t\alpha \wedge \beta$
$t^*\beta$ β
X t Y

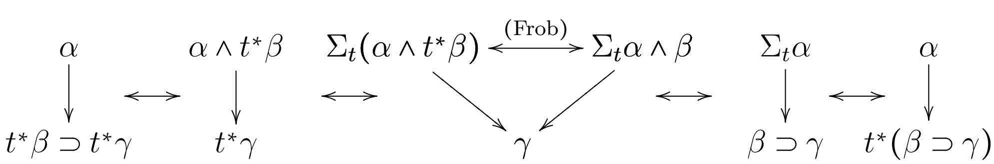

α $\Sigma_t\alpha$
$\Sigma_t\alpha \wedge \beta$
(Frob→)
$\alpha \wedge t^*\beta$ $\Sigma_t(\alpha \wedge t^*\beta)$

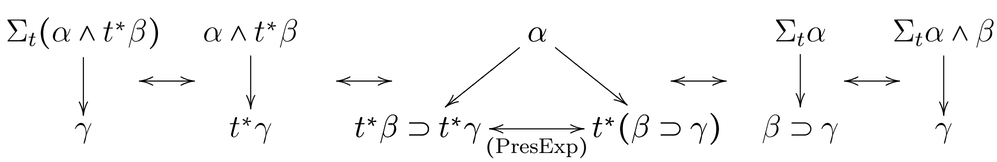

$\mathbf{P}(X)$ Σ_t $\mathbf{P}(Y)$
$\wedge t^*\beta$ $\wedge\beta$
$\mathbf{P}(X)$ Σ_t $\mathbf{P}(Y)$


α $\Sigma_t\alpha$
$\Sigma_t\alpha \wedge \beta$
(Frob)
$\alpha \wedge t^*\beta$ $\Sigma_t(\alpha \wedge t^*\beta)$


$\mathbf{P}(X)$ t^* $\mathbf{P}(Y)$
$t^*\beta\supset$ $\supset\beta$
$\mathbf{P}(X)$ t^* $\mathbf{P}(Y)$


$t^*(\beta \supset \gamma)$ $\beta \supset \gamma$
(PresExp)
$t^*\beta \supset t^*\gamma$
$t^*\gamma$ γ


α $\alpha \wedge t^*\beta$ $\Sigma_t(\alpha \wedge t^*\beta)$ (Frob) $\Sigma_t\alpha \wedge \beta$ $\Sigma_t\alpha$ α
$t^*\beta \supset t^*\gamma$ $t^*\gamma$ γ $\beta \supset \gamma$ $t^*(\beta \supset \gamma)$


$\Sigma_t(\alpha \wedge t^*\beta)$ $\alpha \wedge t^*\beta$ α $\Sigma_t\alpha$ $\Sigma_t\alpha \wedge \beta$
γ $t^*\gamma$ $t^*\beta \supset t^*\gamma$ (PresExp) $t^*(\beta \supset \gamma)$ $\beta \supset \gamma$ γ

§5. Construction 2: Hyperdoctrine → LPCE

(P.523):

We must now make sure that all this is justified. (...) We must check that the introduction and eleimination rules for ∃ and ∀ are satisfied: for example, $(\exists E)$ becomes the assertation...

$$\begin{array}{ccc} \varphi & \longmapsto & \Sigma_{\pi_Y}\varphi \\ {\scriptstyle\alpha}\downarrow & \overset{(\exists E)}{\longmapsto} & \downarrow{\scriptstyle\beta} \\ \pi_Y^*\varphi' & \longleftarrow\!\shortmid & \varphi' \end{array} \qquad \begin{array}{ccc} P(x,y) & \longmapsto & \exists x{:}X.P(x,y) \\ {\scriptstyle\alpha}\downarrow & \overset{(\exists E)}{\longmapsto} & \downarrow{\scriptstyle\beta} \\ Q(y) & \longleftarrow\!\shortmid & Q(y) \end{array} \qquad \dfrac{\exists x{:}X.P(x,y) \qquad \begin{array}{c}[P(x,y)]^1 \\ \vdots\ \alpha \\ Q(y)\end{array}}{Q(y)}\ (\exists E);1$$

$$X \times Y \xrightarrow{\pi_Y} Y \qquad\qquad X \times Y \xrightarrow{\quad\pi_Y\quad} Y$$

$$\begin{array}{ccc} \varphi & \longmapsto & \Sigma_{\pi_Y}\varphi \\ {\scriptstyle\eta???}\downarrow & \overset{(\exists I)}{\longleftarrow\!\shortmid} & \downarrow{\scriptstyle\mathrm{id}} \\ \pi_Y^*\Sigma_{\pi_Y}\varphi & \longleftarrow\!\shortmid & \Sigma_{\pi_Y}\varphi \end{array} \qquad \begin{array}{ccc} P(x,y) & \longmapsto & \exists x{:}X.P(x,y) \\ {\scriptstyle\eta???}\downarrow & \overset{(\exists I)}{\longleftarrow\!\shortmid} & \downarrow{\scriptstyle\mathrm{id}} \\ \exists x{:}X.P(x,y) & \longleftarrow\!\shortmid & \exists x{:}X.P(x,y) \end{array} \qquad \dfrac{P(x,y)}{\exists x{:}X.P(x,y)}\ (\exists I)$$

$$X \times Y \xrightarrow{\pi_Y} Y \qquad\qquad X \times Y \xrightarrow{\quad\pi_Y\quad} Y$$

$$\begin{array}{ccccc} t^*\varphi & \longleftarrow\!\shortmid & \varphi & \longmapsto & \Sigma_{\pi_Y}\varphi \\ \downarrow & \longleftarrow\!\shortmid & \downarrow & \longleftarrow\!\shortmid & \downarrow{\scriptstyle\mathrm{id}} \\ \pi_t^*\pi_Y^*\Sigma_{\pi_Y}\varphi & \longleftarrow\!\shortmid & \pi_Y^*\Sigma_{\pi_Y}\varphi & \longleftarrow\!\shortmid & \Sigma_{\pi_Y}\varphi \\ \updownarrow & & & & \\ {\pi'_Y}^*\Sigma_{\pi_Y}\varphi & & & & \end{array} \qquad \begin{array}{ccccc} P(t'(z),y) & \longleftarrow\!\shortmid & P(x,y) & \longmapsto & \exists x{:}X.P(x,y) \\ \downarrow & \longleftarrow\!\shortmid & \downarrow & \longleftarrow\!\shortmid & \downarrow{\scriptstyle\mathrm{id}} \\ \exists x{:}X.P(x,y) & \longleftarrow\!\shortmid & \exists x{:}X.P(x,y) & \longleftarrow\!\shortmid & \exists x{:}X.P(x,y) \\ \updownarrow & & & & \\ \exists x{:}X.P(x,y) & & & & \end{array}$$

$$Z \times Y \xrightarrow{t} X \times Y \xrightarrow{\pi_Y} Y, \quad Z \times Y \xrightarrow{\pi'_Y} Y \qquad\qquad Z \times Y \xrightarrow{t} X \times Y \xrightarrow{\pi_Y} Y, \quad Z \times Y \xrightarrow{\pi'_Y} Y$$

$$\dfrac{P(t'(z),y)}{\exists x{:}X.P(x,y)}\ (\exists I)$$

(P.523, equality rules):

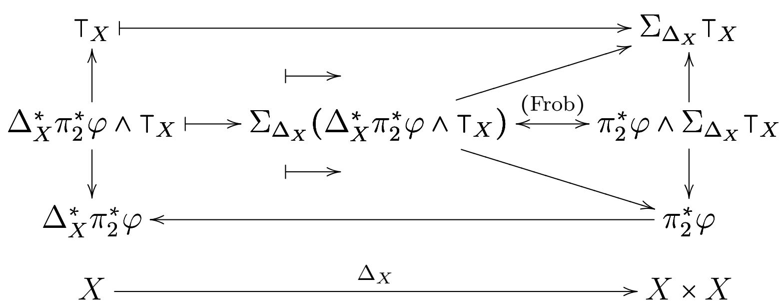
The equality rules follow as immediate consequences of our definition of E_X as $\Sigma_{\Delta_X}\top_X$: (R) asserts the existence of a morphism $\top_X \to \Delta_X^*\Sigma_{\Delta_X}\top_X$, which is the unit...

$$
\begin{array}{ccc}
\top_X & \longmapsto & \Sigma_{\Delta_X}\top_X \\
{\scriptstyle (R)}\big\downarrow & \longleftarrow\!\!\mid & \big\downarrow{\scriptstyle \mathrm{id}} \\
\Delta_X^*\Sigma_{\Delta_X}\top_X & \leftarrow\!\mid & \Sigma_{\Delta_X}\top_X \\
 & & \\
X & \xrightarrow{\Delta_X} & X \times X
\end{array}
\qquad
\begin{array}{ccc}
\top & \longmapsto & x{=}x' \\
{\scriptstyle (R)}\big\downarrow & \longleftarrow\!\!\mid & \big\downarrow{\scriptstyle \mathrm{id}} \\
x{=}x & \longleftarrow\!\!\mid & x{=}x' \\
 & & \\
X & \xrightarrow{\Delta_X} & X \times X
\end{array}
$$

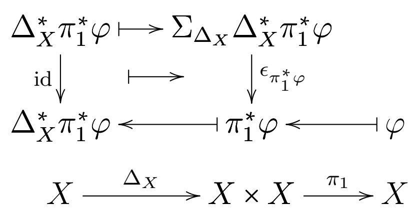
For (sub), note that we have a morphism:

$$\pi_2^*\varphi \wedge \Sigma_{\Delta_X}\top_X \xleftrightarrow{\text{(Frob)}} \Sigma_{\Delta_X}(\Delta_X^*\pi_2^*\varphi \wedge \top_X) \longleftrightarrow \Sigma_{\Delta_X}\varphi \longleftrightarrow \Sigma_{\Delta_X}\Delta_X^*\pi_1^*\varphi \xrightarrow{\epsilon_{\pi_1^*\varphi}} \pi_1^*\varphi$$
$$P(x') \wedge x{=}x' \xleftrightarrow{\text{(Frob)}} x{=}x' \wedge (P(x) \wedge \top) \longleftrightarrow x{=}x' \wedge P(x) \longleftrightarrow x{=}x' \wedge P(x) \xrightarrow{\epsilon_{\pi_1^*\varphi}} P(x)$$

Let's see how to build its parts...



$$\dfrac{\dfrac{\dfrac{}{\Delta_X^*\pi_2^*\varphi \wedge \top \leftrightarrow \Delta_X^*\pi_2^*\varphi} \qquad \dfrac{\dfrac{\dfrac{}{\Delta_X;\pi_2 = \mathrm{id}}}{\Delta_X^*\pi_2^* \cong \mathrm{id}^*} \quad \varphi}{\Delta_X^*\pi_2^*\varphi \leftrightarrow \varphi}}{\Delta_X^*\pi_2^*\varphi \wedge \top \leftrightarrow \varphi}}{\Sigma_{\Delta_X}(\Delta_X^*\pi_2^*\varphi \wedge \top) \leftrightarrow \Sigma_{\Delta_X}\varphi} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{id} = \Delta_X;\pi_1}}{\mathrm{id}^* = \Delta_X^*\pi_1^*} \quad \varphi}{\varphi \leftrightarrow \Delta_X^*\pi_1^*}}{\Sigma_{\Delta_X}\varphi \leftrightarrow \Sigma_{\Delta_X}\Delta_X^*\pi_1^*}$$



The actual form of (sub) can be easily derived from this.

Thanks

...to (in alphabetical order): Ana Luiza Tenório, Caio Mendes, Daniel Almeida and Victor Nascimento for lots of useful questions and comments.

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