Grothendieck Topologies for Children

Eduardo Ochs
http://angg.twu.net/math-b.html#2021-groth-tops

Abstract (1)

The notes in

http://angg.twu.net/math-b.html#favorite-conventions

— I'll refer to them as "[FavC]" from here on — define an extensible diagrammatic language that lets us take complex definitions in Category Theory and then complement them with several kinds of diagrams to lower the level of complexity and abstraction of the original definition. What we usually get after adding these diagrams is the original definition (very abstract, "for adults") drawn side to side with diagrams for particular cases ("for children"), in two parallel diagrams with the same shape; see the introduction of [FavC] for several different overviews of the method, and for several different attempts to define "children" in a useful way.

Abstract (2)

The definition of a Grothendieck topology is quite hard to understand — I found it *impossible* for many years — and in this presentation I will show how the extend the diagrammatic language from [FavC] to handle that. Most of the material that I will present is in

`http://angg.twu.net/LATEX/2021groth-tops-children.pdf`,

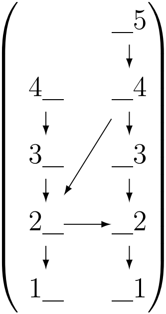
but I need to confess that this is an early draft that I need to rewrite as soon as possible.

The presentation will be in Portuguese, with slides in English.

What we will need:

1) The order topologies/ZHAs generated by 2-column graphs from [PH1, sections 14–17],

$D = (P, A) =$



$\mathcal{O}(D) = \mathcal{O}_A(P) =$

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    45
  44  35
43  34  25
42  33  24
  32  23
    22  13
      12  03
    11  02
  10  01
    00
```

What we will need (2)

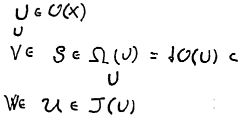
2) This extension to the notations in [PH1]:

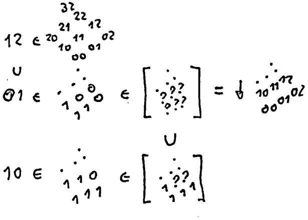
$$
\begin{aligned}
\left[\begin{smallmatrix} & 0 & & & \\ & & 0 & & \\ & 0 & & ? & \\ 0 & & ? & & 1 \\ & 1 & \cdot & 1 & \\ & & 1 & & \end{smallmatrix}\right]
&= \{\, U \in \mathcal{O}(B) \mid U \text{ is of the form } \begin{smallmatrix} & 0 & & & \\ & & 0 & & \\ & 0 & & ? & \\ 0 & & ? & & 1 \\ & 1 & \cdot & 1 & \\ & & 1 & & \end{smallmatrix} \,\} \\
&= \{\, U \in \mathcal{O}(B) \mid \begin{smallmatrix} & 0 & & & \\ & & 0 & & \\ & 0 & & 0 & \\ 0 & & 0 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix} \subseteq U \subseteq \begin{smallmatrix} & 0 & & & \\ & & 0 & & \\ & 0 & & 1 & \\ 0 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix} \,\} \\
&= \left\{ \begin{smallmatrix} & 0 & & & \\ & & 0 & & \\ & 0 & & 0 & \\ 0 & & 0 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix},\ \begin{smallmatrix} & 0 & & & \\ & & 0 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix},\ \begin{smallmatrix} & 0 & & & \\ & & 0 & & \\ & 0 & & 1 & \\ 0 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix} \right\}
\end{aligned}
$$

$$
\begin{aligned}
\left[\begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & \cdot & & ? & \\ \cdot & & ? & & 1 \\ & 1 & \cdot & 1 & \\ & & 1 & & \end{smallmatrix}\right]
&= \{\, U \in \mathcal{O}(12) \mid U \text{ is of the form } \begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & \cdot & & ? & \\ \cdot & & ? & & 1 \\ & 1 & \cdot & 1 & \\ & & 1 & & \end{smallmatrix} \,\} \\
&= \{\, U \in \mathcal{O}(12) \mid \begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & \cdot & & 0 & \\ \cdot & & 0 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix} \subseteq U \subseteq \begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & \cdot & & 1 & \\ \cdot & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix} \,\} \\
&= \left\{ \begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & \cdot & & 0 & \\ \cdot & & 0 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix},\ \begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & \cdot & & 0 & \\ \cdot & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix},\ \begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & \cdot & & 1 & \\ \cdot & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix} \right\} \\
&= \{\, {\downarrow}\{10, 02\},\ {\downarrow}\{11, 02\},\ {\downarrow}12 \,\}
\end{aligned}
$$

What we will need (3)

Several conventions from the beginning of [FavC],
the convention on "functors as objects" from [FavC, sec.7.12],
and a new conventions for drawing diagrams of *names*,
pronounciations and *notations*.





Quotient topologies

Consider this partition of $\mathbb{R}$:

$$P \;=\; \{\underbrace{(-\infty,1)}_{A}, \underbrace{[1,2)}_{B}, \underbrace{[2,3]}_{C}, \underbrace{(3,4]}_{D}, \underbrace{(4,+\infty)}_{E}\}$$

We will say that a subset $U \subseteq \mathbb{R}$ *respects* P iff for every $I \in P$ either $I \subset U$ or $I \cap U = \emptyset$.
For example, $B \cup D = [1,2) \cup (3,4]$ respects P, but $[0.5, 2.34]$ does not.
Let:

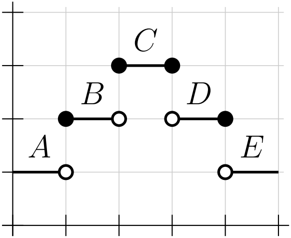
$$\begin{array}{rcl} \mathcal{P}_P(\mathbb{R}) & = & \{\, U \in \mathcal{P}(\mathbb{R}) \mid U \text{ respects } P \,\} \\ \mathcal{O}_P(\mathbb{R}) & = & \{\, U \in \mathcal{O}(\mathbb{R}) \mid U \text{ respects } P \,\} \end{array}$$

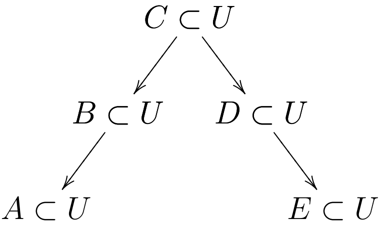
Then $\mathcal{P}_P(\mathbb{R})$ has $2^5 = 32$ elements, and $\mathcal{O}_P \subset \mathcal{P}_P(\mathbb{R})$.

Quotient topologies (2)

Here is another way to draw P and the conditions that an $U \in \mathcal{P}_P(\mathbb{R})$ must obey to obey $U \in \mathcal{O}_P(\mathbb{R})$:

$$P \quad = \quad \{\underbrace{(-\infty,1)}_{A}, \underbrace{[1,2)}_{B}, \underbrace{[2,3]}_{C}, \underbrace{(3,4]}_{D}, \underbrace{(4,+\infty)}_{E}\}$$





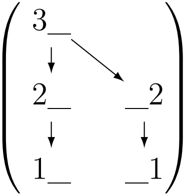
Quotient topologies (3)

Here are the 10 elements of $\mathcal{O}_P(\mathbb{R})$:

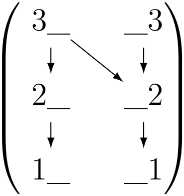
$$\begin{array}{ccccccccc}
 & & ABCDE & & & & & & \\
 & & & & ABDE & & & & \\
 & & ABD & & & & ADE & & \\
AB & & & & AE & & & & DE \\
 & & A & & & & E & & \\
 & & & & \emptyset & & & &
\end{array}$$

2-column graphs and their order topologies

...or: 2CGs and ZHAs

$X = H =$ 

$$\mathcal{O}(X) = \mathcal{O}(H) = \begin{array}{ccccc} & 32 & & & \\ & & 22 & & \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{array}$$

$D = N =$ 

$$\mathsf{D}(D) = \mathsf{D}(N) = \begin{array}{cccccc} & & 33 & & & \\ & 32 & & 23 & & \\ & & 22 & & 13 & \\ & 21 & & 12 & & 03 \\ 20 & & 11 & & 02 & \\ & 10 & & 01 & & \\ & & 00 & & & \end{array}$$

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45 36

44 35 26

43 34 25 16

42 33 24 15 06

41 32 23 14 05

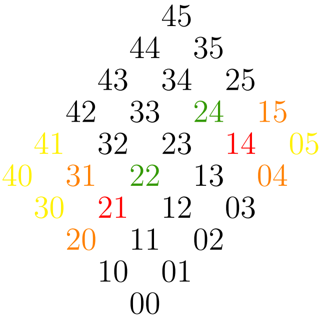
40 31 22 13 04

30 21 12 03

20 11 02

10 01

00


45
44 35
43 34 25
42 33 24 15
41 32 23 14 05
40 31 22 13 04
30 21 12 03
20 11 02
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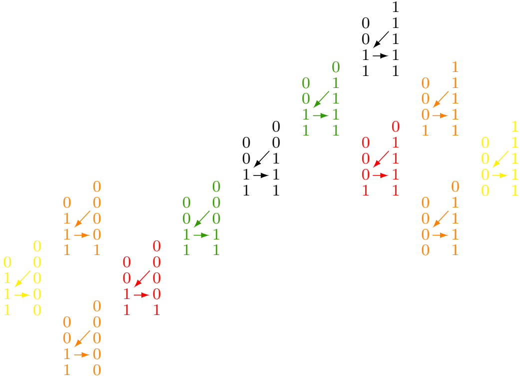

References

[FavC] E. Ochs. "On my favorite conventions for drawing the missing diagrams in Category Theory". http://angg.twu.net/math-b.html#favorite-conventions. 2020.

[PH1] E. Ochs. "Planar Heyting Algebras for Children". In: *South American Journal of Logic* 5.1 (2019). http://angg.twu.net/math-b.html#zhas-for-children-2, pp. 125–164.