Grothendieck Topologies for Children

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Abstract

The paper "Planar Heyting Algebras for Children" ([PH1]) showed how to use Planar Heyting Algebras to visualize the truth-values and the operations of Propositional Calculus in certain toposes; the "...for children" of its title means: "we will start from some motivating examples ('for children') that are easy to visualize, and then go the general case ('for adults') — *but there are precise techniques for working on the case 'for children' and on the case 'for adults' in parallel*". These techniques are described in detail in [FavC]; see also sec.15.

In these notes we will use these techniques to visualize Grothendieck topologies — first in the "archetypal" case of the canonical Grothendieck topology on a certain finite topological space, and then we will generalize that to arbitrary Grothendieck topologies on certain finite posets, that we will treat as "ex-topologies" (sec.11).

This is a working draft! It is still messy and incomplete at many points! I want/need to rewrite several sections of it and reorder everything!

The latest version is at:
`http://angg.twu.net/math-b.html#2021-groth-tops`

Contents

1 Introduction

One of the key ideas for understanding sheaves is *Grothendieck topologies*. They are defined by a long zig-zag of steps in which the 'zig's are like "take this familiar construction", the 'zag's are like "here is the *right* generalization of the previous step", and the reasons for these choices of generalizations only become clear many steps afterwards — when we define sheaves in several different ways. We will use some of the conventions in [FavC] to compare these different definitions of sheaves — mainly these three conventions, from [FavC, sec.2]:

(CPSh) A particular case of a diagram D is drawn with the same shape as D.

(CNSh) A translation of a diagram D to another notation is drawn with the same shape as D.

(CFSh) The image by a functor of a diagram D is drawn with the same shape as D.

Here is one example of a shape that we will use often. Start with a topology $\mathcal{O}(X)$. The topos $\mathbf{Set}^{\mathcal{O}(X)^{\mathrm{op}}}$ has objects Ω and J_{can}, with an inclusion

$J_{\mathrm{can}} \hookrightarrow \Omega$. For each open set $U \in \mathcal{O}(X)$ the set $\Omega(U)$ is the set of "sieves on U", and the subset $J_{\mathrm{can}}(U) \subset \Omega(U)$ is the set of "covering sieves on U". The elements of the set $J_{\mathrm{can}}(U)$ are called "covering sieves (on U)", and denoted by letters like $\mathcal{U}$; each covering sieve is a set of open sets, i.e., of elements of $\mathcal{O}(X)$, and they are denoted by letters like V. We will draw all — or, more honestly, most of — this information as:

$$\begin{array}{ccccccccc}
\mathcal{O}(X) \\
\cup \\
V & \in & \mathcal{U} & \in & J_{\mathrm{can}}(U) & \subset & \Omega(U) \\
\\
\begin{pmatrix}\text{our}\\\text{topology}\end{pmatrix} \\
\cup \\
\begin{pmatrix}\text{an}\\\text{open}\\\text{set}\end{pmatrix} & \in & \begin{pmatrix}\text{a}\\\text{covering}\\\text{sieve}\end{pmatrix} & \in & \begin{pmatrix}\text{all}\\\text{covering}\\\text{sieves}\\\text{on } U\end{pmatrix} & \in & \begin{pmatrix}\text{all}\\\text{sieves}\\\text{on } U\end{pmatrix}
\end{array}$$

In Section 11 we will generalize this to:

$$\begin{array}{ccccccccc}
D \\
\cup \\
v & \in & \mathcal{U} & \in & J(u) & \subset & \Omega(u) \\
\\
\begin{pmatrix}\text{our ex-}\\\text{topology}\end{pmatrix} \\
\cup \\
\begin{pmatrix}\text{an ex-}\\\text{open}\\\text{set}\end{pmatrix} & \in & \begin{pmatrix}\text{a}\\J\text{-covering}\\\text{sieve}\end{pmatrix} & \in & \begin{pmatrix}\text{all}\\J\text{-covering}\\\text{sieves}\\\text{on } u\end{pmatrix} & \in & \begin{pmatrix}\text{all}\\\text{sieves}\\\text{on } u\end{pmatrix}
\end{array}$$

We will also use this shape to compare our notational conventions with the ones in [LM92] and [Lin14], and to show our conventions for drawing particular cases.

Grothendieck Topologies are not only hard to *define*. They are also very hard to *visualize*, even when we start with finite topologies like the Planar Heyting Algebras of [PH1], that *ought to* yield nice archetypal cases — in the sense of [IDARCT, section 16]; see also [Che04]. Let me sketch why.

Take a topological space $(X, \mathcal{O}(X))$. If we follow the ideas in [PH1] this $\mathcal{O}(X)$ will be a planar poset (I'll refrain from mentioning lattices at this point!), with the empty set $\emptyset$ as its bottom element, and the X at its top. Choose an open set $U \in \mathcal{O}(X)$; $\mathcal{O}(U)$ will be a sub-poset of $\mathcal{O}(X)$. A subset $\mathcal{S} \subseteq \mathcal{O}(U)$ is a *sieve on U* when it is closed downwards; if we write $\mathsf{D}(\mathcal{O}(U))$ for

$$\mathsf{D}(\mathcal{O}(U)) \quad = \quad \{\, \mathcal{S} \subseteq \mathcal{O}(U) \mid S \text{ is closed downwards} \,\}$$

then the set of sieves on U is exactly $\mathsf{D}(\mathcal{O}(U))$.

We say that a sieve $\mathcal{S} \in \mathsf{D}(\mathcal{O}(U))$ *covers* U when $\bigcup \mathcal{S} = U$. The set of covering sieves on U contains the top element of $\mathsf{D}(\mathcal{O}(U))$, and is closed upwards and by finite intersections — so the set of covering sieves on U is a filter on $\mathsf{D}(\mathcal{O}(U))$, and if we write $\mathsf{F}(\mathbf{P})$ for the set of filters on a poset $\mathbf{P}$ then the set of covering sieves on U is a element of $\mathsf{F}(\mathsf{D}(\mathcal{O}(U))$. The *canonical Grothendieck topology* on a topological space $(X, \mathcal{O}(X))$ is an operation J_{can} that chooses for each $U \in \mathcal{O}(X)$ a filter $J_{\text{can}}(U) \in \mathsf{F}(\mathsf{D}(\mathcal{O}(U)))$, and to define Grothendieck topologies in general we need to understand some properties of this operation J_{can}, and the generalize them *in the right way*.

We will discuss ways to draw posets, topologies, down-sets, and filters in sections 4 and 5. In section 6 we will see how to draw our first Grothendieck topology, and in sections 7 and 10 we will show how to define Grothendieck topologies.

2 The hierarchy

The definition of a Grothendieck topology on a topological space $(X, \mathcal{O}(X))$ uses elements and subsets a lot. It also uses lots of linguistic constructs like "a J-covering sieve on U is...", and lots of notational conventions on the default meanings and the default types of some letters and fonts. We need to extend the conventions in [FavC] a bit to handle that; here is a first attempt. Let's take this (incomplete) version on the definition:

- Fix a topological space $(X, \mathcal{O}(X))$.
- Take any element $U \in \mathcal{O}(X)$. This U is an open set.
- The set $\Omega(U)$ is the set of *sieves* on U, defined as $\Omega(U) = \mathsf{D}(\mathcal{O}(U))$; i.e., a sieve $\mathcal{S}$ on U is a down-set $\mathcal{S} \in \mathsf{D}(\mathcal{O}(U)) = \Omega(U)$.
- The set $J(U) \subset \Omega(U)$ is the set of J-covering sieves on U. We say that a sieve $\mathcal{U}$ on U is *J-covering* when $\mathcal{U} \in J(U)$.
- Every element V of a sieve $\mathcal{S}$ is an open set of X; every element V of a covering sieve $\mathcal{U}$ is an open set of X.

We can organize this in two parallel diagrams, as:

$$
\begin{array}{ccccccccc}
U & \in & \mathcal{O}(X) \\
 & & \cup \\
V & \in & \mathcal{S} & \in & \Omega(U) & = & \mathsf{D}(\mathcal{O}(U)) & \subset & \mathcal{P}(\mathcal{O}(U)) \\
 & & & & \cup \\
V & \in & \mathcal{U} & \in & J(U)
\end{array}
$$

$$
\begin{array}{ccccccccc}
\begin{pmatrix}\text{an}\\ \text{open}\\ \text{set}\end{pmatrix} & \in & \begin{pmatrix}\text{our}\\ \text{topology}\end{pmatrix} \\
 & & \cup \\
\begin{pmatrix}\text{an}\\ \text{open}\\ \text{set}\end{pmatrix} & \in & \begin{pmatrix}\text{a}\\ \text{sieve}\end{pmatrix} & \in & \begin{pmatrix}\text{all}\\ \text{sieves}\\ \text{on } U\end{pmatrix} & = & \begin{pmatrix}\text{all down-}\\ \text{sets of}\\ \mathcal{O}(U)\end{pmatrix} & \subset & \begin{pmatrix}\text{all}\\ \text{subsets}\\ \text{of } \mathcal{O}(U)\end{pmatrix} \\
 & & & & \cup \\
\begin{pmatrix}\text{an}\\ \text{open}\\ \text{set}\end{pmatrix} & \in & \begin{pmatrix}\text{a}\\ \text{covering}\\ \text{sieve}\end{pmatrix} & \in & \begin{pmatrix}\text{all } J\text{-}\\ \text{covering}\\ \text{sieves}\\ \text{on } U\end{pmatrix}
\end{array}
$$

I *guess* that some experimental extensions to some proof assistants like Agda — that supports unicode characters in symbol names — may have ways to attribute default types for some names, but at this moment I don't even know where to look for. *All hints and pointers are welcome!*

The diagrams above shows a certain hierarchy between our symbols: going right or going upwards in practically all cases means moving to something bigger. We won't use diagrams with that shape much — instead we will use the smaller version below:

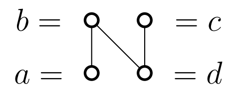
$$\begin{array}{ccccccccc}
\mathcal{O}(X) \\
\cup \\
V & \in & \mathcal{U} & \in & J(U) & \subset & \Omega(U) \\
\\
\left(\begin{array}{c}\text{our}\\\text{topology}\end{array}\right) \\
\cup \\
\left(\begin{array}{c}\text{an}\\\text{open}\\\text{set}\end{array}\right) & \in & \left(\begin{array}{c}\text{a}\\\text{covering}\\\text{sieve}\end{array}\right) & \in & \left(\begin{array}{c}\text{all } J\text{-}\\\text{covering}\\\text{sieves}\\\text{on } U\end{array}\right) & \in & \left(\begin{array}{c}\text{all}\\\text{sieves}\\\text{on } U\end{array}\right)
\end{array}$$

We will call that shape "the hierarchy", and will often use it to show examples of particular cases, by also drawing a copy in which the $\mathcal{O}(X)$ is replaced by a particular topology, U is replaced by a particular open set in $\mathcal{O}(X)$, and so on.

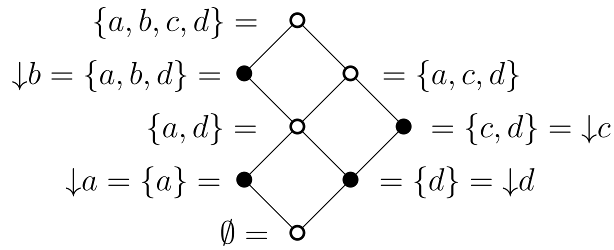
3 Davey and Priestley

The book [DP02] is a standard reference on lattices; I guess that its way of drawing lattices should be considered standard, too. On pages 20–21 it defines and draws *the ordered set* $\mathcal{O}(P)$ *of down-sets of a poset* P like this:

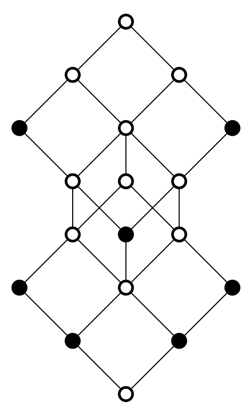
$N \quad =$



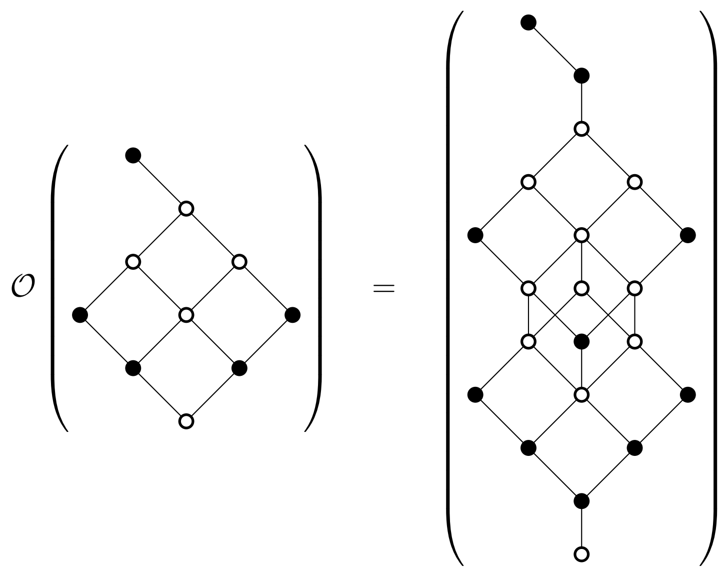
$\mathcal{O}(N) \quad =$



The book usually draws the "join-irreducible" elements of lattices using "shaded dots"; see its page 54. Here is an example from the book:



And here is a bigger example:



We will pronounce "$\mathcal{O}(P)$" as "the order topology on the poset P".

4 Order topologies

Most of the material in [PH1] was tested on "real children", as explained in this part of its introduction:

> This paper can be seen as part of bigger projects in at least the two ways described above, but it was also written to be as readable and as self-contained as possible. In 2016 and 2017 I had the opportunity to test some of the ideas here on "real children", in the sense of "people with little mathematical *knowledge* and little mathetical *maturity*". I gave a seminar course about Logic and λ-calculus that had no prerequisites, and that was mostly based on exercises that the students would try to solve together by discussing on the whiteboard; it was mostly attended by Computer Science students who had just finished a course on Discrete Mathematics (...)

They found the notation for order topologies below much easier to start with.

In section 13 of [PH1] we defined H as the “house” poset at the left here (see also section 1 for the “reading order” and section 2 for the “black pawns moves”):

$$
H = \begin{tikzcd}[column sep=small]
& 1 \arrow[dl] \arrow[dr] & \\
2 \arrow[d] & & 3 \arrow[d] \\
4 & & 5
\end{tikzcd}
\qquad
(\mathcal{O}(H), \subset_1) =
\begin{tikzcd}[column sep=tiny, row sep=small]
& {\begin{smallmatrix} &1& \\ 1&&1 \\ 1&&1 \end{smallmatrix}} & & & \\
& & {\begin{smallmatrix} &0& \\ 1&&1 \\ 1&&1 \end{smallmatrix}} \arrow[ul] & & \\
& {\begin{smallmatrix} &0& \\ 1&&0 \\ 1&&1 \end{smallmatrix}} \arrow[ur] & & {\begin{smallmatrix} &0& \\ 0&&1 \\ 1&&1 \end{smallmatrix}} \arrow[ul] & \\
{\begin{smallmatrix} &0& \\ 1&&0 \\ 1&&0 \end{smallmatrix}} \arrow[ur] & & {\begin{smallmatrix} &0& \\ 0&&0 \\ 1&&1 \end{smallmatrix}} \arrow[ul] \arrow[ur] & & {\begin{smallmatrix} &0& \\ 0&&1 \\ 0&&1 \end{smallmatrix}} \arrow[ul] \\
& {\begin{smallmatrix} &0& \\ 0&&0 \\ 1&&0 \end{smallmatrix}} \arrow[ul] \arrow[ur] & & {\begin{smallmatrix} &0& \\ 0&&0 \\ 0&&1 \end{smallmatrix}} \arrow[ul] \arrow[ur] & \\
& & {\begin{smallmatrix} &0& \\ 0&&0 \\ 0&&0 \end{smallmatrix}} \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

Here we will prefer this slanted version of the house DAG, that is a 2-column graph ([PH1, section 14]):

$$
H = \left(\begin{tikzcd}[column sep=small, row sep=small]
3\_ \arrow[d] \arrow[dr] & \\
2\_ \arrow[d] & \_2 \arrow[d] \\
1\_ & \_1
\end{tikzcd} \right)
\qquad
(\mathcal{O}(H), \subset_1) =
\begin{tikzcd}[column sep=tiny, row sep=tiny]
& 32 & & & \\
& & 22 \arrow[ul] & & \\
& 21 \arrow[ur] & & 12 \arrow[ul] & \\
20 \arrow[ur] & & 11 \arrow[ul] \arrow[ur] & & 02 \arrow[ul] \\
& 10 \arrow[ul] \arrow[ur] & & 01 \arrow[ul] \arrow[ur] & \\
& & 00 \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

but we will sometimes use the unslanted for drawing subsets of H: we will draw

$$
\begin{smallmatrix} &0& \\ 1&&0 \\ 1&&1 \end{smallmatrix} \qquad \text{instead of:} \qquad \begin{tikzcd}[column sep=tiny, row sep=0pt] 0 \arrow[dr] & \\ 1 & 0 \\ 1 & 1 \end{tikzcd} \; .
$$

The ZHA with 10 elements at the right is the “bottle” poset, and we will call it B. So:

$$\mathcal{O}(H) \quad = \quad B.$$

5 Down-sets, up-sets, and filters

We will define "down-sets", "up-sets", and "filters" on posets *almost* in the standard way, i.e., as in [DP02]; but we need to take into account that it H the arrows go down, and in B they go up. So: if $\mathbf{P}$ is a poset with underlying set $\mathbf{P}_0$,

- A subset $D \subseteq \mathbf{P}_0$ is a *down-set* of $\mathbf{P}$ iff

$$\forall p, q \in \mathbf{P}_0.\ (p \text{ above } q) \to \begin{pmatrix} p \in D \\ \downarrow \\ q \in D \end{pmatrix},$$

- A subset $U \subseteq \mathbf{P}_0$ is an *up-set* of $\mathbf{P}$ iff

$$\forall p, q \in \mathbf{P}_0.\ (p \text{ above } q) \to \begin{pmatrix} p \in U \\ \uparrow \\ q \in U \end{pmatrix},$$

- If $\mathbf{P}$ has a top element $\top$ and a binary meet operation '$\wedge$', then a subset $F \subseteq \mathbf{P}_0$ is a *filter* iff: 1) F is an up-set of $\mathbf{P}$, 2) $\top \in F$, 3)

$$\forall p, q \in \mathbf{P}_0.\ \begin{pmatrix} p \in F \text{ and } q \in F \\ \downarrow \\ p \wedge q \in F \end{pmatrix}.$$

We will define $\mathsf{D}(\mathbf{P})$, $\mathsf{U}(\mathbf{P})$, and $\mathsf{F}(\mathbf{P})$ as:

$$\begin{array}{rcl} \mathsf{D}(\mathbf{P}) & = & \{\, D \subseteq \mathbf{P}_0 \mid D \text{ is a down-set of } \mathbf{P} \,\}, \\ \mathsf{U}(\mathbf{P}) & = & \{\, U \subseteq \mathbf{P}_0 \mid U \text{ is an up-set of } \mathbf{P} \,\}, \\ \mathsf{F}(\mathbf{P}) & = & \{\, F \subseteq \mathbf{P}_0 \mid U \text{ is a filter in } \mathbf{P} \,\}, \end{array}$$

and if $S \in \mathbf{P}_0$ we will denote the down-set of $\mathbf{P}$ generated by S as $\downarrow S$ or $\downarrow_{\mathbf{P}} S$, and the up-set of $\mathbf{P}$ generated by S as $\uparrow S$ or $\uparrow_{\mathbf{P}} S$. When $p \in \mathbf{P}_0$ we will sometimes write $\downarrow\{p\}$ as just $\downarrow p$, and $\uparrow\{p\}$ as just $\uparrow p$.

Here are some examples:

$$\begin{smallmatrix} & 0 & \\ 0 & & 0 \\ 1 & & 1 \end{smallmatrix} \in \mathsf{D}(H) \quad \text{but} \quad \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ 1 & & 1 \end{smallmatrix} \notin \mathsf{U}(H),$$

$$\begin{smallmatrix} & 1 & \\ 0 & & 1 \\ 0 & & 0 \end{smallmatrix} \notin \mathsf{D}(H) \quad \text{but} \quad \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ 0 & & 0 \end{smallmatrix} \in \mathsf{U}(H),$$

$$\downarrow \begin{smallmatrix} & 0 & \\ 1 & & 1 \\ 0 & & 0 \end{smallmatrix} = \begin{smallmatrix} & 0 & \\ 1 & & 1 \\ 1 & & 1 \end{smallmatrix}, \qquad \uparrow \begin{smallmatrix} & 0 & \\ 1 & & 1 \\ 0 & & 0 \end{smallmatrix} = \begin{smallmatrix} & 1 & \\ 1 & & 1 \\ 0 & & 0 \end{smallmatrix},$$

$$\begin{smallmatrix} & 1 & & & \\ & & 1 & & \\ & 1 & & 1 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{smallmatrix} \in \mathsf{U}(B) \quad \text{but} \quad \begin{smallmatrix} & 1 & & & \\ & & 1 & & \\ & 1 & & 1 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{smallmatrix} \notin \mathsf{F}(B),$$

$$\begin{smallmatrix} & 1 & & & \\ & & 1 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{smallmatrix} \in \mathsf{F}(B).$$

$$\downarrow_H 2\_ = \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ 1 & & 0 \end{smallmatrix} = 20,$$

$$\downarrow_B \downarrow_H 2\_ = \downarrow_B 20 = \{20, 10, 00\} = \begin{smallmatrix} & 0 & & & \\ & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{smallmatrix},$$

$$20 \cap 11 = \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ 1 & & 0 \end{smallmatrix} \cap \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ 1 & & 1 \end{smallmatrix} = \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ 1 & & 0 \end{smallmatrix} = 10,$$

$$20 \cup 11 = \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ 1 & & 0 \end{smallmatrix} \cap \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ 1 & & 1 \end{smallmatrix} = \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ 1 & & 1 \end{smallmatrix} = 21,$$

$$\bigcup \begin{smallmatrix} & 0 & & & \\ & & 0 & & \\ & 0 & & 0 & \\ 1 & & 1 & & 1 \\ & 0 & & 0 & \\ & & 0 & & \end{smallmatrix} = \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ 1 & & 0 \end{smallmatrix} \cup \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ 1 & & 1 \end{smallmatrix} \cup \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ 0 & & 1 \end{smallmatrix} = \begin{smallmatrix} & 0 & \\ 1 & & 1 \\ 1 & & 1 \end{smallmatrix} = 22,$$

$$\downarrow \bigcup \begin{smallmatrix} & 0 & & & \\ & & 0 & & \\ & 0 & & 0 & \\ 1 & & 1 & & 1 \\ & 0 & & 0 & \\ & & 0 & & \end{smallmatrix} = \downarrow 22 = \begin{smallmatrix} & 0 & & & \\ & & 1 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix},$$

$$\downarrow \begin{smallmatrix} & 0 & & & \\ & & 0 & & \\ & 0 & & 0 & \\ 1 & & 1 & & 1 \\ & 0 & & 0 & \\ & & 0 & & \end{smallmatrix} = \begin{smallmatrix} & 0 & & & \\ & & 0 & & \\ & 0 & & 0 & \\ 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix}.$$

And here are some notations for sets of down-sets:

$$
\begin{array}{rcl}
\left[\begin{smallmatrix} &0&&& \\ &&0&& \\ &0&&?& \\ 0&&?&&1 \\ &1&&1& \\ &&1&& \end{smallmatrix}\right] & = & \{\, U \in \mathcal{O}(B) \mid U \text{ is of the form } \begin{smallmatrix} &0&&& \\ &&0&& \\ &0&&?& \\ 0&&?&&1 \\ &1&&1& \\ &&1&& \end{smallmatrix} \,\} \\
& = & \{\, U \in \mathcal{O}(B) \mid \begin{smallmatrix} &0&&& \\ &&0&& \\ &0&&0& \\ 0&&0&&1 \\ &1&&1& \\ &&1&& \end{smallmatrix} \subseteq U \subseteq \begin{smallmatrix} &0&&& \\ &&0&& \\ &0&&1& \\ 0&&1&&1 \\ &1&&1& \\ &&1&& \end{smallmatrix} \,\} \\
& = & \left\{ \begin{smallmatrix} &0&&& \\ &&0&& \\ &0&&0& \\ 0&&0&&1 \\ &1&&1& \\ &&1&& \end{smallmatrix},\ \begin{smallmatrix} &0&&& \\ &&0&& \\ &0&&0& \\ 0&&1&&1 \\ &1&&1& \\ &&1&& \end{smallmatrix},\ \begin{smallmatrix} &0&&& \\ &&0&& \\ &0&&1& \\ 0&&1&&1 \\ &1&&1& \\ &&1&& \end{smallmatrix} \right\}
\end{array}
$$

$$
\begin{array}{rcl}
\left[\begin{smallmatrix} &\cdot&&& \\ &&\cdot&& \\ &\cdot&&?& \\ \cdot&&?&&1 \\ &1&&1& \\ &&1&& \end{smallmatrix}\right] & = & \{\, U \in \mathcal{O}(12) \mid U \text{ is of the form } \begin{smallmatrix} &\cdot&&& \\ &&\cdot&& \\ &\cdot&&?& \\ \cdot&&?&&1 \\ &1&&1& \\ &&1&& \end{smallmatrix} \,\} \\
& = & \{\, U \in \mathcal{O}(12) \mid \begin{smallmatrix} &\cdot&&& \\ &&\cdot&& \\ &\cdot&&0& \\ \cdot&&0&&1 \\ &1&&1& \\ &&1&& \end{smallmatrix} \subseteq U \subseteq \begin{smallmatrix} &\cdot&&& \\ &&\cdot&& \\ &\cdot&&1& \\ \cdot&&1&&1 \\ &1&&1& \\ &&1&& \end{smallmatrix} \,\} \\
& = & \left\{ \begin{smallmatrix} &\cdot&&& \\ &&\cdot&& \\ &\cdot&&0& \\ \cdot&&0&&1 \\ &1&&1& \\ &&1&& \end{smallmatrix},\ \begin{smallmatrix} &\cdot&&& \\ &&\cdot&& \\ &\cdot&&0& \\ \cdot&&1&&1 \\ &1&&1& \\ &&1&& \end{smallmatrix},\ \begin{smallmatrix} &\cdot&&& \\ &&\cdot&& \\ &\cdot&&1& \\ \cdot&&1&&1 \\ &1&&1& \\ &&1&& \end{smallmatrix} \right\} \\
& = & \{\ {\downarrow}\{10, 02\},\ {\downarrow}\{11, 02\},\ {\downarrow}12\ \}
\end{array}
$$

Note that $\{\ {\downarrow}\{10, 02\},\ {\downarrow}\{11, 02\},\ {\downarrow}12\ \}$ is an up-set, a filter, and a principal filter. More precisely,

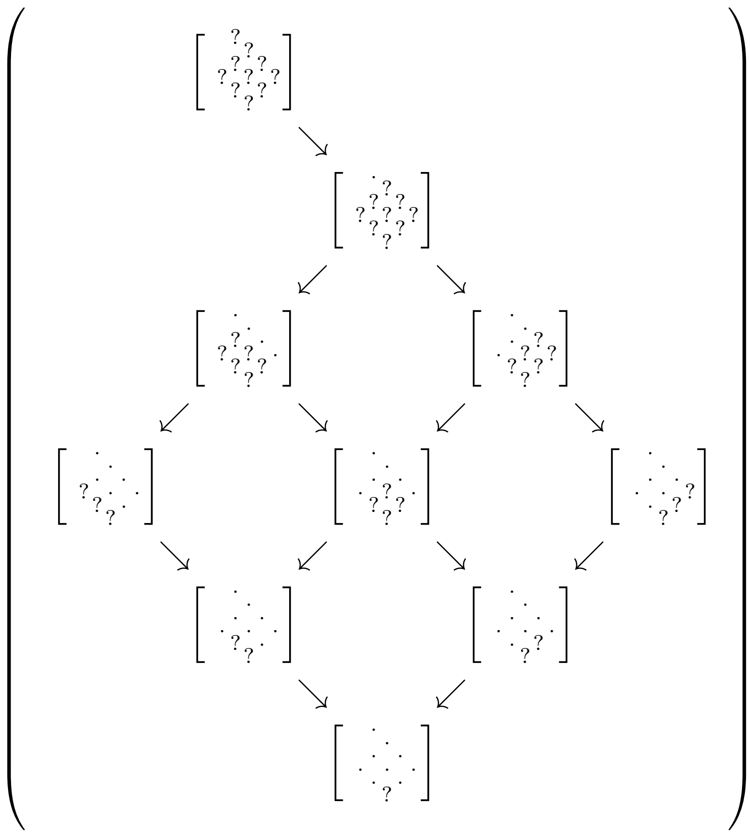
$$
\begin{array}{rcl}
\{\ {\downarrow}\{10, 02\},\ {\downarrow}\{11, 02\},\ {\downarrow}12\ \} & \subseteq & \mathsf{D}(\mathcal{O}(12)), \\
\{\ {\downarrow}\{10, 02\},\ {\downarrow}\{11, 02\},\ {\downarrow}12\ \} & \in & \mathsf{U}(\mathsf{D}(\mathcal{O}(12))), \\
\{\ {\downarrow}\{10, 02\},\ {\downarrow}\{11, 02\},\ {\downarrow}12\ \} & \in & \mathsf{F}(\mathsf{D}(\mathcal{O}(12))), \\
\{\ {\downarrow}\{10, 02\},\ {\downarrow}\{11, 02\},\ {\downarrow}12\ \} & = & {\uparrow}_{\mathsf{D}(\mathcal{O}(12))}({\downarrow}\{10, 02\}).
\end{array}
$$

The '$\cdot$'s in this notation mean "this point is out of the domain". They help to indicate that we are in $\mathsf{D}(\mathcal{O}(12))$, not in $\mathsf{D}(\mathcal{O}(22))$ or $\mathsf{D}(\mathcal{O}(32))$.

6 An example

If you know more than the basics of Topos Theory then the following figures should make a lot of sense... if you don't then it's better to start from the definitions in the next section.

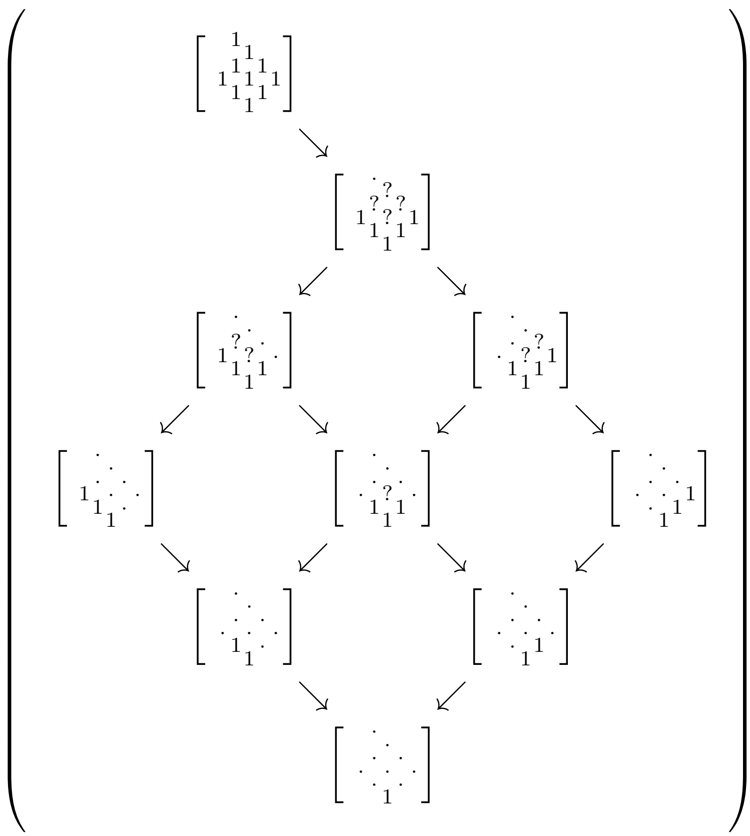
Remember that $\mathcal{O}(H) = B$. The classifier object of the topos $\mathbf{Set}^{\mathcal{O}(H)^{\mathrm{op}}}$ is a functor $\Omega : \mathcal{O}(H)^{\mathrm{op}} \to \mathbf{Set}$ whose action on objects takes each $U \in \mathcal{O}(H)$ to $\mathcal{O}(U)$. Using the notations from the previous section and the trick for "drawing functors as objects" from [FavC, section 7.12], we have:

$\Omega =$ 

(Hint: compare the drawing for Ω above with the one in section 12)

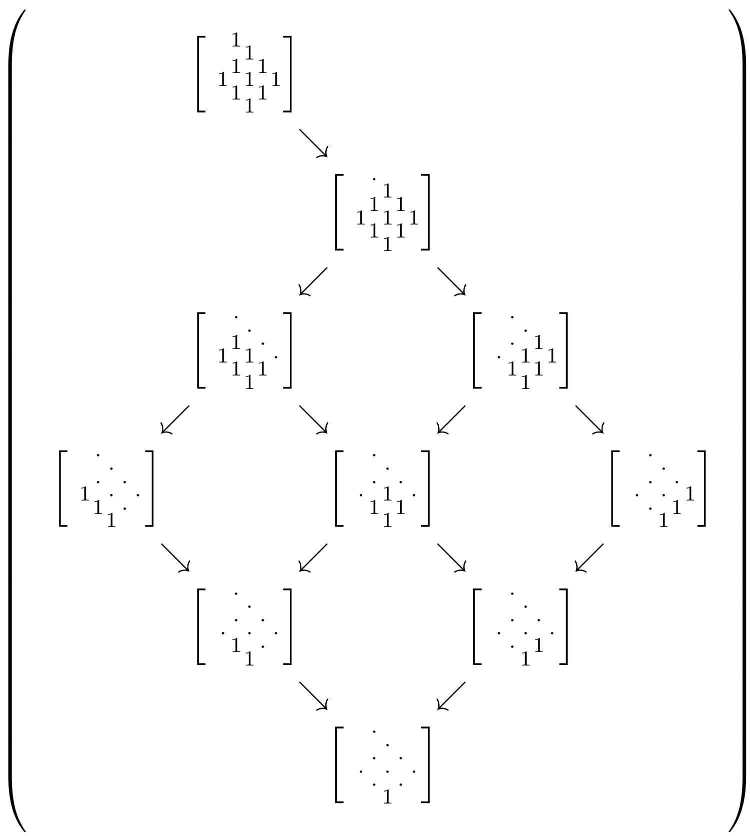
And the canonical Grothendieck topology in $\mathbf{Set}^{\mathcal{O}(H)^{\text{op}}}$ is:

$J_{\text{can}} =$



The morphism $\top : 1 \hookrightarrow \Omega$ in this topos is the inclusion map that goes from this terminal object to the Ω that we have just defined:

$1 =$



The meaning of "inclusion map" here should be obvious, but there is a formal definition of it in these notes: [Och20]. Note that we have these inclusions in $\mathbf{Set}^{\mathcal{O}(H)^{\mathrm{op}}}$:

$$1 \hookrightarrow J_{\mathrm{can}} \hookrightarrow \Omega \,.$$

7 Defining Grothendieck topologies

Here's how to define Grothendieck topologies on a category $\mathcal{O}(X)$.

1. We start with a topological space $(X, \mathcal{O}(X))$, and with the category $\mathcal{O}(X)$ — the poset of open sets of X, in which an arrow $V \to U$ exists iff $V \subseteq U$. We will define Grothendieck topologies "on $\mathcal{O}(X)$" now, and generalize this later.

2. For every $U \in \mathcal{O}(X)$ the *maximal sieve on* U is the set $t(U) = \mathcal{O}(U)$.

3. A *sieve on* U is a subset $\mathcal{S} \subseteq t(U)$ that is closed downwards.

4. We write $\Omega(U)$ for the set of sieves on U.

5. This Ω is a functor $\Omega : \mathcal{O}(X)^{\text{op}} \to \mathbf{Set}$. Its definition by a diagram (in the sense of [FavC, section 5.2]) is:

$$
\begin{tikzcd}
U \arrow[r, mapsto] \arrow[dr, phantom, "\mapsto"] & \Omega(U) \arrow[d] & \mathcal{S} \arrow[d, mapsto] \\
V \arrow[u] \arrow[r, mapsto] & \Omega(V) & \mathcal{S} \cap \mathcal{O}(V) \\
\mathcal{O}(X) & & \\
\mathcal{O}(X)^{\text{op}} \arrow[r] & \mathbf{Set} &
\end{tikzcd}
$$

6. We say that a sieve $\mathcal{S}$ on U *covers* U iff this holds: $\bigcup \mathcal{S} = U$.

7. We write $J_{\text{can}}(U)$ for the set of sieves on U that cover U. Note that $J_{\text{can}}(U) \subseteq \Omega(U)$.

8. This J_{can} is a functor $J_{\text{can}} : \mathcal{O}(X)^{\text{op}}$. Its definition by a diagram is

$$
\begin{tikzcd}
U \arrow[r, mapsto] \arrow[dr, phantom, "\mapsto"] & \Omega(U) \arrow[d] & \mathcal{S} \arrow[d, mapsto] \\
V \arrow[u] \arrow[r, mapsto] & \Omega(V) & \mathcal{S} \cap \mathcal{O}(V) \\
\mathcal{O}(X) & & \\
\mathcal{O}(X)^{\text{op}} \arrow[r] & \mathbf{Set} &
\end{tikzcd}
$$

9. This J_{can} is a *subfunctor* of Ω: we have a natural transformation $i : J_{\mathrm{can}} \hookrightarrow \Omega$ in which each map $i(U) : J_{\mathrm{can}}(U) \hookrightarrow \Omega(U)$ is an inclusion in **Set**.

10. We call this functor $J_{\mathrm{can}} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$ the *canonical Grothendieck topology on* $\mathcal{O}(X)$, or "on (the topological space) X".

11. For each $U \in \mathcal{O}(X)$ the corresponding $J_{\mathrm{can}}(U)$ is a filter. More precisely: each sieve $\mathcal{S}$ on U is a down-set of $\mathcal{O}(U)$; in the notation of section sss, $\mathcal{S} \in \mathsf{D}(\mathcal{O}(U))$; so $J_{\mathrm{can}}(U) \subseteq \mathsf{D}(\mathcal{O}(U))$. And:

    (a) The top element of $\mathsf{D}(\mathcal{O}(U))$, $t(U)$, is a covering sieve on U, so $t(U) \in J_{\mathrm{can}}(U)$.

    (b) If $\mathcal{A}$ and $\mathcal{B}$ are sieves on U and $\mathcal{A}$ covers U, then $\mathcal{B}$ also covers U. This means that $J_{\mathrm{can}}(U)$ is closed upwards, and so $J_{\mathrm{can}}(U)$ is an up-set: $J_{\mathrm{can}}(U) \in \mathsf{U}(\mathsf{D}(\mathcal{O}(U)))$.

    (c) If $\mathcal{A}$ and $\mathcal{B}$ are covering sieves on U then $\mathcal{A} \cap \mathcal{B}$ is also a covering sieve on U. This means that $J_{\mathrm{can}}(U)$ is closed by binary meets, and so $J_{\mathrm{can}}(U)$ obeys all the conditions for being a filter: $J_{\mathrm{can}}(U) \in \mathsf{F}(\mathsf{D}(\mathcal{O}(U)))$.

    (d) The intersection $\bigcap J_{\mathrm{can}}(U)$ is a sieve on U that is "below" all covering sieves $\mathcal{U} \in J_{\mathrm{can}}(U)$; more precisely, for every $\mathcal{U} \in J_{\mathrm{can}}(U)$ we have $\bigcap J_{\mathrm{can}}(U) \subseteq \mathcal{U}$. This means that the filter $J_{\mathrm{can}}(U)$ is contained in the principal filter $\uparrow_{\mathsf{D}(\mathcal{O}(U))} \bigcap J_{\mathrm{can}}(U)$.

    (e) When $\mathcal{O}(X)$ is a finite set all the '$\mathcal{O}(U)$'s and '$\mathsf{D}(\mathcal{O}(U))$'s are finite sets. In this case each $J_{\mathrm{can}}(U)$ is closed by arbitrary meets, and $\bigcap J_{\mathrm{can}}(U) \in J_{\mathrm{can}}(U)$. Each $J_{\mathrm{can}}(U)$ is a principal filter, and $J_{\mathrm{can}}(U) = \uparrow_{\mathsf{D}(\mathcal{O}(U))} \bigcap J_{\mathrm{can}}(U)$.

12. This J_{can} obeys the properties $\mathsf{hasmax}_{J_{\mathrm{can}}}$, $\mathsf{stab}_{J_{\mathrm{can}}}$, and $\mathsf{trans}_{J_{\mathrm{can}}}$, described in section 8.

13. We define the *Grothendieck topologies on* $\mathcal{O}(X)$ as the subfunctors $J \hookrightarrow \Omega$ that obey the properties hasmax_J, stab_J, and trans_J.

14. The properties hasmax_J, stab_J, and trans_J imply that each $J(U)$ contains the top element $t(U)$ and is closed upwards and by binary meets; so each $J(U)$ is a filter on $\mathsf{D}(\mathcal{O}(U))$, i.e., $J(U) \in \mathsf{F}(\mathsf{D}(\mathcal{O}(U)))$.

8 The properties $\mathsf{hasmax}_\mathsf{J}$, stab_J, and $\mathsf{trans}_\mathsf{J}$

This is the definition of a Grothendieck topology on an arbitrary category, taken from [LM92, page 110]:

> **Definition 1.** A *Grothendieck Topology* on a category $\mathbf{C}$ is a function J which assigns to each object C of $\mathbf{C}$ a collection $J(C)$ of sieves on C, in such a way that:
>
> 1. The maximal sieve $t_C = \{ f \mid \mathrm{cod}(f) = C \}$ is in $J(C)$;
> 2. (stability axiom) is $S \in J(C)$, then $h^*(S) \in J(D)$ for any arrow $h : D \to C$;
> 3. (transitivity axiom) if $S \in J(C)$ and R is any sieve on C such that $h^*(S) \in J(D)$ for all $h : D \to C$ in S, then $R \in J(C)$.

It follows these notational conventions:

$$
\begin{array}{ccccccccc}
\mathbf{C} \\
\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}} \\
(\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}})_1 \\
\Cup \\
h & \in & S & \in & J(C) & \subset & \Omega(C) \\
\\
\begin{pmatrix}\text{a small}\\\text{category}\end{pmatrix} \\
\begin{pmatrix}\text{our}\\\text{topos}\end{pmatrix} \\
\begin{pmatrix}\text{its}\\\text{morphisms}\end{pmatrix} \\
\Cup \\
\begin{pmatrix}\text{a}\\\text{morphism}\end{pmatrix} & \in & \begin{pmatrix}\text{a}\\J\text{-covering}\\\text{sieve}\end{pmatrix} & \in & \begin{pmatrix}\text{all}\\J\text{-covering}\\\text{sieves}\\\text{on } C\end{pmatrix} & \in & \begin{pmatrix}\text{all}\\\text{sieves}\\\text{on } C\end{pmatrix}
\end{array}
$$

We will translate that definition to our definition of a Grothendick Topology on a topological space in two steps.

This is the first step of the translation — note that here the sieves are still collections of arrows, not collections of open sets:

> **Definition 1′.** A *Grothendieck Topology* on a category $\mathcal{O}(X)$ is a function J which assigns to each object U of $\mathcal{O}(U)$ a collection $J(U)$ of sieves on U, in such a way that:
>
> 1. The maximal sieve $t(U) = \{\, f \mid \mathrm{cod}(f) = U \,\}$ is in $J(U)$;
> 2. (stability axiom) is $\mathcal{U} \in J(U)$, then $h^*(\mathcal{U}) \in J(V)$ for any arrow $h : V \to U$;
> 3. (transitivity axiom) if $\mathcal{U} \in J(U)$ and $\mathcal{S}$ is any sieve on U such that $h^*(\mathcal{S}) \in J(V)$ for all $h : V \to U$ in $\mathcal{U}$, then $\mathcal{S} \in J(U)$.

The definition above follows these notational conventions:

$$
\begin{array}{ccccccccc}
\mathcal{O}(X) \\
\mathbf{Set}^{\mathcal{O}(X)^{\mathrm{op}}} \\
(\mathbf{Set}^{\mathcal{O}(X)^{\mathrm{op}}})_1 \\
\rotatebox[origin=c]{90}{$\in$} \\
h & \in & \mathcal{U} & \in & J(U) & \subset & \Omega(U)
\end{array}
$$

$$
\begin{array}{ccccccccc}
\begin{pmatrix}\text{a}\\ \text{topology}\end{pmatrix} \\
\begin{pmatrix}\text{our}\\ \text{topos}\end{pmatrix} \\
\begin{pmatrix}\text{its}\\ \text{morphisms}\end{pmatrix} \\
\rotatebox[origin=c]{90}{$\in$} \\
\begin{pmatrix}\text{a}\\ \text{morphism}\end{pmatrix} & \in & \begin{pmatrix}\text{a}\\ J\text{-covering}\\ \text{sieve}\end{pmatrix} & \in & \begin{pmatrix}\text{all}\\ J\text{-covering}\\ \text{sieves}\\ \text{on } C\end{pmatrix} & \in & \begin{pmatrix}\text{all}\\ \text{sieves}\\ \text{on } C\end{pmatrix}
\end{array}
$$

This is the second step of the translation. Here the sieves becomes collections of open subsets, instead of collections of morphisms:

> **Definition 1″.** A *Grothendieck Topology* on a category $\mathcal{O}(X)$ is a function J which assigns to each object U of $\mathcal{O}(U)$ a collection $J(U)$ of sieves on U, in such a way that:
>
> 1. The maximal sieve $t(U) = \{\, V \in \mathcal{O}(X) \mid V \subseteq U \,\}$ is in $J(U)$;
> 2. (stability axiom) is $\mathcal{U} \in J(U)$, then $(V \subseteq U)^*(\mathcal{U}) \in J(V)$ for any open set $V \in \mathcal{O}(U)$;
> 3. (transitivity axiom) if $\mathcal{U} \in J(U)$ and $\mathcal{S}$ is any sieve on U such that $(U \subseteq V)^*(\mathcal{S}) \in J(V)$ for all $V \in \mathcal{O}(U)$ in $\mathcal{U}$, then $\mathcal{S} \in J(U)$.

Here are its notational conventions, in the same shape as before:

$$
\begin{array}{ccccccccc}
\mathcal{O}(X) \\
\cup \\
V & \in & \mathcal{U} & \in & J(U) & \subset & \Omega(U)
\end{array}
$$

$$
\begin{array}{ccccccc}
\begin{pmatrix}\text{our}\\\text{topology}\end{pmatrix} \\
\cup \\
\begin{pmatrix}\text{an}\\\text{open}\\\text{set}\end{pmatrix} & \in & \begin{pmatrix}\text{a}\\\text{covering}\\\text{sieve}\end{pmatrix} & \in & \begin{pmatrix}\text{all}\\\text{covering}\\\text{sieves}\\\text{on } U\end{pmatrix} & \in & \begin{pmatrix}\text{all}\\\text{sieves}\\\text{on } U\end{pmatrix}
\end{array}
$$

Many very important details from Definitions 1, 1′, and 1″ were left out of these L-shaped diagrams, so let's create something more detailed — in two steps. In the first one we will translate the three conditions to something closer to first-order logic, but with some of the implications being drawn vertically to stress what is "above" and what is "below", following that the convention that bigger open sets are above and small ones are below. We get this.

This is Definition 1,

$$
\begin{array}{rcl}
\mathsf{hasmax}_J & := & (\forall C \in \mathbf{C}.\ t(C) \in J(C)) \\
\mathsf{stab}_J & := & \left(\begin{array}{l} \forall h : D \to C. \\ \forall S \in \mathsf{Sieves\_on}(C). \\ \left(\begin{array}{c} S \in J(C) \\ \downarrow \\ h^*(S) \in J(D) \end{array} \right) \end{array} \right) \\
\mathsf{trans}_J & := & \left(\begin{array}{l} \forall C \in \mathbf{C}. \\ \forall S \in J(U). \\ \forall R \in \mathsf{Sieves\_on}(U). \\ \left(\begin{array}{c} (R \in J(C)) \\ \uparrow \\ \forall (h : D \to C) \in S.\ (h^*(R) \in J(D)) \end{array} \right) \end{array} \right)
\end{array}
$$

And this is Definition 1″:

$$
\begin{array}{rcl}
\mathsf{hasmax}_J & := & (\forall U \in \mathcal{O}(X).t(U) \in J(U)) \\
\mathsf{stab}_J & := & \left(\begin{array}{l} \forall V \subseteq U. \\ \forall \mathcal{U} \in \mathsf{Sieves\_on}(U). \\ \left(\begin{array}{c} \mathcal{U} \in J(V) \\ \downarrow \\ (V \subseteq U)^*(\mathcal{U}) \in J(V) \end{array} \right) \end{array} \right) \\
\mathsf{trans}_J & := & \left(\begin{array}{l} \forall U \in \mathcal{O}(X). \\ \forall \mathcal{U} \in J(U). \\ \forall \mathcal{S} \in \mathsf{Sieves\_on}(U). \\ \left(\begin{array}{c} (\mathcal{S} \in J(U)) \\ \uparrow \\ \forall V \in \mathcal{U}.((V \subseteq U)^*(\mathcal{S}) \in J(V)) \end{array} \right) \end{array} \right)
\end{array}
$$

9 Drawing the properties

Here is my (current) favorite way to draw the properties $\mathsf{hasmax}_{\mathsf{J}}$, $\mathsf{stab}_{\mathsf{J}}$, and $\mathsf{trans}_{\mathsf{J}}$ for a Grothendieck topology J on the topological space $(X, \mathcal{O}(X))$:

$$
\begin{tikzcd}
U & & & (\mathcal{O}(U) \in J(U)) \\
U & & & (\mathcal{U} \in J(U)) \arrow[d] \\
V \arrow[u] & & & (\mathcal{U} \cap \mathcal{O}(V) \in J(V)) \\
U & \mathcal{U} & \mathcal{S} \arrow[d, mapsto] & (\mathcal{S} \in J(U)) \\
\forall V \arrow[u] \arrow[ur, phantom, "\in" sloped] & & \mathcal{S} \cap \mathcal{O}(V) & (\forall V {\in} \mathcal{U}.\, \mathcal{S} \cap \mathcal{O}(V) \in J(V)) \arrow[u]
\end{tikzcd}
$$

[Rewrite this paragraph:]

Some details of the formal translation are omitted in that diagram, but most of them can be reconstructed from the fonts and the placements. Actually the main intent of that diagram is to list notational conventions; in the case above, 1) the objects of the category $\mathcal{O}(X)$ have names like U and V; 2) the arrows in $\mathcal{O}(X)$ go upwards, so our topos is $\mathbf{Set}^{\mathcal{O}(X)^{\mathrm{op}}}$; not $\mathbf{Set}^{\mathcal{O}(X)}$; 3) the top element in each $\mathsf{J}(U)$ can be denoted by $\mathcal{O}(U)$; 4) we denote sieves by names like $\mathcal{S}$; 5) our typical name for a covering sieve on U is $\mathcal{U}$; 6) the operation that "restricts sieves" and moves them downward in diagrams like the ones in section 6 is '$\cap\, \mathcal{O}(V)$'.

If I draw the Definition 1 in that shape I get this:

$$
\begin{tikzcd}
C & & & (t_C \in J(C)) \\
C & & & (S \in J(C)) \arrow[d] \\
D \arrow[u, "h"] & & & (h^*(S) \in J(D)) \\
C & S & R \arrow[d, mapsto] & (R \in J(C)) \\
\forall D \arrow[u, "\forall h"] \arrow[ur, phantom, "\in" sloped] & & h^*(R) & (\forall (h : D \to C) \in S.\, h^*(R) \in J(D)) \arrow[u]
\end{tikzcd}
$$

Note that in the first drawing the '$\in$' points to the 'V', and in the second one to the 'h'.

Here is Definition 1 again, followed by its two translations to the more visual ways to represent the properties, that stress that stab_J propagates J-coveringness down and trans_J propagates J-coveringness up:

1. The maximal sieve $t_C = \{\, f \mid \mathrm{cod}(f) = C \,\}$ is in $J(C)$;
2. (stability axiom) is $S \in J(C)$, then $h^*(S) \in J(D)$ for any arrow $h : D \to C$;
3. (transitivity axiom) if $S \in J(C)$ and R is any sieve on C such that $h^*(S) \in J(D)$ for all $h : D \to C$ in S, then $R \in J(C)$.

$$
\begin{array}{rcl}
\mathsf{hasmax}_J & := & (\forall C \in \mathbf{C}.\ t(C) \in J(C)) \\
\mathsf{stab}_J & := & \left(\begin{array}{l} \forall h : D \to C. \\ \forall S \in \mathsf{Sieves\_on}(C). \\ \left(\begin{array}{c} S \in J(C) \\ \downarrow \\ h^*(S) \in J(D) \end{array} \right) \end{array} \right) \\
\mathsf{trans}_J & := & \left(\begin{array}{l} \forall C \in \mathbf{C}. \\ \forall S \in J(U). \\ \forall R \in \mathsf{Sieves\_on}(U). \\ \left(\begin{array}{c} (R \in J(C)) \\ \uparrow \\ \forall (h : D \to C) \in S.\ (h^*(R) \in J(D)) \end{array} \right) \end{array} \right)
\end{array}
$$

$$
\begin{tikzcd}
C & & (t_C \in J(C))
\end{tikzcd}
$$

$$
\begin{tikzcd}
C & & (S \in J(C)) \arrow[d] \\
D \arrow[u, "h"] & & (h^*(S) \in J(D))
\end{tikzcd}
$$

$$
\begin{tikzcd}
C & S & R \arrow[d, mapsto] & (R \in J(C)) \\
\forall D \arrow[u, "\forall h"] & & h^*(R) & (\forall (h : D \to C) \in S.\, h^*(R) \in J(D)) \arrow[u]
\end{tikzcd}
$$

I learned most of what I know about Grothendieck topologies on posets from [Lin14]. He uses the notational conventions below to define Grothendieck topologies on a poset $\mathbf{P}$ (in pages 8–9 of the paper):

$$
\begin{array}{ccc}
p & & ({\downarrow}p \in J(p)) \\[10pt]
\begin{tikzcd}[row sep=small] p \\ q \arrow[u, "q\leq p"] \end{tikzcd} & & \begin{tikzcd}[row sep=small] (S \in J(p)) \arrow[d] \\ (S \cap {\downarrow}q \in J(q)) \end{tikzcd} \\[10pt]
\begin{tikzcd}[row sep=small] p & S \\ \forall q \arrow[u, "q\leq p"] \arrow[ur, phantom, "\in" sloped] & \end{tikzcd} & \begin{tikzcd}[row sep=small] R \arrow[d, mapsto] \\ R \cap {\downarrow}q \end{tikzcd} & \begin{tikzcd}[row sep=small] (R \in J(p)) \\ (\forall q {\in} S.\ R \cap {\downarrow}q \in J(q)) \arrow[u] \end{tikzcd}
\end{array}
$$

I will use notational conventions for grtops on posets that are very close to the first diagram in this section, that showed the properties of a grtop on a topological space $(X, \mathcal{O}(X))$.

In the diagrams of section 6 I showed how to visualize the canonical Grothendieck topology J_{can} on the topological space $(H, \mathcal{O}(H))$; this J_{can} was an object of the topos $\mathbf{Set}^{\mathcal{O}(H)^{\text{op}}}$, but we had this:

$$
\mathcal{O}(H) \quad = \quad B \quad = \quad
\begin{tikzcd}[row sep=small, column sep=small]
 & 32 & & & \\
 & & 22 \arrow[ul] & & \\
 & 21 \arrow[ur] & & 12 \arrow[ul] & \\
20 \arrow[ur] & & 11 \arrow[ul] \arrow[ur] & & 02 \arrow[ul] \\
 & 10 \arrow[ul] \arrow[ur] & & 01 \arrow[ul] \arrow[ur] & \\
 & & 00 \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

so that J_{can} was also an object of $\mathbf{Set}^{B^{\text{op}}}$. Let's see how to define grtops on arbitrary posets D (mnemonic: "DAG"), i.e., in categories $\mathbf{Set}^{D^{\text{op}}}$ or $\mathbf{Set}^{D}$.

Suppose that D is a poset with its arrows pointing up. Our notational conventions for grtops on D will be exactly the same as the ones that we used for grtops on $\mathcal{O}(X)$, with only these changes: 1) elements of D will be denoted by u and v (instead of $U, V \in \mathcal{O}(X)$, where the U and V were in uppercase); 2) we will write ${\downarrow}u$ and ${\downarrow}v$ instead of $\mathcal{O}(U)$ and $\mathcal{O}(V)$; 3) we will write D instead of $\mathcal{O}(X)$. We will still use $\mathcal{S}$ for a sieve and $\mathcal{U}$ for a covering sieve (but now $\mathcal{U}$ covers u, not on U); so $\mathcal{S}$ and $\mathcal{U}$ are down-sets of D, and $\mathcal{S}$,$\mathcal{U} \subseteq D$ (instead of $\mathcal{S}$,$\mathcal{U} \subseteq \mathcal{O}(X)$).

This was our diagram for a grtop on a topological space $(X, \mathcal{O}(X))$:

$$\begin{tikzcd}
U & & & (\mathcal{O}(U) \in J(U)) \\
U & & & (\mathcal{U} \in J(U)) \arrow[d] \\
V \arrow[u] & & & (\mathcal{U} \cap \mathcal{O}(V) \in J(V)) \\
U & \mathcal{U} & \mathcal{S} \arrow[d, mapsto] & (\mathcal{S} \in J(U)) \\
\forall V \arrow[u] \arrow[ur, phantom, sloped, "\in"] & & \mathcal{S} \cap \mathcal{O}(V) & (\forall V{\in}\mathcal{U}.\ \mathcal{S} \cap \mathcal{O}(V) \in J(V)) \arrow[u]
\end{tikzcd}$$

This is our diagram for a grtop on a poset D with its arrows pointing up:

$$\begin{tikzcd}
u & & & ({\downarrow}u \in J(u)) \\
u & & & (\mathcal{U} \in J(u)) \arrow[d] \\
v \arrow[u] & & & (\mathcal{U} \cap {\downarrow}v \in J(v)) \\
u & \mathcal{U} & \mathcal{S} \arrow[d, mapsto] & (\mathcal{S} \in J(u)) \\
\forall v \arrow[u] \arrow[ur, phantom, sloped, "\in"] & & \mathcal{S} \cap {\downarrow}v & (\forall v{\in}\mathcal{U}.\ \mathcal{S} \cap {\downarrow}v \in J(v)) \arrow[u]
\end{tikzcd}$$

This characterizes the subobjects $J \subseteq \Omega$ of a category $\mathbf{Set}^{D^{\mathrm{op}}}$ that are Grothendieck topologies. To define grtops when the arrows in D point down — the 2-column graphs of [PH1, section 14] use arrows pointing down — we just need to switch the direction of the arrows in the left column.

10 Other Grothendieck topologies

In the section 6 we drew a specific Grothendieck topology on $(H, \mathcal{O}(H))$: the "canonical" one, that was a subobject $J_{\text{can}} \hookrightarrow \Omega$ in $\mathbf{Set}^{\mathcal{O}(H)^{\text{op}}}$. Let's start by a way to obtain other grtops on $(H, \mathcal{O}(H))$.

Remember that we use $\mathcal{U}$ and $\mathcal{S}$ to denote subsets of a topology $\mathcal{O}(X)$: $\mathcal{S}$ denotes a sieve, and $\mathcal{U}$ denotes a covering sieve. Let's use the letter $\mathcal{Y}$ to denote an *arbitrary* subset of $\mathcal{O}(X)$. We define the Grothendieck topology $J_{\mathcal{Y}}$ on $\mathcal{O}(X)$ by:

$$\begin{array}{rcl} J_{\mathcal{Y}}(U) & := & \{\, \mathcal{S} \in \Omega(U) \mid \mathcal{Y} \cap \mathcal{O}(U) \subseteq \mathcal{S} \,\} \\ & = & \{\, \mathcal{S} \in \Omega(U) \mid {\downarrow}(\mathcal{Y} \cap \mathcal{O}(U)) \subseteq \mathcal{S} \,\} \\ & = & {\uparrow}_{\mathsf{D}(\Omega(U))} {\downarrow}(\mathcal{Y} \cap \mathcal{O}(U)) \end{array}$$

Let's start with an example. We will work in $(H, \mathcal{O}(H))$, as in section 6, and let $\mathcal{Y} := \{01, 02, 11, 12\}$. So:

$$\mathcal{Y} = \begin{smallmatrix} & 0 & & & \\ & & 0 & & \\ & 0 & & 1 & \\ 0 & & 1 & & 1 \\ & 0 & & 1 & \\ & & 0 & & \end{smallmatrix}$$

Then:

$$\begin{array}{rcl} J_{\mathcal{Y}}(21) & := & \{\, \mathcal{S} \in \Omega(21) \mid {\downarrow}(\mathcal{Y} \cap \mathcal{O}(21)) \subseteq \mathcal{S} \,\} \\ & := & \left\{\, \mathcal{S} \in \left[\begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & ? & & \cdot & \\ ? & & ? & & \cdot \\ & ? & & ? & \\ & & ? & & \end{smallmatrix} \right] \;\middle|\; {\downarrow}\left(\begin{smallmatrix} & 0 & & & \\ & & 0 & & \\ & 0 & & 1 & \\ 0 & & 1 & & 1 \\ & 0 & & 1 & \\ & & 0 & & \end{smallmatrix} \;\cap\; \begin{smallmatrix} & 0 & & & \\ & & 0 & & \\ & 1 & & 0 & \\ 1 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix} \right) \subseteq \mathcal{S} \,\right\} \\ & := & \left\{\, \mathcal{S} \in \left[\begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & ? & & \cdot & \\ ? & & ? & & \cdot \\ & ? & & ? & \\ & & ? & & \end{smallmatrix} \right] \;\middle|\; {\downarrow}\left(\begin{smallmatrix} & 0 & & & \\ & & 0 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 1 & \\ & & 0 & & \end{smallmatrix} \right) \subseteq \mathcal{S} \,\right\} \\ & := & \left\{\, \mathcal{S} \in \left[\begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & ? & & \cdot & \\ ? & & ? & & \cdot \\ & ? & & ? & \\ & & ? & & \end{smallmatrix} \right] \;\middle|\; \left(\begin{smallmatrix} & 0 & & & \\ & & 0 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix} \right) \subseteq \mathcal{S} \,\right\} \\ & := & \left[\begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & ? & & \cdot & \\ ? & & 1 & & \cdot \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix} \right] \end{array}$$

So for this $\mathcal{Y}$

$$\mathcal{Y} \;=\; \{01, 02, 11, 12\} \;=\; \begin{smallmatrix} & 0 & & & \\ & & 0 & & \\ & 0 & & 1 & \\ 0 & & 1 & & 1 \\ & 0 & & 1 & \\ & & 0 & & \end{smallmatrix} \;\subset\; \mathcal{O}(H)$$

we have:

$$J_{\mathcal{Y}} = \left(\begin{tikzcd}[column sep=small, row sep=small]
& \left[\begin{smallmatrix} & ? & & & \\ & & ? & & \\ & ? & & 1 & \\ ? & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix}\right] \arrow[dr] & & & \\
& & \left[\begin{smallmatrix} & \cdot & & & \\ & & ? & & \\ & ? & & 1 & \\ ? & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix}\right] \arrow[dl] \arrow[dr] & & \\
& \left[\begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & ? & & \cdot & \\ ? & & 1 & & \cdot \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix}\right] \arrow[dl] \arrow[dr] & & \left[\begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & \cdot & & 1 & \\ \cdot & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix}\right] \arrow[dl] \arrow[dr] & \\
\left[\begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & \cdot & & \cdot & \\ ? & & \cdot & & \cdot \\ & ? & & \cdot & \\ & & ? & & \end{smallmatrix}\right] \arrow[dr] & & \left[\begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & \cdot & & \cdot & \\ \cdot & & 1 & & \cdot \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix}\right] \arrow[dl] \arrow[dr] & & \left[\begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & \cdot & & \cdot & \\ \cdot & & \cdot & & 1 \\ & \cdot & & 1 & \\ & & 1 & & \end{smallmatrix}\right] \arrow[dl] \\
& \left[\begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot \\ & ? & & \cdot & \\ & & ? & & \end{smallmatrix}\right] \arrow[dr] & & \left[\begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot \\ & \cdot & & 1 & \\ & & 1 & & \end{smallmatrix}\right] \arrow[dl] & \\
& & \left[\begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & \\ & & ? & & \end{smallmatrix}\right] & &
\end{tikzcd} \right)$$

All grtops on a finite poset D are of the form $J_{\mathcal{Y}}$ for some $\mathcal{Y} \subseteq D$ (see [Lin14, page 43]); in particular, our J_{can} is a $J_{\mathcal{Y}}$ for this $\mathcal{Y}$:

$$\mathcal{Y} \;=\; \{{\downarrow}\_1, {\downarrow}\_2, {\downarrow}1\_, {\downarrow}2\_, {\downarrow}3\_\} \;=\; \{01, 02, 10, 20, 32\} \;=\; \begin{smallmatrix} & 1 & & & \\ & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 1 \\ & 1 & & 1 & \\ & & 0 & & \end{smallmatrix} \;.$$

The definition for $J_{\mathcal{Y}}$ given above also works on arbitrary posets, and there is also a way to start with any Grothendieck topology J on an arbitrary poset and obtain the $\mathcal{Y}$ such that $J = J_{\mathcal{Y}}$. Here are the two constructions in the notation of [Lin14, pages 11–12], each one followed by its translations into our two notations:

$$\begin{array}{rcl}
J_X(p) & := & \{\, S \in \mathcal{D}(\downarrow p) \mid X \cap \downarrow p \subseteq S \,\} \\
J_{\mathcal{Y}}(U) & := & \{\, \mathcal{S} \in \Omega(U) \mid \mathcal{Y} \cap \mathcal{O}(U) \subseteq \mathcal{S} \,\} \\
J_{\mathcal{Y}}(u) & := & \{\, \mathcal{S} \in \Omega(u) \mid \mathcal{Y} \cap \downarrow u \subseteq \mathcal{S} \,\} \\
\\
X_J & := & \{\, p \in \mathbf{P} \mid J(p) = \{\downarrow p\} \,\} \\
\mathcal{Y}_J & := & \{\, U \in \mathcal{O}(X) \mid J(U) = \{\downarrow U\} \,\} \\
\mathcal{Y}_J & := & \{\, u \in D \mid J(u) = \{\downarrow u\} \,\}
\end{array}$$

Let's see an example of this. Let $D := H$, the (slanted) house DAG with arrows going downwards, and let:

$$\mathcal{Y} \quad = \quad \{1\_\_,\ \_\_2\} \quad = \quad \begin{smallmatrix} 0 & & \\ 0 & \searrow & 1 \\ 1 & & 0 \end{smallmatrix}$$

This $\mathcal{Y}$ generates this Grothendieck topology, $J_{\mathcal{Y}} \subseteq \Omega$, in the topos $\mathbf{Set}^H$:

$$J_{\mathcal{Y}} \quad = \quad \left(\begin{tikzcd}
\left[\begin{smallmatrix} ? & & \\ ? & \searrow & 1 \\ 1 & & 1 \end{smallmatrix}\right] \arrow[d] \arrow[dr] & \\
\left[\begin{smallmatrix} \cdot & & \\ ? & \searrow & \cdot \\ 1 & & \cdot \end{smallmatrix}\right] \arrow[d] & \left[\begin{smallmatrix} \cdot & & \\ \cdot & \searrow & 1 \\ \cdot & & 1 \end{smallmatrix}\right] \arrow[d] \\
\left[\begin{smallmatrix} \cdot & & \\ \cdot & \searrow & \cdot \\ 1 & & \cdot \end{smallmatrix}\right] & \left[\begin{smallmatrix} \cdot & & \\ \cdot & \searrow & \cdot \\ \cdot & & ? \end{smallmatrix}\right]
\end{tikzcd}\right)$$

11 Ex-Lion Tamer

> The title words are absent from the song itself, which has travelled far from its origins. Newman initially wrote a lyric featuring a lion tamer. “Graham took one look, said, ‘This is rubbish’ and rewrote it. There was a lion tamer involved somewhere.” According to Lewis, “That was the thing I thought was most memorable, but totally irrelevant in the end. I rewrote it and, by the end, it was called ‘Ex-Lion Tamer’ because even the lion tamer had disappeared.” ([Nea08])

Let’s review our archetypal case, its terminology and its notational conventions, and then make some changes to that.

We started with a topological space $\mathcal{O}(X)$, that was usually regarded as a poset; that poset was also denoted by $\mathcal{O}(X)$. In most examples $\mathcal{O}(X)$ was the order topology on the (slanted) house DAG H, that had 5 elements, and $\mathcal{O}(X)$-as-a-poset was the bottle DAG B, with 10 elements (see sec.4). We referred to the elements of $\mathcal{O}(X)$ as “open sets”, and denoted them by letters like U and V. A *sieve* is a subset of $\mathcal{O}(X)$ obeying certain properties; we denoted sieves by calligraphic letters like $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{U}$, where $\mathcal{U}$ was a *covering sieve* (on U). Also, we used the calligraphic letter $\mathcal{Y}$ to denote a subset of $\mathcal{O}(X)$ that did not need to be a sieve. The set of all sieves on an open set $U \in \mathcal{O}(X)$ was denoted by $\Omega(U)$, and the set of all covering sieves on a $U \in \mathcal{O}(X)$ by $J(U)$. We had this:

$$
\begin{array}{ccccccccc}
\mathcal{O}(X) \\
\Cup \\
V & \in & \mathcal{U} & \in & J(U) & \subset & \Omega(U) \\
V & \in & \mathcal{U} & \in & J(21) & \subset & \Omega(21) \\
\\
\begin{smallmatrix} & 32 & & & \\ & & 22 & & \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{smallmatrix} \\
\Cup \\
10 & \in & \begin{smallmatrix} & 0 & & & \\ & & 0 & & \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix} & \in & \left[\begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & ? & & \cdot & \\ ? & & ? & & \cdot \\ & ? & & 1 & \\ & & 1 & & \end{smallmatrix}\right] & \subset & \left[\begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & ? & & \cdot & \\ ? & & ? & & \cdot \\ & ? & & ? & \\ & & ? & & \end{smallmatrix}\right] \\
& & \| \\
& & \left\{\begin{smallmatrix} 10, & & 01, \\ & 00 & \end{smallmatrix}\right\}
\end{array}
$$

Note that this diagram includes particular cases — in:

$$\begin{array}{ccccccc} V & \in & \mathcal{U} & \in & J(U) & \subset & \Omega(U) \\ V & \in & \mathcal{U} & \in & J(21) & \subset & \Omega(21) \end{array}$$

the bottom line specializes the top line to the case $U = 21$. It also includes examples, like "in this context 10 is a possible value for V".

So: we first learned how to define Grothendieck topologies on a topological space $\mathcal{O}(X)$, with this $\mathcal{O}(X)$ being regarded as a poset, and then we saw how to define Grothendieck topologies on posets that do not need to come from topologies. We will use Wire's trick, and refer to these "posets that no longer need to be topologies" as "ex-topologies", and to their elements as "ex-open sets". Our notational conventions for them will change: the topology $\mathcal{O}(X)$ will become the ex-topology D (a poset), and the open sets $U, V \in \mathcal{O}(X)$ will become ex-open sets $u, v \in D$. The rest will be kept unchanged: for example, sieves will still be called sieves, and will be denoted by the same calligraphic letters as before.

11.1 Ex-topologies: sieves as two-digit numbers

We will also change something else. In the case of Grothendieck topologies on topological spaces we used this:

$$X = H = \left(\begin{array}{ccc} 3\_ & & \\ \downarrow & \searrow & \\ 2\_ & & \_2 \\ \downarrow & & \downarrow \\ 1\_ & & \_1 \end{array} \right) \qquad \mathcal{O}(X) = \mathcal{O}(H) = \begin{array}{ccccc} & 32 & & & \\ & & 22 & & \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{array}$$

There the two-digit numbers, like 10, denoted open sets.

In the new setting, i.e., in the case of Grothendieck topologies on ex-topological spaces, we will switch to this other convention:

$$D = N = \left(\begin{array}{ccc} 3\_ & & \_3 \\ \downarrow & \searrow & \downarrow \\ 2\_ & & \_2 \\ \downarrow & & \downarrow \\ 1\_ & & \_1 \end{array} \right) \qquad \mathsf{D}(D) = \mathsf{D}(N) = \begin{array}{ccccccc} & & & 33 & & & \\ & & 32 & & 23 & & \\ & & & 22 & & 13 & \\ & 21 & & 12 & & 03 & \\ 20 & & 11 & & 02 & & \\ & 10 & & 01 & & & \\ & & 00 & & & & \end{array}$$

Here it is our ex-topological space, D, that is a 2-column graph. This will let us draw the set $\mathsf{D}(D)$ of all down-sets on D, i.e., the set of all *sieves* in this context, as the ZHA associated to the 2-column graph D, as above; see sec.4, and [PH1, sec.15] for the details. This will give us an alternative way to represent sieves and sets of sieves: sieves are now two-digit numbers, and sets of sieves are sets of two-digit numbers.

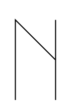
Here is a translation of the previous diagram to this new case:

$$
\begin{array}{ccccccc}
D \\
\rotatebox{90}{$\in$} \\
v & \in & \mathcal{U} & \in & J(u) & \subset & \Omega(u) \\
v & \in & \mathcal{U} & \in & J(3\_) & \subset & \Omega(3\_)
\end{array}
$$

$$
\left(\begin{tikzcd}[sep=small]
3\_ \arrow[d] \arrow[dr] & \_3 \arrow[d] \\
2\_ \arrow[d] & \_2 \arrow[d] \\
1\_ & \_1
\end{tikzcd}\right)
$$

$$
\begin{array}{ccccccc}
\rotatebox{90}{$\in$} \\
2\_ & \in & \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) & \in & \left[\begin{smallmatrix} ? & \cdot \\ 1 & ? \\ 1 & 1 \end{smallmatrix}\right] & \subset & \left[\begin{smallmatrix} ? & \cdot \\ ? & ? \\ ? & ? \end{smallmatrix}\right] \\
 & & \| & & \| & & \| \\
 & & 21 & \in & \left(\begin{smallmatrix} & & \cdot & & & \\ & 32 & & \cdot & & \\ & & 22 & & \cdot & \\ & 21 & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot & \\ & \cdot & & \cdot & & \\ & & \cdot & & & \end{smallmatrix}\right) & \subset & \left(\begin{smallmatrix} & & \cdot & & & \\ & 32 & & \cdot & & \\ & & 22 & & \cdot & \\ & 21 & & 12 & & \cdot \\ 20 & & 11 & & 02 & \\ & 10 & & 01 & & \\ & & 00 & & & \end{smallmatrix}\right) \\
 & & \| \\
 & & \left\{\begin{smallmatrix} 2\_, & \\ 1\_, & \_1 \end{smallmatrix}\right\}
\end{array}
$$

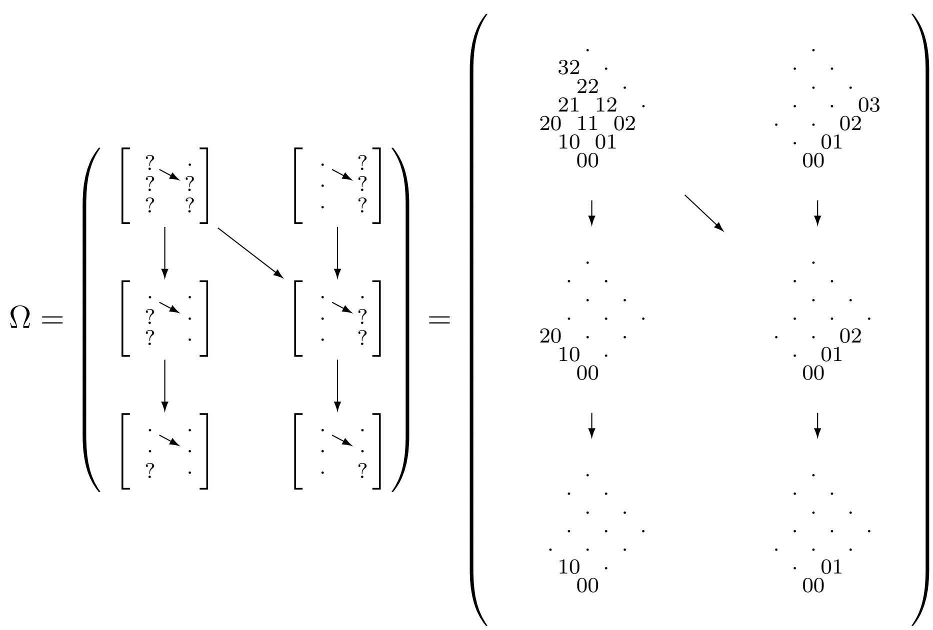
Note that here we have $v = 2\_$ and $u = 3\_$ (two ex-open sets), $\mathcal{U} = 21$ (a sieve), $\Omega(u) = \Omega(3\_) = {\downarrow\!\downarrow}3\_ = {\downarrow}32$, and $J(3\_) = {\uparrow_{\Omega(3\_)}}21$. The 6-node poset that plays the role of D in this example will sometimes be called by its proper name: "Art Déco N", or just "N", because in some Art Déco fonts the uppercase N is drawn like this:



12 Sieves as truth-values

From this point onwards all our examples will use the Art Déco N of the end of the last section as our ex-topology; in other words, our examples will use $D := N$. Also, from here onwards N will always stand for the Art Déco N, not for the traditional N with four nodes of section 3. We will write D and N as $\mathbf{D}$ and $\mathbf{N}$ when we regard them as categories; their arrows will point downwards.

This is the subobject classifier of the topos $\mathbf{Set}^{\mathbf{N}}$ (compare it with the Ω in section 6):



Note that the maximal sieves on 3_ and _3 are:

$$t_{3\_} = {\downarrow}3\_ = 32 = \left(\begin{smallmatrix} 1 & \searrow & \cdot \\ 1 & & 1 \\ 1 & & 1 \end{smallmatrix}\right)$$

$$\text{and} \qquad t_{\_3} = {\downarrow}\_3 = 03 = \left(\begin{smallmatrix} \cdot & \searrow & 1 \\ \cdot & & 1 \\ \cdot & & 1 \end{smallmatrix}\right) ,$$

so here we have sieves that are not in any of the ‘$\Omega(u)$’s. The set of all sieves on the ex-topology N is:

$$\mathsf{D}(N) = \left(\begin{array}{cccccc} & & 33 & & & \\ & 32 & & 23 & & \\ & & 22 & & 13 & \\ & 21 & & 12 & & 03 \\ 20 & & 11 & & 02 & \\ & 10 & & 01 & & \\ & & 00 & & & \end{array} \right),$$

that is exactly the set of subobjects of the terminal object $1 \in \mathbf{Set}^{\mathbf{N}}$. So:

$$\mathrm{Sub}(1_{\mathbf{Set}^{\mathbf{N}}}) = \left[\begin{array}{ccc} ? & & ? \\ & \searrow & \\ ? & & ? \\ ? & & ? \end{array} \right]$$

This holds for any poset D. More precisely: our way of speaking of Grothendieck topologies on a poset D requires treating D as an ex-topology and working on the topos $\mathbf{Set}^{\mathbf{D}}$; in the terminology that we are using the set of truth-values of that topos, $\mathrm{Sub}(1_{\mathbf{Set}^{\mathbf{D}}})$, is essentially the same thing as the set of sieves on D, $\mathsf{D}(D)$.

To be coherent with our notational conventions we will use for truth-values the same letters that we use for sieves: $\mathcal{R}, \mathcal{S}, \mathcal{U}, \mathcal{V}$. When D is a (finite, acyclic) 2-column graph the set of truth-values of $\mathbf{Set}^{\mathbf{D}}$ will be a Planar Heyting Algebra — a ZHA. The paper [PH1] is about how to visualize the operations of Propositional Calculus on ZHAs, and we will reuse all its ideas and notations, except that here we will use letters like $\mathcal{R}, \mathcal{S}, \mathcal{U}, \mathcal{V}$ to denote truth-values, instead of P, Q, R, S. Also, very important:

> From this point onwards the letter H will denote the Heyting Algebra $H = \mathrm{Sub}(1_{\mathbf{Set}^{\mathbf{D}}}) = \mathsf{D}(D)$, instead of the house DAG. This Heyting Algebra $H = \mathsf{D}(D)$ will be a ZHA when D is a (finite, acyclic) 2-column graph.

13 The union as a modality

Let's go back to the case of Grothendieck topologies on a topological space $\mathcal{O}(X)$ for a moment. We defined the canonical topology there by saying that a sieve $\mathcal{S}$ in $\mathcal{O}(U)$ "covers" U iff $\bigcup \mathcal{S} = U$. This is equivalent to saying that a sieve $\mathcal{S}$ in $\mathcal{O}(U)$ covers U iff $\downarrow \bigcup \mathcal{S} = \downarrow U$, and if we define an operation $(\cdot)^*$ on sieves by:

$$\mathcal{S}^* := \downarrow \bigcup \mathcal{S}$$

then we can redefine the notion of canonical Grothendieck topology in terms of it (details soon!). This operation $(\cdot)^* : H \to H$, where H is the Heyting Algebra of sieves-a.k.a.-truth-values obeys, for all sieves $\mathcal{R}$ and $\mathcal{S}$,

$$\mathcal{S} \subseteq \mathcal{S}^* = \mathcal{S}^{**} \qquad \text{and} \qquad (\mathcal{R} \cap \mathcal{S})^* = \mathcal{R}^* \cap \mathcal{S}^* \ ,$$

which are exactly the conditions on J-operators in [PH2], except that J-operators act on ZHAs, not in Heyting Algebras in general.

An operation $(\cdot)^* : H \to H$ on a Heyting Algebra H that obeys $\mathcal{S} \subseteq \mathcal{S}^* = \mathcal{S}^{**}$ and $(\mathcal{R} \cap \mathcal{S})^* = \mathcal{R}^* \cap \mathcal{S}^*$ is called a *nucleus* on H. In [PH2] there are lots of examples of nuclei and ways to visualize them, but the operation $\mathcal{S}^* := {\downarrow}\bigcup\mathcal{S}$ doesn't appear there. In the next section we will see how to convert nuclei to Grothendick topologies, and vice-versa.

Note that in the page of examples in sec.5 the operation $\mathcal{S}^* := {\downarrow}\bigcup\mathcal{S}$ appears at the end (...not yet! fix this!)

14 Nuclei, congruences, and sub-HAs

In section 10 we saw a way to take a set $\mathcal{Y}$ of sieves and generate a Grothendieck topology $J_\mathcal{Y}$ from it, and how to take a Grothendieck topology J and generate a set of sieves $\mathcal{Y}_J$ from it. Now we will do something much more powerful. We will use a shorter notation: we will use J to denote a Grothendieck topology, $\mathcal{Y}$ to denote a set of sieves, $\sim$ to denote a congruence, $(\cdot)^*$ to denote a nucleus, and H' to denote a sub-ZHA (of our planar Heyting Algebra H); note that we are using indefinite articles everywhere — they imply that there is a "space of all 'J's", a "space of all '$\sim$'s", and so on. From these space of 'J's, of '$\sim$'s, etc, we can build the spaces of all functions between them, and when we say "a function $(J \mapsto\sim)$" it will be implicit that it will be a function that receives 'J's and returns '$\sim$'s — an element of that space of functions.

Fix a two-column graph D; let H be the planar Heyting Algebra of the down-sets of D. This $H = \mathsf{D}(D)$ is also the algebra of truth-values of the topos $\mathbf{Set^D}$: $H = \mathrm{Sub}(1_{\mathbf{Set^D}})$. Let Ω be the (canonical) subobject classifier of the topos $\mathbf{Set^D}$. We will need the 10 definitions below, that suppose that D, H, and Ω are fixed:

1. a *proto-Grothendieck topology* J is an operation that takes each $u \in D$ to a subset $J(u) \subset \Omega(u)$, and a *Grothendieck topology* J is a proto-

Grothendieck topology J plus the assurance (in the sense of [FavC, sections 6.4 and 7.5]) that this J obeys hasmax_J, trans_J, and stab_J,

2. a *proto-congruence* $\sim$ is a relation $\sim\, \subset H \times H$, and a *congruence* $\sim$ is a proto-congruence $\sim$ plus the assurance that this $\sim$ is an equivalence relation on H,

3. a *proto-nucleus* $(\cdot)^*$ is a function $(\cdot)^* : H \to H$, and a *nucleus* $(\cdot)^*$ is a proto-nucleus $(\cdot)^*$ plus the assurance that for all $\mathcal{R}, \mathcal{S} \in H$ the conditions $\mathcal{R} \leq \mathcal{R}^* = \mathcal{R}^{**}$ and $(\mathcal{R} \wedge \mathcal{S})^* = \mathcal{R}^* \wedge \mathcal{S}^*$ hold,

4. a *proto-sub-HA* is a subset $H' \subset H$, and a *sub-HA* is a H' that is a Heyting Algebra (obs: with operations $\vee_{H'}$ and $\to_{H'}$ defined in a funny way; see [Lin14, Definition B.10]. TODO: explain that!),

5. a *proto-set of sieves* $\mathcal{Y}$ is a subset $\mathcal{Y} \subseteq D$, and a *set of sieves* is the same thing as a proto-set of sieves.

In some contexts "an operation $(J \mapsto\, \sim)$" will be a function that receives proto-Grothendieck topologies and returns proto-congruences, and in other contexts it will be a function that receives (non-proto-)Grothendieck topologies and returns (non-proto-)congruences — the notation won't change. We will always say very clearly if the current context is "everything is proto" or "nothing is proto".

We will always draw our $(\cdot)^*$s, H's, $\sim$s, Js and $\mathcal{Y}$s in this position:

$$
\begin{array}{ccc}
(\cdot)^* & & H' \\
& \mathcal{Y} & \\
\sim & & J
\end{array}
\qquad\qquad
\begin{array}{ccc}
\text{(nucleus)} & & \text{(sub-HA)} \\
& \text{(set of sieves)} & \\
\text{(congruence)} & & \text{(Groth.top)}
\end{array}
$$

Our first twelve conversion functions will be these ones:

$$\begin{array}{rrl}
(\mathcal{Y} \mapsto J) & J(u) & = \{\, \mathcal{S} \in \Omega(u) \mid \mathcal{Y} \cap {\downarrow}u \subseteq \mathcal{S} \,\} \\
(J \mapsto \mathcal{Y}) & \mathcal{Y} & = \{\, u \in D \mid J(u) = \{{\downarrow}u\} \,\} \\
\\
((\cdot)^* \mapsto H') & H' & = \{\, \mathcal{S} \in H \mid \mathcal{S}^* = \mathcal{S} \,\} = H^* \\
(H' \mapsto (\cdot)^*) & \mathcal{S}^* & = \bigcap\{\, \mathcal{T} \in H' \mid \mathcal{S} \subseteq \mathcal{T} \,\} \\
\\
((\cdot)^* \mapsto \sim) & \sim & = \{\, (\mathcal{R}, \mathcal{S}) \in H^2 \mid \mathcal{R}^* = \mathcal{S}^* \,\} \\
(\sim \mapsto (\cdot)^*) & \mathcal{S}^* & = \bigcup\{\, \mathcal{R} \in H \mid \mathcal{R} \sim \mathcal{S} \,\} \\
\\
((\cdot)^* \mapsto J) & J(u) & = \{\, \mathcal{S} \in \Omega(u) \mid u \in \mathcal{S}^* \,\} \\
(J \mapsto (\cdot)^*) & \mathcal{S}^* & = \{\, u \in D \mid \mathcal{S} \cap {\downarrow}u \in J(u) \,\} \\
\\
(H' \mapsto J) & J(u) & = \{\, \mathcal{S} \in \Omega(u) \mid \forall \mathcal{T} \in H'.\, (\mathcal{S} \subseteq \mathcal{T} \Rightarrow u \in \mathcal{T}) \,\} \\
(J \mapsto H') & H' & = \{\, \mathcal{S} \in H \mid \forall u \in D.\, (\mathcal{S} \cap {\downarrow}u \in J(u) \Rightarrow u \in \mathcal{S}) \,\} \\
\\
(\sim \mapsto J) & J(u) & = \{\, \mathcal{S} \in \Omega(u) \mid \mathcal{S} \sim {\downarrow}u \,\} \\
(J \mapsto \sim) & \sim & = \{\, (\mathcal{R}, \mathcal{S}) \in H^2 \mid \forall u \in D.\, (\mathcal{R} \cap {\downarrow}u \in J(u) \leftrightarrow \mathcal{S} \cap {\downarrow}u \in J(u)) \,\}
\end{array}$$

They correspond to the arrows in these diagrams,

$$\begin{tikzcd}
\mathcal{Y} \arrow[dr, mapsto, shift left] & & (\cdot)^* \arrow[r, mapsto, shift left] \arrow[d, mapsto, shift left] \arrow[dr, mapsto, shift left] & H' \arrow[l, mapsto, shift left] \arrow[d, mapsto, shift left] \\
& J \arrow[ul, mapsto, shift left] & \sim \arrow[u, mapsto, shift left] \arrow[r, mapsto, shift left] & J \arrow[l, mapsto, shift left] \arrow[u, mapsto, shift left] \arrow[ul, mapsto, shift left]
\end{tikzcd}$$

The right half of that diagram corresponds to this diagram from [Lin14], theorem B.25, page 64:

$$\begin{tikzcd}
\mathrm{Nuc}(\mathcal{D}(\mathbf{P})) \arrow[r, "j \mapsto \mathcal{M}_j"] \arrow[d, "j \mapsto \theta_j"'] \arrow[dr, "j \mapsto J_j"] & \mathrm{Sub}(\mathcal{D}(\mathbf{P}))^{\mathrm{op}} \arrow[d, "\mathcal{M} \mapsto J_{\mathcal{M}}"] \\
\mathrm{Con}(\mathcal{D}(\mathbf{P})) \arrow[r, "\theta \mapsto J_\theta"'] & \mathcal{G}(\mathbf{P})
\end{tikzcd}$$

but Lindenhovius uses ‘j’s for nuclei, ‘$\mathcal{M}$’s for subframes (we use sub-HAs instead of subframes), and ‘θ’s for congruences.

Lindenhovius presents the conversions in the way that is standard in mathematical texts: he starts by defining each of the four corners of the diagram in its non-proto version, first as a set, and then as a poset — see the references in the diagram below,

$$
\begin{tikzcd}
\underset{\text{(def. B.6)}}{\mathrm{Nuc}(\mathcal{D}(\mathbf{P}))} \arrow[r, "\text{(prop. B.12)}"] \arrow[d, "\text{(prop. B.23)}"'] \arrow[dr, "\text{(prop. B.8)}" description] & \underset{\text{(def. B.10)}}{\mathrm{Sub}(\mathcal{D}(\mathbf{P}))^{\mathrm{op}}} \arrow[d, "\text{(thm. B.25)}"] \\
\underset{\text{(def. B.15)}}{\mathrm{Con}(\mathcal{D}(\mathbf{P}))} \arrow[r, "\text{(thm. B.25)}"'] & \underset{\text{(def. 2.1)}}{\mathcal{G}(\mathbf{P})}
\end{tikzcd}
$$

and then he defines each of the five arrows above as a "non-proto" order isomorphism, and this takes at least a half page of his paper in each case. Let me explain what I mean by "proto" and "non-proto" here, and explain how we can change the order of the definitions and leave everything that is not "proto" to a second stage ("for adults"). This idea comes from sections 12 and 19 of [IDARCT].

Look at this diagram:

$$
\begin{tikzcd}
A \arrow[d, shift right, "f"'] & A' \arrow[d, dashed, shift right, "f'"'] & = \{\, a \in A \mid P(a) \,\} \\
B \arrow[u, shift right, "g"'] & B' \arrow[u, dashed, shift right, "g'"'] & = \{\, b \in A \mid Q(a) \,\}
\end{tikzcd}
$$

Suppose that A is our set of proto-'a's, A' is our set of 'a's, B is our set of proto-'b's, and B' is our set of 'b's. Also, P is a proposition on A, and Q is a proposition on B.

When we define a bijection between sets A and B in a proof assistant or in a type theory we define it as a 4-uple, made of an arrow $f : A \to B$, an arrow $g : B \to A$, and assurances that $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$. A *proto-bijection* is just the pair of arrows (f, g), without the assurances (i.e., without the "proof-terms"). In order to show that this bijection restricts to a bijection between A' and B' we need two other proof terms: one for $\forall a \in A.(P(a) \to Q(f(a)))$ and another for $\forall b \in B.(Q(b) \to P(g(b)))$. Lindenhovius uses this order of definition,

$$\begin{array}{l}
A, P, B, Q, \\
f, (\forall a \in A.(P(a) \to Q(f(a)))), \\
g, (\forall b \in B.(Q(b) \to P(g(b)))), \\
(\forall a \in A.(P(a) \to a = g(f(a)))), \\
(\forall b \in B.(Q(b) \to b = f(g(b)))),
\end{array}$$

and on top of that he also defines ordering on its sets A' and B', and proves that his 'f's and 'g's preserve these orderings.

Our intent here is to show that to understand Grothendieck topologies on (finite) posets we *can*, and we *should*, start by just the "proto" part,

$$A, B, f, g$$

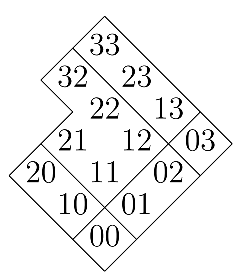
that in this case will be the definitions of proto-Grothendieck topology, proto-congruence, proto-nucleus, proto-sub-HA and proto-set of sieves from the beginning of this section, plus the 12 conversion functions in this diagram:

$$\begin{tikzcd}
 & & (\cdot)^* \arrow[r, shift left, mapsto] \arrow[d, shift left, mapsto] \arrow[dr, shift left, mapsto] & H' \arrow[l, shift left, mapsto] \arrow[d, shift left, mapsto] \\
\mathcal{Y} \arrow[dr, shift left, mapsto] & & \sim \arrow[u, shift left, mapsto] \arrow[r, shift left, mapsto] & J \arrow[l, shift left, mapsto] \arrow[u, shift left, mapsto] \arrow[ul, shift left, mapsto] \\
 & J \arrow[ul, shift left, mapsto] & &
\end{tikzcd}$$

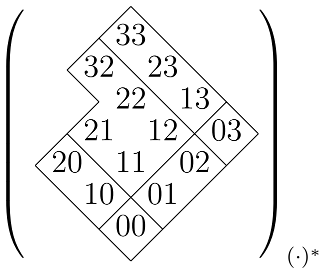
Let's fix a two-column graph D; we will use the Art Déco N of section 11.1 as our D. Remember that its algebra of truth-values is this ZHA:

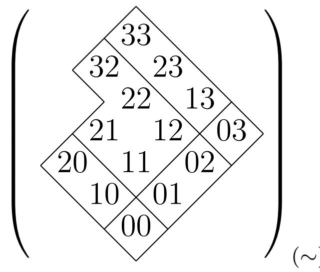
$$H = \mathsf{D}(N) = \left(\begin{array}{cccccc}
 & & 33 & & & \\
 & 32 & & 23 & & \\
 & & 22 & & 13 & \\
 & 21 & & 12 & & 03 \\
20 & & 11 & & 02 & \\
 & 10 & & 01 & & \\
 & & 00 & & &
\end{array}\right)$$

In sections 1 and 2 of [PH2] we defined "slashings" on a ZHA. They looked like this:



but they could be interpreted in many ways. We will add subscripts to them to disambiguate:

$(\cdot)^* =$ 

$\sim =$ 

Here the subscript $(\cdot)^*$ in the first slashing indicates that we are interpreting it as the operation that takes all elements in each region to the top element of that region, and the subscript $(\sim)$ in the second one indicate that we are interpreting it as the equivalence relation that says that $\mathcal{R} \sim \mathcal{S}$ is true iff $\mathcal{S}$ and $\mathcal{S}$ are both in the same region. The "$(\cdot)^*$=" and the "$\sim =$" mean "let $(\cdot)^*$ be this function from H to H" and "let $\sim$ be this equivalence relation on H".

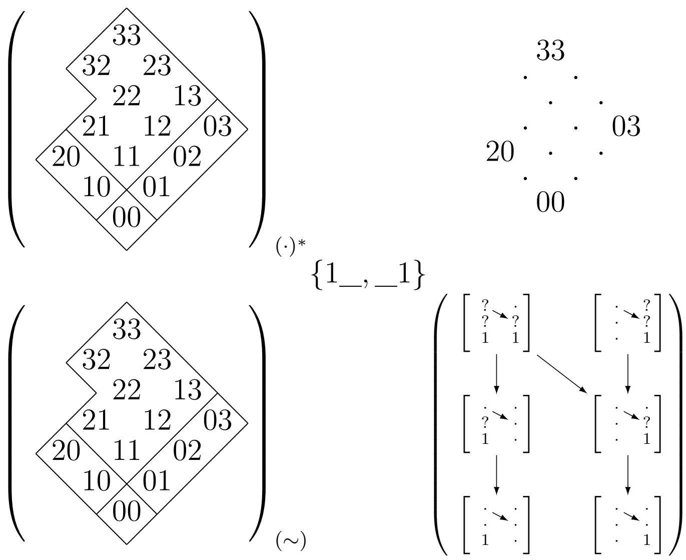
The *X-shaped diagram* associated to a given set of sieves $\mathcal{Y}$ is built as this: 1) take this $\mathcal{Y}$, and use the twelve conversions in any order you like to build the $(\cdot)^*$, the H', the $\sim$, and the J associated to it; the theorems in [Lin14] indicate that the results will be will-defined and don't depend on the order used to define them; 2) draw them in this position,

$$\begin{array}{ccc} (\cdot)^* & & H' \\ & \mathcal{Y} & \\ \sim & & J \end{array}$$

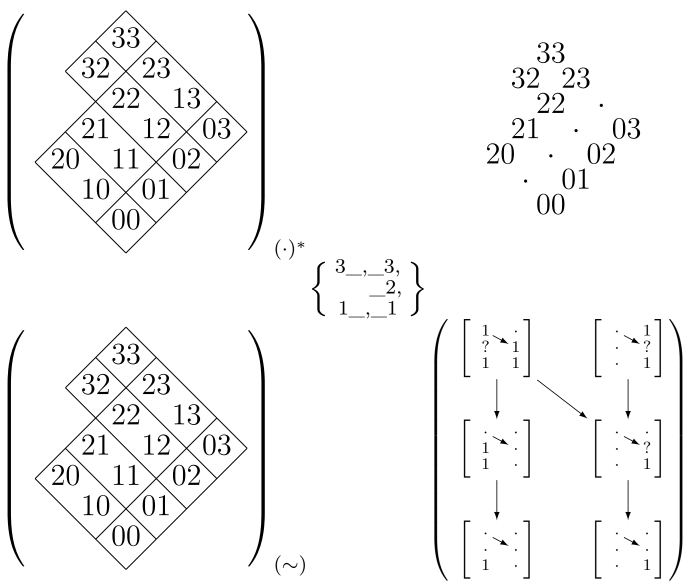
replacing each symbol by its corresponding value.

Here are two examples of X-shaped diagrams.

1. For $\mathcal{Y} = \{1\_, \_1\}$ we get this — compare it with the double negation modality of [PH2, sec.6]:



2. For $\mathcal{Y} = \left\{ \begin{smallmatrix} 3\_, \_3, \\ \_2, \\ 1\_, \_1 \end{smallmatrix} \right\}$ we get this — the "forcing modality" $(\land 21 \land \to 12)$ of [PH2, sec.6]):



15 Extracting meaning from pictures

(This section is a mess)

After calculating a few X-shaped diagrams by brute force we get — or, let me be more honest: *I got* — some visual intuition on what the twelve conversions from section 14 "mean"... in particular, I became able to reconstruct the definitions of the twelve conversions from vague memories of what they should do. My memory is especially bad, and I have to reconstruct definitions and proofs all the time; I wrote a bit about that in [IDARCT, section 10] and [FavC, section 6].

This is an excerpt from a long blog post by Kevin Buzzard ([Buz21]):

> **Mathematicians think in pictures**
>
> I have a picture of the real numbers in my head. It's a straight line. This picture provides a great intuition as to how the real numbers work. I also have a picture of what the graph of a differentiable function looks like. It's a wobbly line with no kinks in. This is by no means a perfect picture, but it will do in many cases. For example: If someone asked me to prove or disprove the existence of a strictly increasing infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(37) = 0$ and $f''(37) < 0$ then I would start by considering a picture of a graph of a strictly increasing function (monotonically increasing as we move from left to right), and a second picture of a function whose derivative at $x = 37$ is zero and whose second derivative is negative (a function with a local maximum). I then note that there are features in these pictures which make them incompatible with each other. Working with these pictures in mind, I can now follow my intuition and write down on paper a picture-free proof that such a function cannot exist, and this proof would be acceptable as a model solution to an exam question. My perception is that other working mathematicians have the same pictures in their head when presented with the same problem, and would go through roughly the same process if they were asked to write down a sketch proof of this theorem.

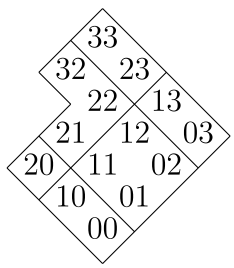
A good part of [Lin14] is about nice properties that we have when our posets are finite, or Artinian, and that do not need to hold in more complex cases — and I don't have a way to discuss them with diagrams yet.

(To do: mention Badiou, and how the non-mathematicians that study Badiou that I know try to understand his uses of toposes)

16 Lawvere-Tierney topologies

It is easy to connect what we saw in the previous sections with Lawvere-Tierney topologies. Let me state some facts without proof.

1. Start with a nucleus $(\cdot)^*$ on a ZHA, and calculate both its associated set of sieves $\mathcal{Y}$ and its 2-column graph with question marks (as in [PH2, section 1.1]). The elements of $\mathcal{Y}$ are exactly the points of the 2CG that do not have question marks. For example, here $\mathcal{Y} = \{3\_, 1\_, \_1, \_2, \_3\}$:



$$\longleftrightarrow \quad \left(\begin{tikzcd}[row sep=small] ?\ 3\_ \arrow[d] \arrow[dr] & \_3 \arrow[d] \\ 2\_ \arrow[d] & \_2\ ? \arrow[d] \\ ?\ 1\_ & \_1 \end{tikzcd}\right)$$

2. Start with a nucleus $(\cdot)^*$ on a ZHA, and calculate its associated Grothendieck topology J and its Lawvere-Tierney topology j. This j is the characteristic map of the inclusion $J \hookrightarrow \Omega$ in the pullback diagram below:

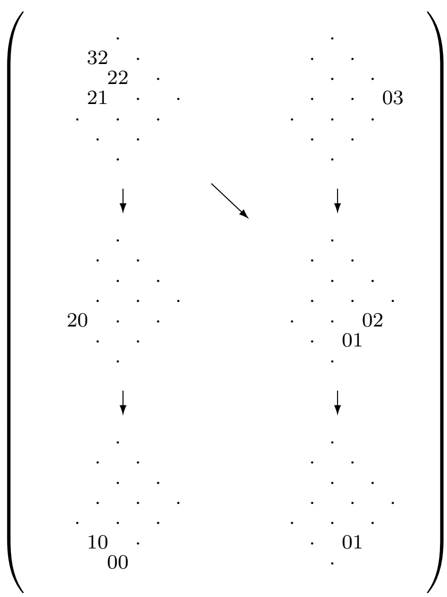
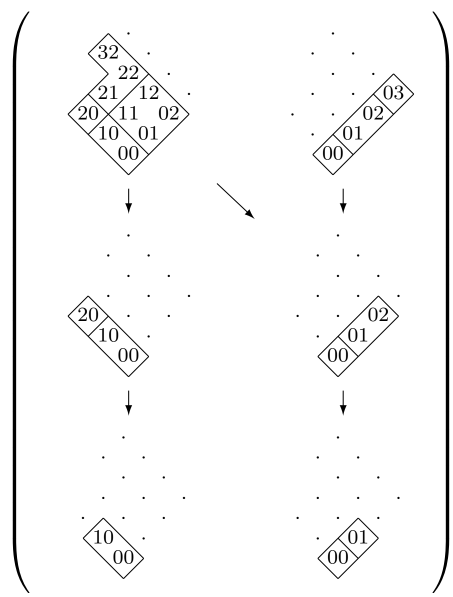
$$\begin{tikzcd} J \arrow[r, "!"] \arrow[d, hook, "i"'] & 1 \arrow[d, hook, "\top"] \\ \Omega \arrow[r, "j:=\chi_i"'] & \Omega \end{tikzcd}$$

Our big diagram in section 11.1 hinted at an alternative way to draw each $J(u)$ and each $\Omega(u)$, in this part:

$$\begin{array}{ccc}
\left[\begin{tikzcd}[row sep=tiny, column sep=small] ? \arrow[dr] & \cdot \\ 1 & ? \\ ? & 1 \end{tikzcd}\right] & \subset & \left[\begin{tikzcd}[row sep=tiny, column sep=small] ? \arrow[dr] & \cdot \\ ? & ? \\ ? & ? \end{tikzcd}\right] \\
\| & & \| \\
\left(\begin{array}{c} \cdot \\ 32\ \ \cdot \\ 22\ \ \cdot \\ 21\ \ \cdot\ \ \cdot \\ \cdot\ \ \cdot\ \ \cdot \\ \cdot\ \ \cdot \\ \cdot \end{array}\right) & \subset & \left(\begin{array}{c} \cdot \\ 32\ \ \cdot \\ 22\ \ \cdot \\ 21\ \ 12\ \ \cdot \\ 20\ \ 11\ \ 02 \\ 10\ \ 01 \\ 00 \end{array}\right)
\end{array}$$

Let's take this alternative way further. For each ex-open set $u \in D$ the map $j(u) : \Omega(u) \to \Omega(u)$ can be drawn in a very nice way as a slashing on just $\Omega(u)$; we draw the points of H that are not in $\Omega(u)$ as dots to indicate that they are outside of the domain, as we did in section 5.

We can calculate the J and the j for the nucleus $(\cdot)^*$ above by brute force, and draw them. We get this:

$J =$  $j =$ 

We have: a) $j(u)(\mathcal{S}) = \mathcal{S}^* \wedge {\downarrow}u$, and b) $J(u) = \{\, \mathcal{S} \in \Omega(u) \mid \mathcal{S}^* = {\downarrow}u \,\}$, i.e., $J(u)$ is made of the points of $\Omega(u)$ in the topmost region of $j(u)$. It is possible to prove that these properties hold in general, and they mean that if we start from a nucleus $(\cdot)^*$ it is very easy to draw the j and the J associated to it: each $j(u)$ is made by "restricting $(\cdot)^*$ to ${\downarrow}u$", and each $J(u)$ in this notation is made of the elements of the topmost region of the corresponing $j(u)$; to draw each $J(u)$ in the other notation we have to erase everything above ${\downarrow}u$ and write the ex-open sets with question marks as '?'s and the other ones as '1'. For example:

$$\left(\begin{array}{ccc} ?\ 3\_ & & \_3 \\ \downarrow & \searrow & \downarrow \\ 2\_ & & \_2\ ? \\ \downarrow & & \downarrow \\ ?\ 1\_ & & \_1 \end{array}\right) \quad \rightsquigarrow \quad \left[\begin{array}{cc} ? & \cdot \\ 1 \searrow & ? \\ ? & 1 \end{array}\right] = J(3\_)$$

17 Etc

TODO: closure operators, sheaves, the sheafification functor. Clean up [Och20] and [PH2]. The functor '+' of [LM92, p.129].

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