Planar Heyting Algebras for Children 2: J-Operators, Slashings, and Nuclei

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Abstract

Every topos admits several "notions of sheafness" on it; for example, one is associated to booleanizing its logic, and for any propositions P and Q on it there is one associated to "forcing $P \to Q$ to be true". How can we visualize them? Or, better: how can we visualize them when we know very little Topos Theory?

Let $\mathbf{D}$ be a finite 2-column graph. Let $\mathbf{E}$ be the topos $\mathbf{Set}^{\mathbf{D}}$, and let H be its Heyting Algebra of truth-values: $H = \mathrm{Sub}(1_{\mathbf{E}})$. Then H is a finite Planar Heyting Algebra (a "ZHA"), and [PH1] shows how to use these ZHAs to visualize how Intuitionistic Propositional Logic works. A *nucleus* on a Heyting Algebra H is an operation $\cdot^* : H \to H$ that obeys $P \leq P^* = P^{**}$ and $(P \wedge Q)^* = P^* \wedge Q^*$; we will show to visualize these nuclei on Heyting Algebras that are ZHAs, and how to use that as a first step towards understanding the bijection between nuclei and notions of sheafness.

We will use the term *J-operator* for a nucleus that acts on a ZHA, and the first sections of this paper will be dedicated to: seeing in elementary terms how these J-operators work, proving that each J-operator on a ZHA H corresponds to a way to slash H by diagonal cuts that do not stop midway, and seeing how to visualize some famous J-operators, like booleanization and forcing.

In the last two sections we will see how to start from this knowledge of J-operators to learn some Topos Theory. If the reader is willing to believe a small list of (provable) statements then he will be able to convert any J-operator $(\cdot)^*$ to the a Lawvere-Tierney topology j and a sheafification functor on our topos $\mathbf{E} = \mathbf{Set}^{\mathbf{D}}$, and to visualize in particular cases what many theorems in, say, [Bel88], "mean" — and to understand the theory by working on a particular case and on the general case in parallel, using the techniques for doing "categories for children" explained in [FavC].

Contents

Status of these notes: this is my n-th attempt to rewrite these notes to make them publishable as a journal article, or at least publishable on Arxiv. The sections 0–6.3 are practically in final form; most of them are stable since 2019, but I rewrote the sections 0–1.3 and 3.1 in August 2021 and I need to revise them. Everything from the section 7 onwards will be totally rewritten to make this paper work a complement to [Och20] and [FavC] and as a preparation and motivation to them for people who know very little Category Theory.

For the most recent version see:
`http://angg.twu.net/math-b.html#zhas-for-children-2`

0 Background

One of the main constructions of [PH1] is a bijection between (proper) 2-column graphs and Planar Heyting Algebras. For example, in

$$(P,A) = \left(\begin{tikzcd} & \_5 \arrow[d] \\ & \_4 \arrow[d] \arrow[ddl] \\ 3\_ \arrow[d] & \_3 \arrow[d] \\ 2\_ \arrow[d] \arrow[r] & \_2 \arrow[d] \\ 1\_ & \_1 \end{tikzcd} \right) \qquad H = \mathcal{O}_A(P) = \begin{array}{ccccc} &&& 35 & \\ && 34 && 25 \\ & 33 && 24 & \\ 32 && 23 && \\ & 22 && 13 & \\ && 12 && 03 \\ & 11 && 02 & \\ 10 && 01 && \\ & 00 &&& \end{array}$$

the 2-column graph (P,A) has left height 3 and right height 5; its set of points is

$$P \;\; = \;\; \{3\_, \ldots, 1\_, \; \_1, \ldots, \_5\}$$

and its set of arrows A is made of all the vertical, or intra-column, arrows going one step down plus two intercolumn arrows: $2\_ \leftarrow \_4$ and $2\_ \to \_2$. We will use the 2-column graph (P,A) above in all examples in this section.

A *pile* is a subset of P of the form:

$$\mathsf{pile}(ab) \;\; = \;\; \{a\_, \ldots, 1\_, \; \_1, \ldots, \_b\}$$

We say that a subset $U \subseteq P$ *obeys* an arrow $v \to w$ when it obeys $v \in U \to w \in U$; for example, $\mathsf{pile}(14)$ obeys $2\_ \to \_2$ but *violates* $2\_ \leftarrow \_4$. The subsets of P that obey all the vertical arrows are exactly the piles.

The *order topology* $\mathcal{O}_A(P)$ is the set:

$$\begin{array}{rcl} \mathcal{O}_A(P) & = & \{\, U \subseteq P \mid \forall (v \to w) \in A.\, U \text{ obeys } v \to w \,\} \\ & = & \{\, U \subseteq P \mid U \text{ obeys all arrows in } A \,\} \end{array}$$

In contexts in which a 2CG (P,A) is defined the letter H will always denote the order topology $\mathcal{O}_A(P)$ regarded as a Heyting Algebra.

Section 3 of [PH1] defines a way to interpret each $ab \equiv \mathsf{pile}(ab)$ as a point of $\mathbb{Z}^2$, by:

$$ab \;\; \equiv \;\; (0,0) + a\overrightarrow{(-1,1)} + b\overrightarrow{(1,1)}$$

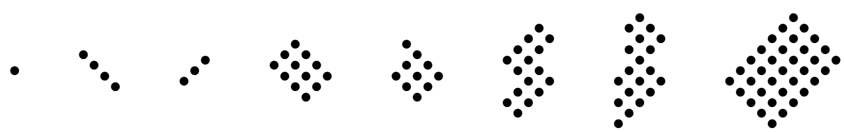
i.e.: start at $(0,0)$, then walk a steps northwest and b steps northeast. The '$\equiv$' here is pronounced "can be interpreted as".

It is easy to see that $22 \in \mathcal{O}_A(P)$ but $21 \notin \mathcal{O}_A(P)$; the arrow $2\_ \to \_2$ forbids 21, and all the piles that can be obtained by walking southwest and northwest from 21 — let's denote the set of those piles by $\mathsf{swnw}(21)$ — are also forbidden. Similarly, 24 is open but 14 is not, and $2\_ \leftarrow \_4$ forbids all piles in the set $\mathsf{sene}(14)$. If we draw all piles and then erase the ones forbidden by $2\_ \to \_2$ and $2\_ \leftarrow \_4$ we get exactly all the piles in the $\mathcal{O}_A(P)$ of the example above, and this holds in general.

We say that two piles ab and cd in $\mathcal{O}_A(P)$ are *neighbors* — notation: $ab \sim_1 cd$ — when they differ by exactly one element; for example, $24 \sim_1 25$ but $34 \not\sim_1 25$ and $25 \not\sim_1 25$. Let $\sim_1^*$ be the transitive-reflexive closure of $\sim_1$, and let's say that $\mathcal{O}_A(P)$ is $\sim_1^*$*-connected* if all piles of $\mathcal{O}_A(P)$ are $\sim_1^*$-equivalent. Section 15 of [PH1] shows an $\mathcal{O}_A(P)$ is $\sim_1^*$-connected iff (P, A) is acyclic.

We will say that a 2-column graph (P, A) is *proper* iff it is finite and acyclic. A *Planar Heyting Algebra* (or: a "ZHA") is a finite subset of $\mathbb{Z}^2$ that "is" the order topology for a proper 2CG. From here onwards all our 2CGs will be implicitly proper.

Here are some examples of ZHAs (drawn with bullets insted of with 2-digit numbers):

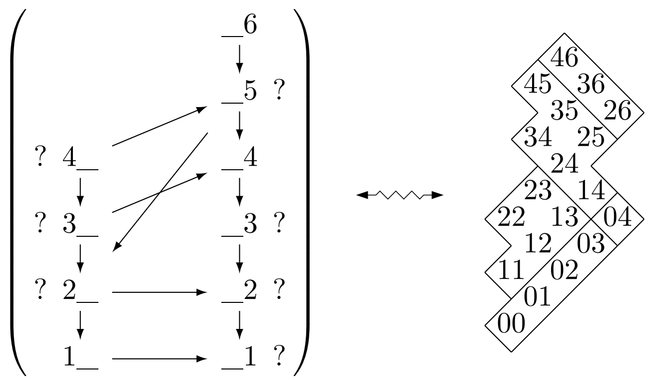


The two *basic* themes in [PH1] are that we can interpret Intuitionistic Propositional Logic on ZHAs and that we can use ZHAs to develop *visual intuition* about IPL. Here we will take that one step ahead. Take IPL and add a modal operator $\cdot^*$ to it, with axioms that assert that $P \le P^* \le P^{**}$ and $(P \land Q)^* = P^* \land Q^*$. Call this new logic IPL*. Here we will see how to use ZHAs with slashings to develop visual intuition about IPL*, and in the last sections we will see how to *use* this visual intuition to learn some ideas about toposes and sheaves, and we will see how to *formalize* what this "visual intuition" works.

1 Question marks and slashings

A *set of question marks* on a 2CG (P, A) is a subset $Q \subseteq P$. We write a 2CG with question marks as $((P, A), Q)$, and we represent this Q graphically by writing a '?' close to each element of P that belongs to Q, as in the figure below. The intended meaning of these question marks is that we want to forget the information on them and then see which elements of $\mathcal{O}_A(P)$ become indistinguishable after this forgetting: two elements $ab, cd \in H$ are *Q-equivalent*, written as $ab \sim_Q cd$, iff $\mathsf{pile}(ab) \backslash Q = \mathsf{pile}(cd) \backslash Q$. In the $((P, A), Q)$ of the figure below we have $23 \sim_Q 13 \not\sim_Q 14$.

A *slashing* S on a ZHA H is a set of diagonal cuts on H "that do not stop midway". These cuts are interpreted as fences that divide H in separate regions, and two elements $ab, cd \in H$ are *S-equivalent*, written as $ab \sim_S cd$, if they belong to the same region. In the slashing at the right in the figure below we have $11 \sim_S 23 \not\sim_S 14$.



In [PH1] we used the notation $(P, A) \leftsquigarrow\!\rightsquigarrow H$ to say that H is the ZHA associated to the 2CG (P, A); this "is associated to" was interpreted formally as $\mathcal{O}_A(P) = H$. We are now extending this to $((P, A), Q) \leftsquigarrow\!\rightsquigarrow (H, S)$ — a 2CG with question marks $((P, A), Q)$ is associated to the ZHA with slashing (H, S) when we have $\mathcal{O}_A(P) = H$ and the equivalence relations $\sim_Q, \sim_S \subseteq H \times H$ coincide. Note that the two '$\leftsquigarrow\!\rightsquigarrow$'s are both pronounced as "is associated to", but they have different formal meanings.

1.1 Piccs and slashings

A picc ("partition into contiguous classes") of a "discrete interval" $I = \{0, \ldots, n\}$ is a partition P of I that obeys this condition ("picc-ness"):

$$\forall a, b, c \in \{0, \ldots, n\}. \, (a < b < c \land a \sim_P c) \to (a \sim_P b \land b \sim_P c).$$

So $P = \{\{0\}, \{1, 2, 3\}, \{4, 5\}\}$ is a picc of $\{0, \ldots, 5\}$, and

$$P' = \{\{0\}, \{1, 2, 4, 5\}, \{3\}\}$$

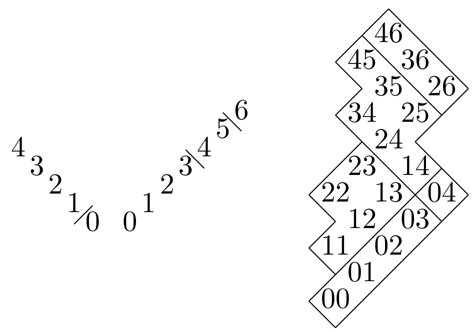
is a partition of $\{0, \ldots, 5\}$ that is not a picc.

A short notation for piccs is this:

$$0|123|45 \equiv \{\{0\},\{1,2,3\},\{4,5\}\}$$

we list all digits in the (discrete) interval in order, and we put bars to indicate where we change from one equivalence class to another.

We will represent a slashing S formally as pairs of piccs, one for the left digit and one for the right digit. Our notation for slashings as pairs will be based on this figure:



The slashing S that we are using in our examples will be represented as:

$$\begin{array}{rcl}
S & = & (L,R) \\
 & = & (\{\{0\},\{1,2,3,4\}\},\ \{\{0,1,2,3\},\{4,5\},\{6\}\}) \\
 & = & (0|1234, 0123|45|6) \\
 & = & (4321/0,\ 0123\backslash 45\backslash 6)
\end{array}$$

We use '/'s and '\'s instead of '|'s to remind us of the direction of the cuts: the '/'s correspond to cuts that go northeast and the '\'s to cuts that go northwest.

We can now define the equivalence relation $\sim_S$ formally: if $S = (L,R)$ then $ab \sim_S cd$ iff $a \sim_L c$ and $c \sim_R d$.

The expression "$S = (L,R)$ is a slashing on H" will mean: H is a ZHA, L is a picc on $\{0,\dots,l\}$, and R a picc on $\{0,\dots,r\}$, where lr is the top element of H. The domain of the equivalence relation $\sim_S$ will be considered to be H, not $\{0,\dots,l\} \times \{0,\dots,r\}$.

1.2 Slash-operators

When $S = (L,R)$ is a slashing on H we will use the notations $[\cdot]^L$, $[\cdot]^R$, $[\cdot]^S$ for the equivalence classes of L, R, S and the notations $\cdot^L$, $\cdot^R$, $\cdot^S$ for the highest element in those equivalence classes. In our example we have $[2]^L = \{1,2,3,4\}$, $[2]^R = \{0,1,2,3\}$, $[22]^S = \{11,12,13,22,23\}$, $2^L = 4$, $2^R = 4$, $2^S = 23$. Note that $[a]^L \times [b]^R$, that we define as

$$[a]^L \times [b]^R \quad = \quad \{\, cd \mid c \in [a]^L, d \in [b]^R \,\}$$

is a rectangle (tilted 45°) that may contain piles that are not open; for example, $04 \in [2]^L \times [2]^R$, and $[22]^S \subsetneq [2]^L \times [2]^R$.

A *slash-operator* on a ZHA H is a function $\cdot^F : H \to H$ that is equal to some $\cdot^S$. Let's do that more explicitly. A function $\cdot F : H \to H$ if a slash-operator iff there exists a slashing S on the ZHA H such that $\cdot^F = \cdot^S$.

Supppose that we have a ZHA H and an arbitrary function $\cdot^F : H \to H$ on it. Suppose that the top element of H is lr. We can define $\sim_L$, $\sim_R$, L, R from that $\cdot^F$ in the following way.

First we define a relation $L_F \subset \{1,\ldots,l\}^2$ in which $a\,L_F\,c$ is true if and only if there are $ab, cd \in H$ with $ab^F = cd$. We then define $\sim_L$ as the transitive-reflexive closure of L_F, and we define the partition L of $\{1,\ldots,l\}$ as the set of equivalence classes of $\sim_L$. We do the same to define $R_F \subset \{1,\ldots,r\}^2$, $\sim_R$, and R. If this L is not a picc on $\{1,\ldots,l\}$, or if this R is not a picc on $\{1,\ldots,r\}$, we stop: our original $\cdot^F$ is not a slash-operator. If both L and R are piccs, we define $\cdot^L$, $\cdot^R$, and $\cdot^S$ as in the beginning of the section, and we test if this $\cdot^S$ if equal to our original $\cdot^F$. If they are equal then our $\cdot^F$ is a slash-operator; if $\cdot^S \neq \cdot^F$ then our $\cdot^F$ is not a slash-operator.

1.3 From slashings to question marks and vice-versa

Let's write $A \bigtriangleup B$ for the symmetric difference between two sets, and H^2_u for the subset of H^2 formed by the pairs of neighboring points of H whose difference is exactly u:

$$H^2_u = \{\, (ab, cd) \in H^2 \mid ab \bigtriangleup cd = \{u\} \,\}$$

There are several ways to convert a slashing to question marks and vice-versa. They are all based on this idea: if one pair $(ab, cd) \in H^2_u$ is S-equivalent, then all the other pairs will also be, and this means that all these pairs have to be Q-equivalent — which means $u \in Q$. So:

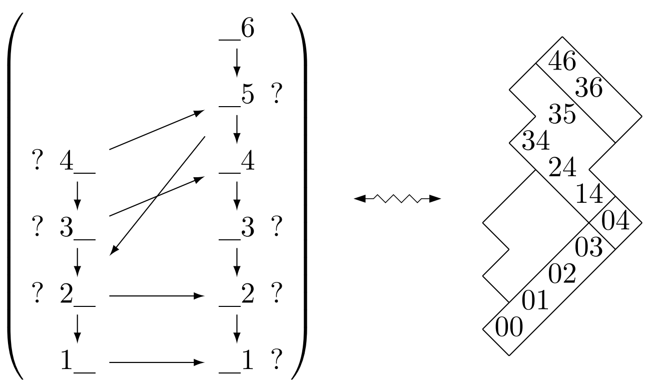
$$\begin{array}{rcl} Q & = & \{\, u \in P \mid \exists (ab, cd) \in H^2_u .\ ab \sim_S cd \,\} \\ & = & \{\, u \in P \mid \forall (ab, cd) \in H^2_u .\ ab \sim_S cd \,\} \end{array}$$

Here is a simple way to do that conversion visually. Choose any path from the bottom element of the ZHA to its top element that is made of one unit steps northwest or northeast — for example, this one:

$$(a_0b_0, a_1b_1, \ldots a_{10}b_{10}) = (00, 01, 02, 03, 04, 14, 24, 34, 35, 36, 46)$$

If we are converting from a slashing to question marks, then for each step from one element of the ZHA to the next one, say, from ab to cd, check if we have crossed one of the cuts of the slashing; if we haven't then we've moved between two S-equivalent points, and as they should also be Q-equivalent we add their difference $ab \bigtriangleup cd$ to Q. If we are converting from question marks to slashings then every time that we move from a point ab to a point cd and their difference $ab \bigtriangleup cd$ is not a question mark point then we draw a cut separating ab and cd.

In a diagram:



To convert from slashings to question marks:
We have $00 \sim_S 01 \sim_S 02 \sim_S 03$, so $\_1, \_2, \_3 \in Q$.
We have $14 \sim_S 24 \sim_S 34$ and $34 \sim_S 35$, so $2\_, 3\_ \in Q$ and $\_5 \in Q$.
We have $36 \sim_S 46$, so $\_4 \in Q$.

To convert from question marks to slashings:
We have $03 \not\sim_Q 04$, so we draw a cut between 03 and 04 (3\4).
We have $04 \not\sim_Q 14$, so we draw a cut between 04 and 14 (1/0).
We have $35 \not\sim_Q 36$, so we draw a cut between 35 and 36 (5\6).
Our slashing is 4321/0 0123\45\6.

2 J-operators

A *J-operator* on a Heyting Algebra $H \equiv (H, \leq, \top, \bot, \land, \lor, \to, \leftrightarrow, \neg)$ is a function $J : H \to H$ that obeys the axioms J1, J2, J3 below; we usually write J as $\cdot^* : H \to H$, and write the axioms as rules.

$$\frac{}{P \leq P^*}\ \mathsf{J1} \qquad \frac{}{P^* = P^{**}}\ \mathsf{J2} \qquad \frac{}{(P \land Q)^* = P^* \land Q^*}\ \mathsf{J3}$$

J1 says that the operation $\cdot^*$ is non-decreasing.
J2 says that the operation $\cdot^*$ is idempotent.
J3 is a bit mysterious but will have interesting consequences.

A J-operator induces an equivalence relation and equivalence classes on H, like slashings do:

$$\begin{array}{rcl} P \sim_J Q & \text{iff} & P^* = Q^* \\ [P]^J & := & \{\, Q \in H \mid P^* = Q^* \,\} \end{array}$$

The equivalence classes of a J-operator J are called *J-regions*.

The axioms J1, J2, J3 have many consequences. The first ones are listed in Figure 1 as derived rules, whose names mean:
Mop (monotonicity for products): a lemma used to prove Mo,
Mo (monotonicity): $P \leq Q$ implies $P^* \leq Q^*$,
Sand (sandwiching): all truth values between P and P^* are equivalent,
EC&: equivalence classes are closed by '&',
EC∨: equivalence classes are closed by '∨',
ECS: equivalence classes are closed by sandwiching,

Take a J-equivalence class, $[P]^J$, and list its elements: $[P]^J = \{P_1, \ldots, P_n\}$. Let $P_\land := ((P_1 \land P_2) \land \ldots) \land P_n$ and $P_\lor := ((P_1 \lor P_2) \lor \ldots) \lor P_n$. Clearly $P_\land \leq P_i \leq P_\lor$ for each i, so $[P]^J \subseteq [P_\land, P_\lor]$. We will use the interval notation $[P, R]$ to mean the set of all elements of H obeying $P \leq Q \leq R$:

$$[P, R] \;=\; \{\, Q \in H \mid P \leq Q \leq R \,\}.$$

Using EC& and EC∨ several times we see that:

$$\begin{array}{rr} P_1 \land P_2 \sim_J P & P_1 \lor P_2 \sim_J P \\ (P_1 \land P_2) \land P_3 \sim_J P & (P_1 \lor P_2) \lor P_3 \sim_J P \\ \vdots & \vdots \\ ((P_1 \land P_2) \land \ldots) \land P_n \sim_J P & ((P_1 \lor P_2) \lor \ldots) \lor P_n \sim_J P \\ P_\land \sim_J P & P_\lor \sim_J P \\ P_\land \in [P]^J & P_\lor \in [P]^J \end{array}$$

$$\dfrac{}{(P \wedge Q)^* \leq Q^*}\,\mathsf{Mop} \quad := \quad \dfrac{\dfrac{}{(P \wedge Q)^* = P^* \wedge Q^*}\,\mathsf{J3} \qquad \dfrac{}{P^* \wedge Q^* \leq Q^*}}{(P \wedge Q)^* \leq Q^*}$$

$$\dfrac{P \leq Q}{P^* \leq Q^*}\,\mathsf{Mo} \quad := \quad \dfrac{\dfrac{\dfrac{P \leq Q}{P = P \wedge Q}}{P^* = (P \wedge Q)^*} \qquad \dfrac{}{(P \wedge Q)^* \leq Q^*}\,\mathsf{Mop}}{P^* \leq Q^*}$$

$$\dfrac{P \leq Q \leq P^*}{P^* = Q^*}\,\mathsf{Sand} \quad := \quad \dfrac{\dfrac{P \leq Q}{P^* \leq Q^*}\,\mathsf{Mo} \qquad \dfrac{\dfrac{Q \leq P^*}{Q^* \leq P^{**}}\,\mathsf{Mo} \qquad \dfrac{}{P^{**} = P^*}\,\mathsf{J2}}{Q^* \leq P^*}}{P^* = Q^*}$$

$$\dfrac{P^* = Q^*}{P^* = Q^* = (P \wedge Q)^*}\,\mathsf{EC\&} \quad := \quad \dfrac{\dfrac{P^* = Q^*}{P^* = Q^* = P^* \wedge Q^*} \qquad \dfrac{}{P^* \wedge Q^* = (P \wedge Q)^*}\,\mathsf{J3}}{P^* = Q^* = (P \wedge Q)^*}$$

$$\dfrac{P^* = Q^*}{P^* = Q^* = (P \vee Q)^*}\,\mathsf{EC\vee} \quad := \quad \dfrac{P^* = Q^* \qquad \dfrac{\dfrac{\dfrac{}{P \leq P \vee Q} \qquad \dfrac{\dfrac{}{P \leq P^*}\,\mathsf{J1} \qquad \dfrac{\dfrac{}{Q \leq Q^*}\,\mathsf{J1} \qquad \dfrac{P^* = Q^*}{Q^* = P^*}}{Q \leq P^*}}{P \vee Q \leq P^*}}{P \leq P \vee Q \leq P^*}}{P^* = (P \vee Q)^*}\,\mathsf{Sand}}{P^* = Q^* = (P \vee Q)^*}$$

$$\dfrac{P \leq Q \leq R \qquad P^* = R^*}{P^* = Q^* = R^*}\,\mathsf{ECS} \quad := \quad \dfrac{\dfrac{\dfrac{P \leq Q \leq R \qquad \dfrac{}{R \leq R^*}\,\mathsf{J1} \qquad \dfrac{P^* = R^*}{R^* = P^*}}{P \leq Q \leq P^*}}{P^* = Q^*}\,\mathsf{Sand} \qquad P^* = R^*}{P^* = Q^* = R^*}$$

Figure 1: J-operators: basic derived rules

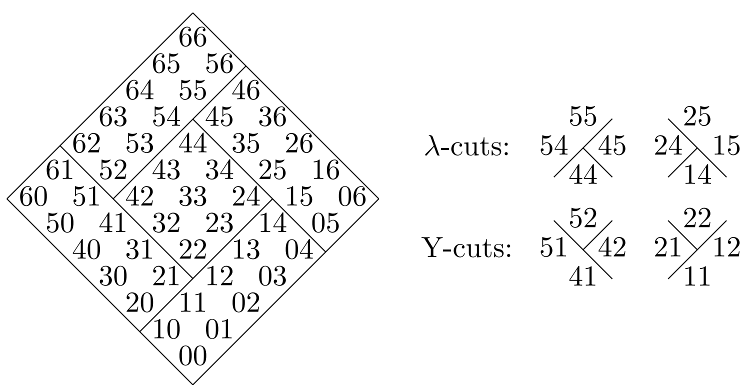
and using ECS we can see that all elements between $P_\wedge$ and $P_\vee$ are J-equivalent to P:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P_\wedge \leq Q \leq P_\vee} \quad \dfrac{\dfrac{\dfrac{P_\wedge \sim_J P}{{P_\wedge}^* = P^*} \quad \dfrac{P_\vee \sim_J P}{{P_\vee}^* = P^*}}{{P_\wedge}^* = {P_\vee}^*}}{}}{{P_\wedge}^* = Q^* = {P_\vee}^*}\,\text{ECS} \quad {P_\vee}^* = P^*}{Q^* = P^*}}{}}{Q \sim_J P}$$

so $[P_\wedge, P_\vee] \subseteq [P]^J$. This means that *$J$-regions are intervals.*

3 Cuts stopping midway

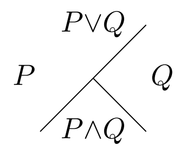
Look at the figure at the left below, that shows a partition of a ZHA $A = [00, 66]$ into five regions, each region being an interval; this partition does not come from a slashing, as it has (four) cuts that stop midway. They are detailed at the right; the ones in which the cuts look like a 'Y' will be called Y-cuts, and the ones that look like 'λ's will be called λ-cuts. Define an operation '$\cdot^*$' on A, that works by taking each truth-value P in it to the top element of its region; for example, $30^* = 61$.

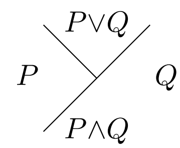


It is easy to see that '$\cdot^*$' obeys J1 and J2; however, it does *not* obey J3 — we will prove that in sec.3.1. As we will see, *the partitions of a ZHA into intervals that obey* J1, J2, J3 *ae exactly the slashings;* or, in other words, *every J-operator comes from a slashing.*

3.1 The are no Y-cuts and no λ-cuts

Let's start with these particular cases of a λ-cut and a Y-cut:

λ-cut: 

Y-cut: 

One way to prove that λ-cuts can't happen when J1, J2, and J3 all hold is to show a proof of $(P \sim_J P{\vee}Q) \to (P{\wedge}Q \sim_J Q)$ that uses only J1, J2, J3 and the axioms of Heyting Algebras; and similarly, we can prove that Y-cuts can't happen by showing a proof of $(P \sim_J P{\vee}Q) \leftarrow (P{\wedge}Q \sim_J Q)$. Here are the proofs, with the proof of "λ-cuts can't happen" first:

$$\dfrac{\dfrac{\dfrac{P^* = (P \vee Q)^*}{P^* \wedge Q^* = (P \vee Q)^* \wedge Q^*}}{(P \wedge Q)^* = ((P \vee Q) \wedge Q)^*}\;\mathsf{J3} \qquad \dfrac{}{(P \vee Q) \wedge Q = Q}}{(P \wedge Q)^* = Q^*}$$

$$\dfrac{\dfrac{}{P = P \vee (P \wedge Q)} \qquad \dfrac{\dfrac{\dfrac{(P \wedge Q)^* = Q^*}{P \vee (P \wedge Q)^* = P \vee Q^*}}{(P \vee (P \wedge Q)^*)^* = (P \vee Q^*)^*}}{(P \vee (P \wedge Q))^* = (P \vee Q)^*}\;\circledvee_6 = \circledvee_4}{P^* = (P \vee Q)^*}$$

The expansion of the double bar labeled '$\circledvee_6 = \circledvee_4$' uses (twice) a derived rule with that name, that can be obtained from the '$\circledvee$-cubes' of sec.4.

4 How J-operators interact with connectives

The axiom J3 says that $(P \land Q)^* = P^* \land Q^*$ — it says something about how '$\cdot^*$' interacts with '$\land$'. Let's introduce a shorter notation. There are eight ways to replace each of the '?'s in $(P^? \land Q^?)^?$ by either nothing or a star. We establish that the three '?'s in $(P^? \land Q^?)^?$ are "worth" 1, 2 and 4 respectively, and we use $P \owedge_n Q$ to denote $(P^? \land Q^?)^?$ with the bits "that belong to n" replaced by stars. So:

$$\begin{array}{rclrcl}
\owedge_0 & = & P \land Q, & \owedge_4 & = & (P \land Q)^*, \\
\owedge_1 & = & P^* \land Q, & \owedge_5 & = & (P^* \land Q)^*, \\
\owedge_2 & = & P \land Q^*, & \owedge_6 & = & (P \land Q^*)^*, \\
\owedge_3 & = & P^* \land Q^*, & \owedge_7 & = & (P^* \land Q^*)^*.
\end{array}$$

We omit the arguments of $\owedge_n$ when they are P and Q — so we can rewrite $(P \land Q)^* = P^* \land Q^*$ as $\owedge_4 = \owedge_3$. These conventions also hold for $\ovee$ and $\oimp$.

It is easy to prove each one of the arrows in the cubes below ($A \longrightarrow B$ means $A \leq B$):

$$\begin{tikzcd}
& (P^*\land Q^*)^* & \\
(P^*\land Q)^* \arrow[ur] & P^*\land Q^* \arrow[u] & (P\land Q^*)^* \arrow[ul] \\
P^*\land Q \arrow[u] \arrow[ur] & (P\land Q)^* \arrow[ul] \arrow[ur] & P\land Q^* \arrow[u] \arrow[ul] \\
& P\land Q \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& (P^*\lor Q^*)^* & \\
(P^*\lor Q)^* \arrow[ur] & P^*\lor Q^* \arrow[u] & (P\lor Q^*)^* \arrow[ul] \\
P^*\lor Q \arrow[u] \arrow[ur] & (P\lor Q)^* \arrow[ul] \arrow[ur] & P\lor Q^* \arrow[u] \arrow[ul] \\
& P\lor Q \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& (P^*\to Q^*)^* \arrow[dr] & \\
(P^*\to Q)^* \arrow[ur] \arrow[dr] & P^*\to Q^* \arrow[u] \arrow[dr] & (P\to Q^*)^* \\
P^*\to Q \arrow[u] \arrow[ur] \arrow[dr] & (P\to Q)^* \arrow[ur] & P\to Q^* \arrow[u] \\
& P\to Q \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Let's write their sets of elements as $\owedge_{0\ldots7} := \{\owedge_0, \ldots, \owedge_7\}$, $\ovee_{0\ldots7} := \{\ovee_0, \ldots, \ovee_7\}$, and $\oimp_{0\ldots7} := \{\oimp_0, \ldots, \oimp_7\}$. The cubes above — we will call them the "obvious and-cube", the "obvious or-cube", and the "obvious implication-cube" — can be interpreted as directed graphs $(\owedge_{0\ldots7}, \mathsf{OCube}_\land)$, $(\ovee_{0\ldots7}, \mathsf{OCube}_\lor)$, $(\oimp_{0\ldots7}, \mathsf{OCube}_\to)$.

The "extended cubes" will be the directed graphs with the arrows above plus the ones coming from these derived rules:

$$\frac{}{(P^* \land Q^*)^* = P^* \land Q^* = (P \land Q)^*}\ \owedge_7 = \owedge_3 = \owedge_4 \quad := \quad \dfrac{\dfrac{\dfrac{}{P^{**} = P^*}\ \mathsf{J2} \quad \dfrac{}{Q^{**} = Q^*}\ \mathsf{J2}}{(P^* \land Q^*)^* = P^{**} \land Q^{**} = P^* \land Q^* = (P \land Q)^*}\ \mathsf{J3}}{(P^* \land Q^*)^* = P^* \land Q^* = (P \land Q)^*}$$

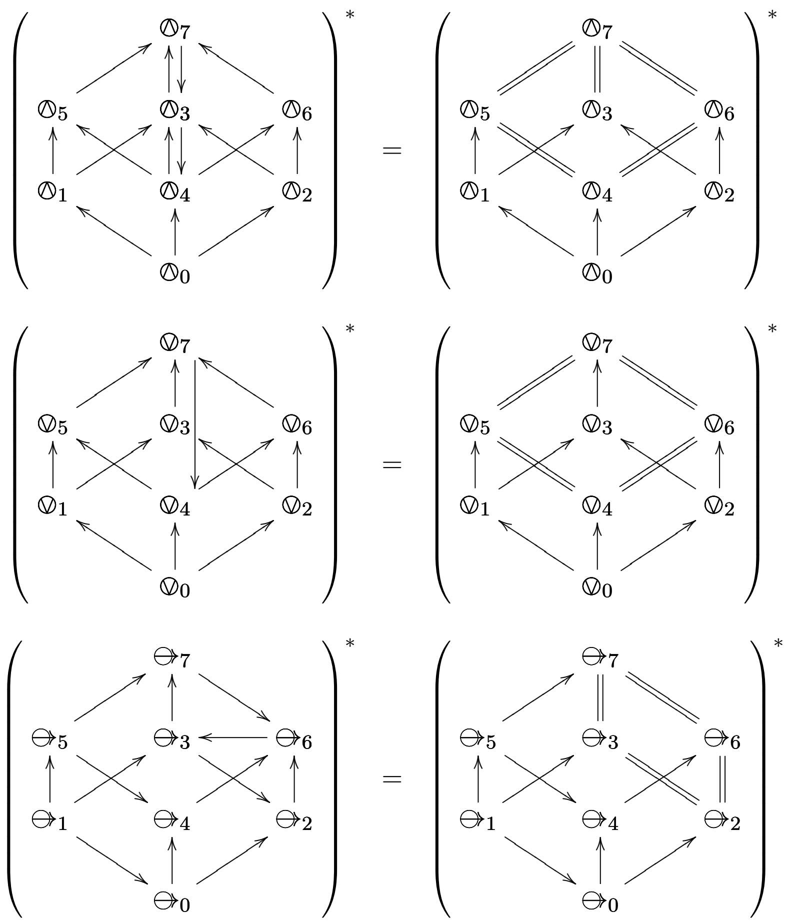
$$\frac{}{(P^* \lor Q^*)^* \leq (P \lor Q)^*}\ \ovee_7 \leq \ovee_3 \quad := \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \leq P \lor Q}}{P^* \leq (P \lor Q)^*}\ \mathsf{Mo} \quad \dfrac{\dfrac{}{Q \leq P \lor Q}}{Q^* \leq (P \lor Q)^*}\ \mathsf{Mo}}{P^* \lor Q^* \leq (P \lor Q)^*}}{(P^* \lor Q^*)^* \leq (P \lor Q)^{**}}\ \mathsf{Mo}}{(P^* \lor Q^*)^* \leq (P \lor Q)^*}\ \mathsf{J2}$$

$$\frac{}{(P \to Q^*)^* \leq P^* \to Q^*}\ \ominus_6 \leq \ominus_3 \quad := \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \to Q^* \leq P \to Q^*}}{(P \to Q^*) \land P \leq Q^*}}{((P \to Q^*) \land P)^* \leq Q^{**}}\ \mathsf{Mo}}{((P \to Q^*) \land P)^* \leq Q^*}\ \mathsf{J2}}{(P \to Q^*)^* \land P^* \leq Q^*}\ \mathsf{J3}}{(P \to Q^*)^* \leq P^* \to Q^*}$$

where $\owedge_7 = \owedge_3 = \owedge_4$ will be interpreted as these arrows:

$$(P^* \land Q^*)^* \rightleftarrows P^* \land Q^* \leftrightarrows (P \land Q)^*$$

The directed graphs of these "extended cubes" will be called $(\owedge_{0...7}, \mathsf{ECube}_\land)$, $(\ovee_{0...7}, \mathsf{ECube}_\lor)$, $(\ominus_{0...7}, \mathsf{ECube}_\to)$. We are interested in the (non-strict) partial orders that they generate, and we want an easy way to remember these partial orders. The figure below shows these extended cubes at the left, and at the right the "simplified cubes", $\mathsf{SCube}_\land$, $\mathsf{SCube}_\lor$, and $\mathsf{SCube}_\to$, that generate the same partial orders that the extended cubes.



From these cubes it is easy to see, for example, that we can prove $\circledvee_5 = \circledvee_6$ (as a derived rule).

5 Valuations

Let $H_\odot$ and $J_\odot$ be a ZHA and a J-operator on it, and let $v_\odot$ be a function from the set $\{P, Q\}$ to H. By an abuse of language $v_\odot$ will also denote the triple $(H_\odot, J_\odot, v_\odot)$ — and by a second abuse of language $v_\odot$ will also denote the obvious extension of $v_\odot : \{P, Q\} \to H$ to the set of all valid expressions formed from P, Q, $\cdot^*$, $\top$, $\bot$, and the connectives.

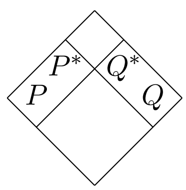
Let $i, j \in \{0, \ldots, 7\}$. Then $(\owedge_i, \owedge_j) \in \mathsf{SCube}^*_\wedge$ means that $\owedge_i \leq \owedge_j$ is a theorem, and so $v_\odot(\owedge_i) \leq v_\odot(\owedge_j)$ holds; i.e.,

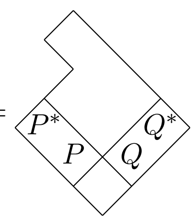
$$\mathsf{SCube}^*_\wedge \subseteq \{\, (\owedge_i, \owedge_j) \mid i, j \in \{0, \ldots, 7\},\ v_\odot(\owedge_i) \leq v_\odot(\owedge_j) \,\}$$

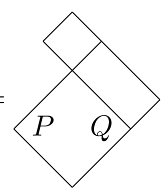
and the same for:

$$\begin{array}{r}
\mathsf{SCube}^*_\vee \subseteq \{\, (\ovee_i, \ovee_j) \mid i, j \in \{0, \ldots, 7\},\ v_\odot(\ovee_i) \leq v_\odot(\ovee_j) \,\} \\
\mathsf{SCube}^*_\to \subseteq \{\, (\ominus\!\!\to_i, \ominus\!\!\to_j) \mid i, j \in \{0, \ldots, 7\},\ v_\odot(\ominus\!\!\to_i) \leq v_\odot(\ominus\!\!\to_j) \,\}
\end{array}$$

Some valuations that turn these '$\subseteq$'s into '$=$'. Let

$(H_\wedge, J_\wedge, v_\wedge) =$ 

$(H_\vee, J_\vee, v_\vee) =$ 

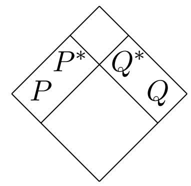
$(H_\to, J_\to, v_\to) =$ 

then

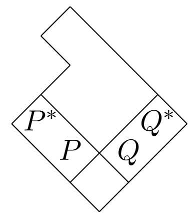
$$\begin{array}{r}
\mathsf{SCube}^*_\wedge = \{\, (\owedge_i, \owedge_j) \mid i, j \in \{0, \ldots, 7\},\ v_\wedge(\owedge_i) \leq v_\wedge(\owedge_j) \,\} \\
\mathsf{SCube}^*_\vee = \{\, (\ovee_i, \ovee_j) \mid i, j \in \{0, \ldots, 7\},\ v_\vee(\ovee_i) \leq v_\vee(\ovee_j) \,\} \\
\mathsf{SCube}^*_\to = \{\, (\ominus\!\!\to_i, \ominus\!\!\to_j) \mid i, j \in \{0, \ldots, 7\},\ v_\to(\ominus\!\!\to_i) \leq v_\to(\ominus\!\!\to_j) \,\}
\end{array}$$

or, in more elementary terms:

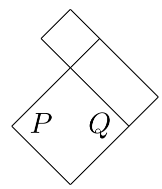
A very important fact. For any i and j,

$\owedge_i \leq \owedge_j$ is a theorem iff it is true in 

,

$\ovee_i \leq \ovee_j$ is a theorem iff it is true in 

,

$\ominus\!\!\!\!\to_i \leq \ominus\!\!\!\!\to_j$ is a theorem iff it is true in 

.

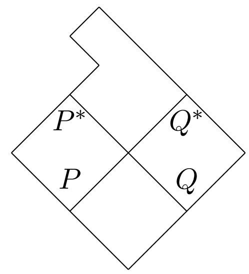
The very important fact, and the valuations $v_\wedge$, $v_\vee$, $v_\to$, give us:

- a way to *remember* which sentences of the forms $\owedge_i \leq \owedge_j$, $\ovee_i \leq \ovee_j$, $\ominus\!\!\!\!\to_i \leq \ominus\!\!\!\!\to_j$ are theorems;
- countermodels for all the sentences of these forms not in $\mathsf{SCube}_\wedge$, $\mathsf{SCube}_\vee$, $\mathsf{SCube}_\to$. For example, $\ovee_7 \leq \ovee_4$ is not in $\mathsf{SCube}_\vee$; and $v_\vee(\ovee_7) \leq v_\vee(\ovee_4)$, which shows that $\ovee_7 \leq \ovee_4$ can't be a theorem.

An observation. I arrived at the cubes $\mathsf{ECube}^*_\wedge$, $\mathsf{ECube}^*_\vee$, $\mathsf{ECube}^*_\to$ by taking the material in the corollary 5.3 of chapter 5 in [Bel88] and trying to make it fit into less mental space (as discussed in [IDARCT]); after that I wanted to be sure that each arrow that is not in the extended cubes has a countermodel, and I found the countermodels one by one; then I wondered if I could find a single countermodel for all non-theorems in $\mathsf{ECube}^*_\wedge$ (and the same for $\mathsf{ECube}^*_\vee$ and $\mathsf{ECube}^*_\to$), and I tried to start with a valuation that distinguished *some* equivalence classes in $\mathsf{ECube}^*_\wedge$, and change it bit by bit, getting valuations that distinguished more equivalence classes at every step. Eventually I arrived at $v_\wedge$, $v_\vee$ and at $v_\to$, and at the — surprisingly nice — "very important fact" above.

Note that this valuation

$$(H_{\wedge\vee}, J_{\wedge\vee}, v_{\wedge\vee}) \quad =$$



distinguishes all equivalence classes in $\mathsf{ECube}^*_\wedge$ and in $\mathsf{ECube}^*_\vee$, but not in $\mathsf{ECube}^*_\to$... it "thinks" that $P \to Q$ and $P^* \to Q$ are equal.

6 Polynomial J-operators

It is not hard to check that for any Heyting Algebra H and any $Q, R \in H$ the operations $(\neg\neg)$, ..., $(\vee Q \wedge \to R)$ below are J-operators:

$$\begin{array}{rcl}
(\neg\neg)(P) & = & \neg\neg P \\
(\to\to R)(P) & = & (P\to R)\to R \\
(\vee Q)(P) & = & P \vee Q \\
(\to R)(P) & = & P\to R \\
(\vee Q \wedge \to R)(P) & = & (P\vee Q) \wedge (P\to R)
\end{array}$$

Checking that they are J-operators means checking that each of them obeys $\mathsf{J1}$, $\mathsf{J2}$, $\mathsf{J3}$. Let's define formally what are $\mathsf{J1}$, $\mathsf{J2}$ and $\mathsf{J3}$ "for a given $F : H \to H$":

$$\begin{array}{rcc}
\mathsf{J1}_F & := & (P \leq F(P)) \\
\mathsf{J2}_F & := & (F(P) = F(F(P)) \\
\mathsf{J3}_F & := & (F(P \wedge P') = F(P) \wedge F(P'))
\end{array}$$

and:

$$\mathsf{J123}_F \quad := \quad \mathsf{J1}_F \wedge \mathsf{J2}_F \wedge \mathsf{J3}_F.$$

Checking that $(\neg\neg)$ obeys $\mathsf{J1}$, $\mathsf{J2}$, $\mathsf{J3}$ means proving $\mathsf{J123}_{(\neg\neg)}$ using only the rules from intuitionist logic from section 10 of [PH1]; we will leave the proof of this, of and $\mathsf{J123}_{(\to\to R)}$, $\mathsf{J123}_{(\vee Q)}$, and so on, to the reader.

The J-operator $(\vee Q \wedge \to R)$ is a particular case of building more complex J-operators from simpler ones. If $J, K : H \to H$, we define:

$$(J \wedge K) := \lambda P{:}H.(J(P){\wedge}K(P))$$

it not hard to prove $\mathsf{J123}_{(J\wedge K)}$ from $\mathsf{J123}_J$ and $\mathsf{J123}_K$ using only the rules from intuitionistic logic.

The J-operators above are the first examples of J-operators in Fourman and Scott's "Sheaves and Logic" ([FS79]); they appear in pages 329–331, but with these names (our notation for them is at the right):

(i) *The closed quotient,*

$$J_a p = a \vee p \qquad J_Q = (\vee Q).$$

(ii) *The open quotient,*

$$J^a p = a \to p \qquad J^R = (\to R).$$

(iii) *The Boolean quotient.*

$$B_a p = (p \to a) \to a \qquad B_R = (\to\to R).$$

(iv) *The forcing quotient.*

$$(J_a \wedge J^b)p = (a \vee p) \wedge (b \to p) \qquad (J_Q \wedge J^R) = (\vee Q \wedge \to R).$$

(vi) *A mixed quotient.*

$$(B_a \wedge J^a)p = (p \to a) \to p \qquad (B_Q \wedge J^Q) = (\to\to Q \wedge \to Q).$$

The last one is tricky. From the definition of B_a and J^a what we have is

$$(B_a \wedge J^a)p = ((p \to a) \to a) \wedge (a \to p),$$

but it is possible to prove

$$((p \to a) \to a) \wedge (a \to p) \quad \leftrightarrow \quad ((p \to a) \to p)$$

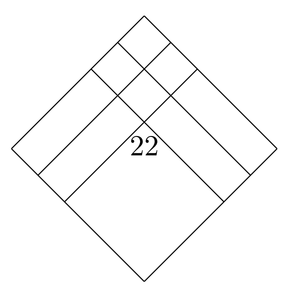
intuitionistically.

The operators above are "polynomials on $P, Q, R, \to, \wedge, \vee, \bot$" in the terminology of Fourman/Scott: "If we take a polynomial in $\to, \wedge, \vee, \bot$, say, $f(p, a, b, \ldots)$, it is a decidable question whether for all $a, b, \ldots$ it defines a J-operator" (p.331).

When I started studying sheaves I spent several years without any visual intuition about the J-operators above. I was saved by ZHAs and brute force — and the brute force method also helps in testing if a polynomial (in the sense above) is a J-operator in a particular case. For example, take the operators $\lambda P{:}H.(P \wedge 22)$ and $(\vee 22)$ on $H = [00, 44]$:

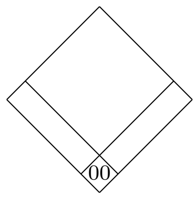
$$\lambda P{:}H.(P \wedge 22) \quad = \quad \begin{array}{ccccccccc}
 & & & & 22 & & & & \\
 & & & 22 & & 22 & & & \\
 & & 22 & & 22 & & 22 & & \\
 & 21 & & 22 & & 22 & & 12 & \\
20 & & 21 & & 22 & & 12 & & 02 \\
 & 20 & & 21 & & 12 & & 02 & \\
 & & 20 & & 11 & & 02 & & \\
 & & & 10 & & 01 & & & \\
 & & & & 00 & & & &
\end{array}$$

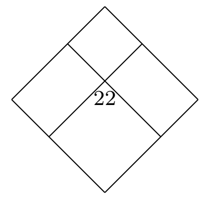
$$(\vee 22) \quad = \quad \begin{array}{ccccccccc}
 & & & & 44 & & & & \\
 & & & 43 & & 34 & & & \\
 & & 42 & & 33 & & 24 & & \\
 & 42 & & 32 & & 23 & & 24 & \\
42 & & 32 & & 22 & & 23 & & 24 \\
 & 32 & & 22 & & 22 & & 23 & \\
 & & 22 & & 22 & & 22 & & \\
 & & & 22 & & 22 & & & \\
 & & & & 22 & & & &
\end{array} \quad =$$

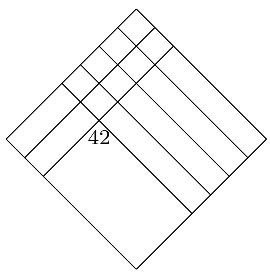


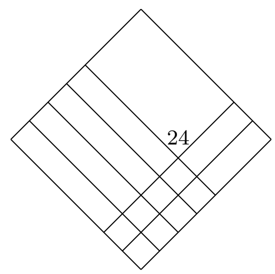
The first one, $\lambda P{:}H.(P \wedge 22)$, is not a J-operator; one easy way to see that is to look at the region in which the result is 22 — its top element is 44, and this violates the conditions on slash-operators in sec.1.2. The second operator, $(\vee 22)$, is a slash operator and a J-operator; at the right we introduce a convenient notation for visualizing the action of a polynomial slash-operator, in which we draw only the contours of the equivalence classes and the constants that appear in the polynomial.

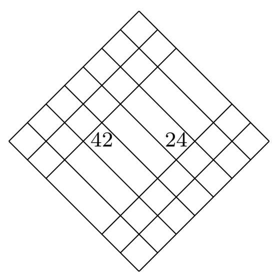
Using this new notation, we have:

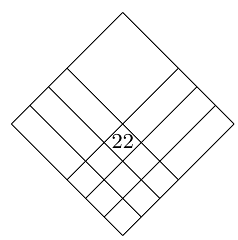
$(\neg\neg) \quad = \quad (\to\to 00) \quad =$ 

$(\to\to 22) \quad =$ 

$(\vee 42) \quad =$ 

$(\to 24) \quad =$ 

$(\vee 42 \wedge \to 24) \quad =$ 

$(\to\to 22 \wedge \to 22) \quad =$ 

Note that the slashing for $(\vee 42 \wedge \to 24)$ has all the cuts for $(\vee 42)$ plus all the cuts for $(\to 24)$, and $(\vee 42 \wedge \to 24)$ "forces $42 \leq 24$" in the following sense: if $P^* = (\vee 42 \wedge \to 24)(P)$ then $42^* \leq 24^*$.

6.1 An algebra of piccs

We saw in the last section a case in which $(J \wedge K)$ has all the cuts from J plus all the cuts from K; this suggests that we *may* have an operation dual to that, that behaves as this: $(J \vee K)$ has exactly the cuts that are both in J and in K:

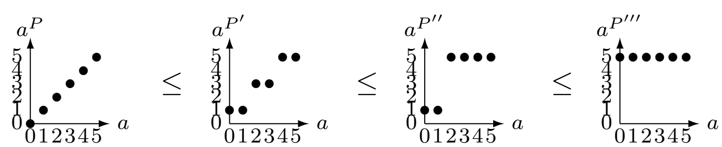
$$\begin{array}{rcl} \mathsf{Cuts}(J \wedge K) & = & \mathsf{Cuts}(J) \cup \mathsf{Cuts}(K) \\ \mathsf{Cuts}(J \vee K) & = & \mathsf{Cuts}(J) \cap \mathsf{Cuts}(K) \end{array}$$

And it $J_1, \ldots, J_n$ are all the slash-operators on a given ZHA, then

$$\begin{array}{rcccl} \mathsf{Cuts}(J_1 \wedge \ldots \wedge J_n) & = & \mathsf{Cuts}(J_1) \cup \ldots \cup \mathsf{Cuts}(J_k) & = & \text{(all cuts)} \\ \mathsf{Cuts}(J_1 \vee \ldots \vee J_n) & = & \mathsf{Cuts}(J_1) \cap \ldots \cap \mathsf{Cuts}(J_k) & = & \text{(no cuts)} \end{array}$$

yield the minimal element and the maximal element, respectively, of an algebra of slash-operators; note that the slash-operator with “all cuts” is the identity map $\lambda P{:}H.P$, and the slash-operator with “no cuts” is the one that takes all elements to $\top$: $\lambda P{:}H.\top$. This yields a lattice of slash-operators, in which the partial order is $J \leq K$ iff $\mathsf{Cuts}(J) \supseteq \mathsf{Cuts}(K)$. This is somewhat counterintuitive if we think in terms of cuts — the order seems to be reversed — but it makes a lot of sense if we think in terms of piccs (sec.1.1) instead.

Each picc P on $\{0, \ldots, n\}$ has an associated function $\cdot^P$ that takes each element to the top element of its equivalence class. If we define $P \leq P'$ to mean $\forall a \in \{0, \ldots, n\}.\, a^P \leq a^{P'}$, then we have this:



$$\begin{array}{ccccccc} 0|1|2|3|4|5 & \leq & 01|23|45 & \leq & 01|2345 & \leq & 012345 \\ P & \leq & P' & \leq & P'' & \leq & P''' \end{array}$$

This yields a partial order on piccs, whose bottom element is the identity function $0|1|2|\ldots|n$, and the top element is $012\ldots n$, that takes all elements to n.

The piccs on $\{0, \ldots, n\}$ form a Heyting Algebra, where $\bot = 0|1|\ldots|n$, $\top = 01\ldots n$, and ‘$\wedge$’ and ‘$\vee$’ are the operations that we have discussed above; it is possible to define a ‘$\to$’ there, but this ‘$\to$’ is not going to be useful for us and we are mentioning it just as a curiosity. We have, for example:

$$
\begin{tikzcd}[column sep=small]
& 01234 & \\
& 01|234 \arrow[u] & \\
0|1|234 \arrow[ur] & & 01|2|34 \arrow[ul] \\
& 0|1|2|34 \arrow[ul] \arrow[ur] & \\
& 0|1|2|3|4 \arrow[u] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& \top & \\
& P \vee Q \arrow[u] & \\
P \arrow[ur] & & Q \arrow[ul] \\
& P \wedge Q \arrow[ul] \arrow[ur] & \\
& \bot \arrow[u] &
\end{tikzcd}
$$

6.2 An algebra of J-operators

Fourman and Scott define the operations $\wedge$ and $\vee$ on J-operators in pages 325 and 329 ([FS79]), and in page 331 they list ten properties of the algebra of J-operators:

$$
\begin{array}{rlcl@{\qquad}l}
\text{(i)} & J_a \vee J_b & = & J_{a \vee b} & (\vee 21) \vee (\vee 12) = (\vee 22) \\
\text{(ii)} & J^a \vee J^b & = & J^{a \wedge b} & (\to 32) \vee (\to 23) = (\to 22) \\
\text{(iii)} & J_a \wedge J_b & = & J_{a \wedge b} & (\vee 21) \wedge (\vee 12) = (\vee 11) \\
\text{(iv)} & J^a \wedge J^b & = & J^{a \vee b} & (\to 32) \wedge (\to 23) = (\to 33) \\
\text{(v)} & J_a \wedge J^a & = & \bot & (\vee 22) \wedge (\to 22) = (\bot) \\
\text{(vi)} & J_a \vee J^a & = & \top & (\vee 22) \vee (\to 22) = (\top) \\
\text{(vii)} & J_a \vee K & = & K \circ J_a & \\
\text{(viii)} & J^a \vee K & = & J^a \circ K & \\
\text{(ix)} & J_a \vee B_a & = & B_a & \\
\text{(x)} & J^a \vee B_b & = & B_{a \to b} &
\end{array}
$$

The first six are easy to visualize; we won't treat the four last ones. In the right column of the table above we've put a particular case of (i), ..., (vi) in our notation, and the figures below put all together.

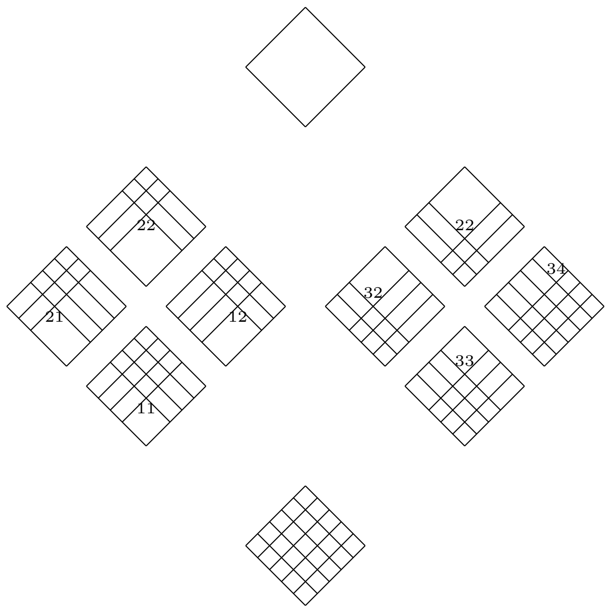
In Fourman and Scott's notation,

$$
\begin{tikzcd}
 & & & J_\top = \top = J^\perp & & & \\
 & J_{22} \arrow[urr] & & & & J^{22} \arrow[ull] & \\
J_{21} \arrow[ur] & & J_{12} \arrow[ul] & & J^{32} \arrow[ur] & & J^{23} \arrow[ul] \\
 & J_{11} \arrow[ul] \arrow[ur] & & & & J^{11} \arrow[ul] \arrow[ur] & \\
 & & & J_\perp = \perp = J^\top \arrow[ull] \arrow[urr] & & &
\end{tikzcd}
$$

in our notation,

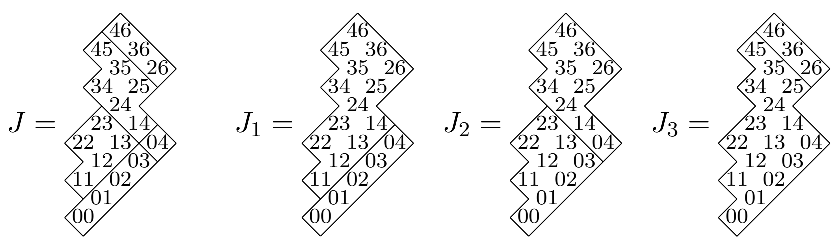
$$
\begin{tikzcd}
 & & & (\top\vee) = (\lambda P.\top) = (\perp\to) & & & \\
 & (22\vee) \arrow[urr] & & & & (22\to) \arrow[ull] & \\
(21\vee) \arrow[ur] & & (12\vee) \arrow[ul] & & (32\to) \arrow[ur] & & (23\to) \arrow[ul] \\
 & (11\vee) \arrow[ul] \arrow[ur] & & & & (33\to) \arrow[ul] \arrow[ur] & \\
 & & & (\perp\vee) = (\lambda P.P) = (\top\to) \arrow[ull] \arrow[urr] & & &
\end{tikzcd}
$$

and drawing the polynomial J-operators as in sec.6:

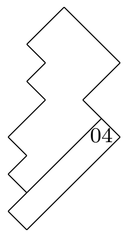


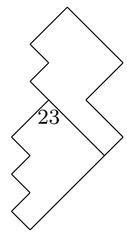
6.3 All slash-operators are polynomial

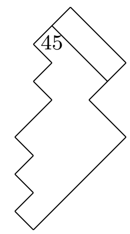
Here is an easy way to see that all slashings — i.e., J-operators on ZHAs — are polynomial. Every slashing J has only a finite number of cuts; call them $J_1, \ldots, J_n$. For example:



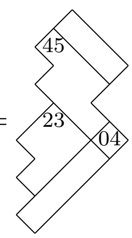
Each cut J_i divides the ZHA into an upper region and a lower region, and $J_i(00)$ yields the top element of the lower region. Also, $(\to\to J_i(00))$ is a polynomial way of expressing that cut:

$J_1 =$
$(\to\to 04) =$ 

$J_2 =$
$(\to\to 23) =$ 

$J_3 =$
$(\to\to 45) =$ 

The conjunction of these '$(\to\to J_i(00))$'s yields the original slashing:

$(\to\to 04) \wedge (\to\to 23) \wedge (\to\to 45) =$ 

$= J$

7 Toposes

Everything from here onwards will be rewritten!

7.1 Some bijections

7.2 A particular case

[TODO: rewrite everything from this point onwards]

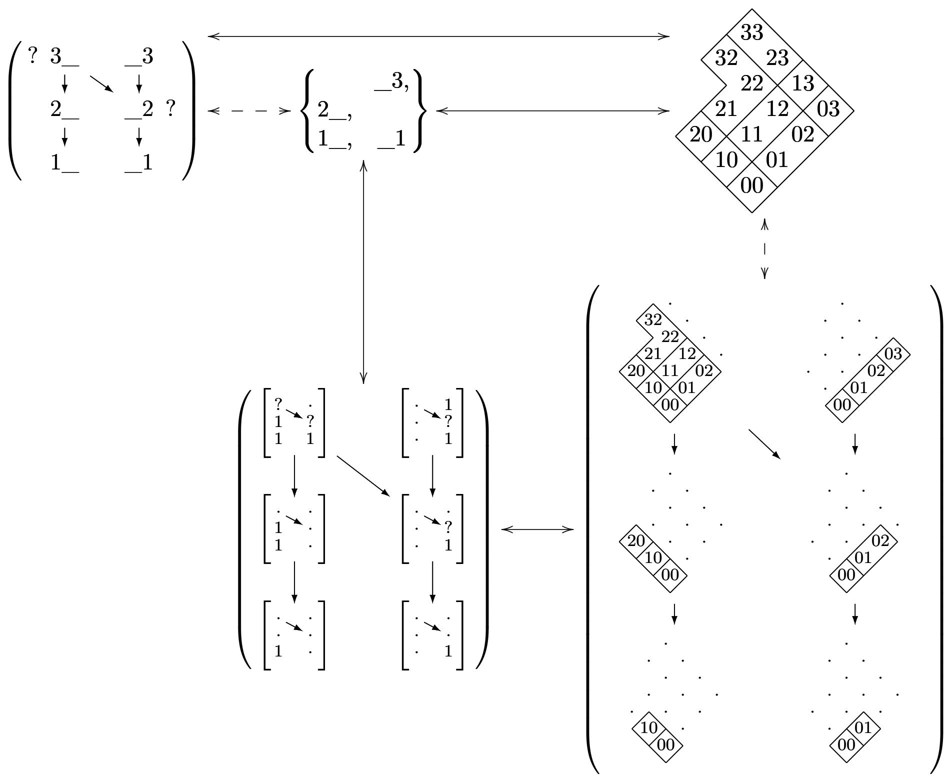
Fix a 2-column graph (P, A), and let $\mathbf{D}$ be the DAG (P, A) regarded as a posetal category. Let $\mathbf{E}$ be the topos $\mathbf{Set}^{\mathbf{D}}$, and let H be the (planar) Heyting Algebra of truth-values of $\mathbf{E}$: $H = \mathsf{CanSub}(1_{\mathbf{E}})$. From here on $\mathbf{D}$, $\mathbf{E}$, and H will be our default 2CG, our default topos, and our default ZHA. Let's take this idea of "defaults" a bit further.

$$\begin{tikzcd}
\begin{pmatrix}\text{a set of}\\ \text{question}\\ \text{marks}\\ Q\subseteq\mathbf{D}_0\end{pmatrix} \arrow[rr, leftrightarrow, "\text{[PH2]}"] \arrow[r, leftrightarrow, dashed] & \begin{pmatrix}\text{a subset}\\ \mathcal{Y}\subseteq\mathbf{D}_0\end{pmatrix} \arrow[r, leftrightarrow, "\text{[Lin14] Thm C4, p.74}"'] \arrow[dd, leftrightarrow, "\text{[Lin14] Thm C4, p.74}"'] & \begin{pmatrix}\text{a nucleus}\\ (\cdot)^*:H\to H\end{pmatrix} \arrow[d, leftrightarrow, dashed] \\
 & & \begin{pmatrix}\text{a closure operator:}\\ \text{for every } E\in\mathbf{E} \text{ a}\\ \overline{(\cdot)}_E:\mathsf{Incs}(E)\to\mathsf{Incs}(E)\end{pmatrix} \arrow[d, leftrightarrow, "\text{[McL92, sec.21], [Och20, sec.2.6]}"] \\
 & \begin{pmatrix}\text{a Grothendieck}\\ \text{topology}\\ J\subset\Omega\end{pmatrix} \arrow[r, leftrightarrow, "\text{[LM92] Sec V.4 Thm 1 p.233}"'] & \begin{pmatrix}\text{a Lawvere-}\\ \text{Tierney}\\ \text{topology}\\ j:\Omega\to\Omega\end{pmatrix}
\end{tikzcd}$$

In sections 2–4 we saw a bijection that converts each set of question marks Q to a J-operator $(\cdot)^*$ and vice-versa. The theorem C4 of [Lin14] defines a bijection that converts every J-operator $(\cdot)^*$ to a subset $\mathcal{Y} \subset \mathcal{D}_0$ and vice-versa, and another bijection that converts each $\mathcal{Y} \subset \mathcal{D}_0$ to a Grothendieck topology J in $\mathbf{Set}^{\mathbf{D}}$. Section V.4 of [LM92] how to convert each J to a Lawvere-Tierney topology j and vice-versa, and [McL92] and [Och20] show how to convert each j to a closure operator $\overline{(\cdot)}$ and vice-versa. Let's refer to the operations that perform the conversions as $(Q \mapsto (\cdot)^*)$, $((\cdot)^* \mapsto Q)$, and so on; for example,

$$
\begin{array}{rcl}
(\mathcal{Y} \mapsto J) & = & \lambda \mathcal{Y} \subset \mathbf{D}_0.\ \lambda u \in \mathbf{D}_0.\ \{\, \mathcal{S} \in \Omega(u) \mid \mathcal{Y} \cap {\downarrow} u \subset \mathcal{S} \,\} \\
(J \mapsto \mathcal{Y}) & = & \lambda J \in \mathsf{GrTops}(\mathbf{E}).\ \{\, u \in \mathbf{D}_0 \mid J(u) = \{{\downarrow} u\} \,\}
\end{array}
$$

This means that once we've chosen a value for Q, or for $(\cdot)^*$, $\mathcal{Y}$, J, j, or $\overline{(\cdot)}$ the default values for the other ones become automatically determined. Here is an example. If we choose Q as in the top left below we get this:



Note that I have dropped the $\overline{(\cdot)}$ from the diagram. This is because I don't have (yet) a good way to draw closure operators.

This diagram — of a particular case! — suggests that the points in $\mathcal{Y}$ are exactly the points of Q without question marks, and that each $j(u) : \Omega(u) \to \Omega(u)$ is the slashing $(\cdot)^*$ restricted to ${\downarrow}u$, or, more precisely, that $j(u)(v) = v^* \wedge u$. If we remake that diagram for the other 63 'Q's, we see that this still holds. If we do the same for some other 2CGs and for all 'Q's in them, we will see that the same patterns still hold — but there infinitely many 2CGs. We *can* obtain direct proofs that the '$\mathcal{Y}$'s are always the points of $\mathbf{D}_0$ without question marks, and that the '$(\cdot)^*$'s are exactly the slashings and that the 'j's are obtained by restricting the '$(\cdot)^*$'s, but some calculations may be hairy.

Remember that we are looking for "visual intuition" on what are the '$(\cdot)^*$'s and their associates j's and J. Are these fully formalized proofs really necessary? Answer: not if we formalize "visual intuition" in the way that we do in the next section.

$$\begin{array}{rcl}
(\cdot)^* & := & \lambda \mathcal{S} \in H.\ \bigcup\{\, \mathcal{R} \in H \mid \mathcal{R} \cap \mathcal{Y} = \mathcal{S} \cap \mathcal{Y} \,\} \\
\mathcal{Y} & := & \{\, u \in \mathbf{D}_0 \mid {\downarrow^-} u \in H^* \,\} \\
& = & \{\, u \in \mathbf{D}_0 \mid ({\downarrow^-} u)^* = {\downarrow^-} u \,\} \\
& = & \{\, u \in \mathbf{D}_0 \mid ({\downarrow} u \backslash \{u\})^* = ({\downarrow} u \backslash \{u\}) \,\} \\
J & := & \lambda u \in \mathbf{D}_0.\ \{\, \mathcal{S} \in \Omega(u) \mid (\mathcal{Y} \cap {\downarrow} u) \subseteq \mathcal{S} \,\} \\
\mathcal{Y} & := & \{\, u \in \mathbf{D}_0 \mid J(u) = \{{\downarrow} u\} \,\} \\
J & := & \lambda u \in \mathbf{D}_0.\ \{\, \mathcal{S} \in \Omega(u) \mid {\downarrow} u = \mathcal{S}^* \cap {\downarrow} u \,\} \\
& = & \lambda u \in \mathbf{D}_0.\ \{\, \mathcal{S} \in \Omega(u) \mid {\downarrow} u \subset \mathcal{S}^* \,\} \\
& = & \lambda u \in \mathbf{D}_0.\ \{\, \mathcal{S} \in \Omega(u) \mid u \in \mathcal{S}^* \,\} \\
(\cdot)^* & := & \lambda \mathcal{S} \in H.\ \{\, u \in \mathbf{D}_0 \mid \mathcal{S} \cap {\downarrow} u \in J(u) \,\} \\
j & := & \lambda u \in \mathbf{D}_0.\ \lambda \mathcal{S} \in \Omega(u).\mathsf{CST}(\{\, v \in {\downarrow} u \mid \Omega \left(\begin{smallmatrix} u \\ \downarrow \\ v \end{smallmatrix}\right) (\mathcal{S}) \in J(v) \,\}) \\
& = & \lambda u \in \mathbf{D}_0.\ \lambda \mathcal{S} \in \Omega(u).\mathsf{CST}(\{\, v \in {\downarrow} u \mid \mathcal{S} \cap {\downarrow} v \in J(v) \,\}) \\
J & := & \lambda u \in \mathbf{D}_0.\ \{\, \mathcal{S} \in \Omega(u) \mid j(u)(\mathcal{S}) = \top(u)(*) \,\} \\
& = & \lambda u \in \mathbf{D}_0.\ \{\, \mathcal{S} \in \Omega(u) \mid j(u)(\mathcal{S}) = {\downarrow} u \,\}
\end{array}$$

Hypotheses:

$$
\begin{array}{rcl}
J(u) & := & \{\, \mathcal{S} \in \Omega(u) \mid \mathcal{S}^* \cap {\downarrow}u = {\downarrow}u \,\} \\
j(u)(\mathcal{S}) & := & \mathcal{S}^* \cap {\downarrow}u \\
u \in Q & := & ?
\end{array}
$$

$$
\begin{array}{rcl}
J_j(p) & = & \{S \in D({\downarrow}p) : p \in j(S)\} \\
J_j(u) & = & \{S \in \Omega(u) : u \in j(S)\} \\
J(u) & = & \{\mathcal{S} \in \Omega(u) : u \in \mathcal{S}^*\} \\
 & = & \{\mathcal{S} \in \Omega(u) : {\downarrow}u \subset \mathcal{S}^*\} \\
 & = & \{\mathcal{S} \in \Omega(u) : {\downarrow}u = \mathcal{S}^* \cap {\downarrow}u\} \\
J & := & \lambda u \in \mathbf{D}_0.\, \{\, \mathcal{S} \in \Omega(u) \mid {\downarrow}u = \mathcal{S}^* \cap {\downarrow}u \,\} \\
 & = & \lambda u \in \mathbf{D}_0.\, \{\, \mathcal{S} \in \Omega(u) \mid {\downarrow}u \subset \mathcal{S}^* \,\} \\
 & = & \lambda u \in \mathbf{D}_0.\, \{\, \mathcal{S} \in \Omega(u) \mid u \in \mathcal{S}^* \,\}
\end{array}
$$

$$
\begin{array}{rcl}
j_J(A) & = & \{p \in P : A \cap \downarrow p \in J(p)\} \\
\mathcal{S}^* & = & \{\, u \in \mathbf{D}_0 \mid \mathcal{S} \cap {\downarrow}u \in J(u) \,\} \\
(\cdot)^* & = & \lambda \mathcal{S} \in H.\, \{\, u \in \mathbf{D}_0 \mid \mathcal{S} \cap {\downarrow}u \in J(u) \,\}
\end{array}
$$

8 Visual intuition

This is an excerpt from a long blog post by Kevin Buzzard ([Buz21]):

> **Mathematicians think in pictures**
>
> I have a picture of the real numbers in my head. It's a straight line. This picture provides a great intuition as to how the real numbers work. I also have a picture of what the graph of a differentiable function looks like. It's a wobbly line with no kinks in. This is by no means a perfect picture, but it will do in many cases. For example: If someone asked me to prove or disprove the existence of a strictly increasing infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(37) = 0$ and $f''(37) < 0$ then I would start by considering a picture of a graph of a strictly increasing function (monotonically increasing as we move from left to right), and a second picture of a function whose derivative at $x = 37$ is zero and whose second derivative is negative (a function with a local maximum). I then note that there are features in these pictures which make them incompatible with each other. Working with these pictures in mind, I can now follow my intuition and write down on paper a picture-free proof that such a function cannot exist, and this proof would be acceptable as a model

> solution to an exam question. My perception is that other working mathematicians have the same pictures in their head when presented with the same problem, and would go through roughly the same process if they were asked to write down a sketch proof of this theorem.

Fulano talks of starting from visual intuition, and from that producing conjectures and formal proofs; in sections 1 to 6 we developed visual intuition for a well-known part of basic Topos Theory. How can we put these two things in the same framework.

$$
\begin{tikzcd}
Q \arrow[rr, <->, bend left] & \mathcal{Y} \arrow[r, <->] \arrow[dd, <->] & (\cdot)^* \\
& & \overline{(\cdot)} \arrow[d, <->] \\
& J \arrow[r, <->] & j
\end{tikzcd}
$$

References

[Bel88] J. L. Bell. *Toposes and Local Set Theories*. Oxford Logic Guides 14. Oxford University Press, 1988.

[Buz21] K. Buzzard. "Formalising mathematics: an introduction". `https://xenaproject.wordpress.com/2021/01/21/formalising-mathematics-an-introduction/`. 2021.

[FavC] E. Ochs. "On my favorite conventions for drawing the missing diagrams in Category Theory". `http://angg.twu.net/math-b.html#favorite-conventions`. 2020.

[FS79] M.P. Fourman and D.S. Scott. "Sheaves and Logic". In: *Applications of Sheaves: Proceedings of the Research Symposium on Applications of Sheaf Theory to Logic, Algebra and Analysis - Durham, july 9-21, 1977*. Ed. by M.P. Fourman, D.J. Mulvey, and D.S. Scott. Lecture Notes in Mathematics 753. Springer, 1979, pp. 302–401.

[IDARCT] E. Ochs. "Internal Diagrams and Archetypal Reasoning in Category Theory". In: *Logica Universalis* 7.3 (Sept. 2013). `http://angg.twu.net/math-b.html#idarct`, pp. 291–321.

[Lin14] A.J. Lindenhovius. "Grothendieck Topologies on Posets". `https://arxiv.org/pdf/1405.4408v2.pdf`. 2014.

[LM92] S. Mac Lane and I. Moerdijk. *Sheaves in geometry and logic: a first introduction to topos theory*. Springer, 1992.

[McL92] C. McLarty. *Elementary Categories, Elementary Toposes*. Oxford Logic Guides 21. Oxford University Press, 1992.

[Och20] E. Ochs. "Each closure operator induces a topology and vice-versa". `http://angg.twu.net/math-b.html#clops-and-tops`. 2020.

[PH1] E. Ochs. "Planar Heyting Algebras for Children". In: *South American Journal of Logic* 5.1 (2019). `http://angg.twu.net/math-b.html#zhas-for-children-2`, pp. 125–164.

[PH2] E. Ochs. "Planar Heyting Algebras for Children 2: Local J-Operators, Slashings, and Nuclei". `http://angg.twu.net/math-b.html#zhas-for-children-2`. 2021.