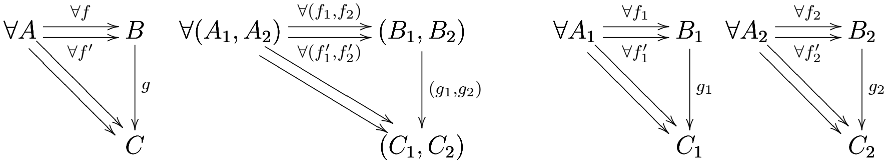
Proposition: a morphism $g = (g_1, g_2)$ in $\mathbf{Set}^2$ is a monic if and only if its two components, g_1 and g_2, are monics in $\mathbf{Set}$.

With the right abbreviations, this becomes:
$((g_1, g_2) \text{ is monic}) \leftrightarrow (g_1 \text{ is monic}) \wedge (g_2 \text{ is monic})$

We will use the diagrams below.



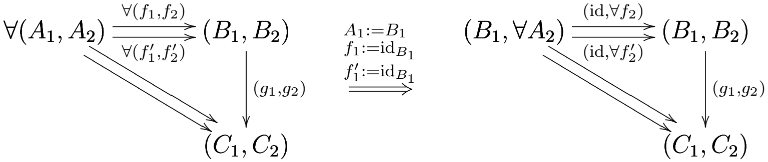
[5pt] Def: $B_1 \xrightarrow{g_1} C_1$ is a monic in **Set**
means $\underbrace{\forall A_1.\,\forall f_1, f_1'.\,(\underbrace{g_1 \circ f_1 = g_1 \circ f_1'}_{\alpha_1} \to \underbrace{f_1 = f_1'}_{\beta_1})}_{\text{“}g_1\text{ is monic”}}$

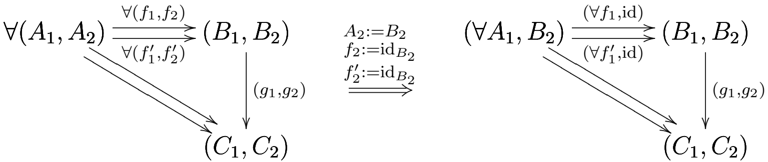
Def: $B_2 \xrightarrow{g_2} C_2$ is a monic in **Set**
means $\underbrace{\forall A_2.\,\forall f_2, f_2'.\,(\underbrace{g_2 \circ f_2 = g_2 \circ f_2'}_{\alpha_2} \to \underbrace{f_2 = f_2'}_{\beta_2})}_{\text{“}g_2\text{ is monic”}}$

Def: $B \xrightarrow{g} C$ is a monic in $\mathbf{Set}^2$,
i.e, $(B_1, B_2) \xrightarrow{(g_1,g_2)} (C_1, C_2)$ is a monic in $\mathbf{Set}^2$,
means $\forall A.\,\forall f, f'.\,(\underbrace{g \circ f = g \circ f'}_{\alpha} \to \underbrace{f = f'}_{\beta})$,
i.e., $\forall (A_1, A_2).\,\forall (f_1, f_2), (f_1', f_2').\,(\underbrace{(g_1, g_2) \circ (f_1, f_2) = (g_1, g_2) \circ (f_1', f_2')}_{\alpha_{12} = \alpha} \to \underbrace{(f_1, f_2) = (f_1', f_2')}_{\beta_{12} = \beta})$,
i.e., $\forall (A_1, A_2).\,\forall (f_1, f_2), (f_1', f_2').\,((g_1 \circ f_1, g_2 \circ f_2) = (g_1 \circ f_1', g_2 \circ f_2') \to (f_1, f_2) = (f_1', f_2'))$,
i.e., $\underbrace{\forall A_1, A_2.\,\forall f_1, f_2, f_1', f_2'.\,((\underbrace{g_1 \circ f_1 = g_1 \circ f_1'}_{\alpha_1}) \land (\underbrace{g_2 \circ f_2 = g_2 \circ f_2'}_{\alpha_2}) \to (\underbrace{f_1 = f_1'}_{\beta_1}) \land (\underbrace{f_2 = f_2'}_{\beta_2}))}_{\text{“}(g_1, g_2)\text{ is monic”}}$.

We want to prove this: $((g_1, g_2)$ is monic$) \leftrightarrow (g_1$ is monic$) \land (g_2$ is monic$)$

Trick 1: make f_1 and f_1' identities. We will need $A_1 := B_1$.
Trick 2: make f_2 and f_2' identities. We will need $A_2 := B_2$.





If we specialize "(g_1, g_2) is monic"
by doing $A_1 := B_1$, $f_1 := \mathrm{id}$, $f_1' := \mathrm{id}$,
we get this:

$$\underbrace{\underbrace{\forall \underbrace{A_1}_{B_1}, A_2. \forall \underbrace{f_1}_{\mathrm{id}}, f_2, \underbrace{f_1'}_{\mathrm{id}}, f_2'. \underbrace{(\underbrace{\underbrace{g_1 \circ \underbrace{f_1}_{\mathrm{id}}}_{g_1} = \underbrace{g_1 \circ \underbrace{f_1'}_{\mathrm{id}}}_{g_1}}_{\mathbf{T}}) \land (\underbrace{g_2 \circ f_2 = g_2 \circ f_2'}_{\alpha_2})}_{\alpha_2} \to \underbrace{(\underbrace{\underbrace{f_1}_{\mathrm{id}} = \underbrace{f_1'}_{\mathrm{id}}}_{\mathbf{T}}) \land (\underbrace{f_2 = f_2'}_{\beta_2})}_{\beta_2}}_{\forall A_2. \forall f_2, f_2'. \alpha_2 \to \beta_2}}_{g_2 \text{ is monic}}$$

so $((g_1, g_2)$ is monic$) \to (g_2$ is monic$)$.
The proof of $((g_1, g_2)$ is monic$) \to (g_1$ is monic$)$ is similar,
but with $A_2 := B_2$, $f_2 := \mathrm{id}$, $f_2' := \mathrm{id}$.
So: $((g_1, g_2)$ is monic$) \to (g_1$ is monic$) \land (g_2$ is monic$)$.

This is a proof of
$((g_1, g_2) \text{ is monic}) \leftarrow (g_1 \text{ is monic}) \wedge (g_2 \text{ is monic})$
in Natural Deduction:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\alpha_1 \wedge \alpha_2]^1}{\alpha_1} \quad \dfrac{[A_1, f_1, f_1']^2 \quad g_1 \text{ is monic}}{\alpha_1 \to \beta_1}}{\beta_1} \quad \dfrac{\dfrac{[\alpha_1 \wedge \alpha_2]^1}{\alpha_1} \quad \dfrac{[A_2, f_2, f_2']^2 \quad g_2 \text{ is monic}}{\alpha_2 \to \beta_2}}{\beta_2}}{\dfrac{\beta_1 \wedge \beta_2}{\alpha_1 \wedge \alpha_2 \to \beta_1 \wedge \beta_2}\,1}}{\dfrac{\forall A_1, A_2, f_1, f_2, f_1', f_2'. (\alpha_1 \wedge \alpha_2 \to \beta_1 \wedge \beta_2)}{(g_1, g_2) \text{ is monic}}}\,2$$

Page 26:

Each object C of $\mathbf{C}$ gives rise to a presheaf $\mathbf{y}(C)$ on $\mathbf{C}$, defined on an object D of $\mathbf{C}$ by

$$\mathbf{y}(C)(D) = \mathrm{Hom}_{\mathbf{C}}(D, C) \tag{3}$$

and on a morphism $D' \xrightarrow{\alpha} D$, for $u\colon D \to C$, by

$$\begin{gathered}\mathbf{y}(C)(\alpha)\colon \mathrm{Hom}_{\mathbf{C}}(D, C) \to \mathrm{Hom}_{\mathbf{C}}(D', C)\\ \mathbf{y}(C)(\alpha)(u) = u \circ \alpha;\end{gathered} \tag{4}$$

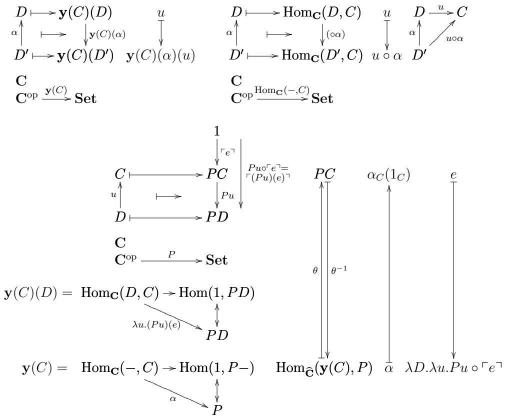
or briefly, $\mathbf{y}(C) = \mathrm{Hom}_{\mathbf{C}}(-, C)$ is the contravariant Hom-functor. Presheaves which, up to isomorphism, are of this form are called *representable presheaves* or *representable functors*. If $f\colon C_1 \to C_2$ is a morphism in $\mathbf{C}$, there is a natural transformation $\mathbf{y}(C_1) \to \mathbf{y}(C_2)$ obtained by composition with f. This makes $\mathbf{y}$ into a functor

$$\mathbf{y}\colon \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}, \qquad C \mapsto \mathrm{Hom}_{\mathbf{C}}(-, C) \tag{5}$$

from $\mathbf{C}$ to the *contravariant* functors on $\mathbf{C}$ (hence the exponent $\mathbf{C}^{\mathrm{op}}$). It is called the *Yoneda embedding*. The Yoneda embedding is a full and faithful functor. This fact is a special case of the so-called *Yoneda lemma*, which asserts for an arbitrary presheaf P on C that there is a bijective correspondence between natural transformations $\mathbf{y}(C) \to P$ and elements of the set $P(C)$:

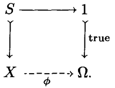
$$\theta\colon \mathrm{Hom}_{\widehat{\mathbf{C}}}(\mathbf{y}(C), P) \xrightarrow{\ \sim\ } P(C), \tag{6}$$

defined for $\alpha\colon \mathbf{y}(C) \to P$ by $\theta(\alpha) = \alpha_C(1_C)$ (see [**CWM**, p. 61]).



Page 32:

Definition. *In a category* **C** *with finite limits, a subobject classifier is a monic,* true: $1 \rightarrowtail \Omega$, *such that to every monic* $S \rightarrowtail X$ *in* **C** *there is a unique arrow* ϕ *which, with the given monic, forms a pullback square*

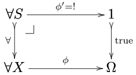


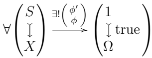
(3)

In other words, every subobject is uniquely a pullback of a "universal" monic true.

This property amounts to saying that the subobject functor is representable (i.e., isomorphic to a Hom-functor). In detail, a subobject of an object X in any category **C** is an equivalence class of monics $m\colon S \rightarrowtail X$ to X (cf. the preliminaries). By a familiar abuse of language, we say that the subobject *is* S or *is* m, meaning always the equivalence class of m. Then, $\mathrm{Sub}_{\mathbf{C}}\, X$ is the set of all subobjects of X in the category **C**; this set is partially ordered under inclusion. The category **C** is said to be *well-powered* when $\mathrm{Sub}_{\mathbf{C}}\, X$ is isomorphic to a small set for all X; all of our typical categories are well-powered. Now given an arrow $f\colon Y \to X$ in **C**, the pullback of any monic $m\colon S \rightarrowtail X$ along f is a monic $m'\colon S' \rightarrowtail Y$, and the assignment $m \mapsto m'$ defines a function $\mathrm{Sub}_{\mathbf{C}} f\colon \mathrm{Sub}_{\mathbf{C}}\, X \to \mathrm{Sub}_{\mathbf{C}}\, Y$; when **C** is well-powered, this makes $\mathrm{Sub}_{\mathbf{C}}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$ a functor to **Sets**. Briefly, Sub is a functor "by pullback".

In the category of monic and pullbacks,



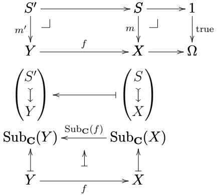


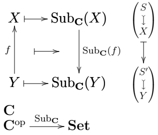
and $\phi' =!$, so $\exists!\phi$.

$\begin{pmatrix} S \\ \downarrow \\ X \end{pmatrix}$ is an element of $\mathrm{Sub}_{\mathbf{C}}(X)$.

$\begin{pmatrix} S \\ \downarrow \\ X \end{pmatrix}$ is an equivalence class,

and may not be a set.



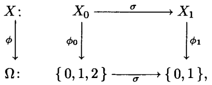


Pages 35 and 36:

In **Sets** × **Sets**, an arrow is a pair of functions $f\colon Y \to X$, $f'\colon Y' \to X'$. The pair of subsets $(1 \subset 2\,,\, 1 \subset 2)$ is a subobject classifier, and the characteristic arrow of any subobject $(S \subset X\,,\; S' \subset X')$ is evidently just the pair of characteristic functions $(\phi_S\colon X \to 2\,,\; \phi_{S'}\colon X' \to 2)$ from the category **Sets**. Thus, there are, in 2×2, four "truth-values". The corresponding subobject classifier for $\mathbf{Sets}^n$ has 2^n truth-values; as we shall see, it is the Boolean algebra of all 2^n subsets of n.

For the arrow category **2** and $\mathbf{Sets}^{\mathbf{2}}$, a subset $(S_0 \xrightarrow{\sigma} S_1) \rightarrowtail (X_0 \xrightarrow{\sigma} X_1)$ is a pair of subsets $S_0 \subset X_0$, $S_1 \subset X_1$ with $\sigma S_0 \subset S_1$. Relative to this subset S there are three sorts of elements x of X_0: Those x in S_0,

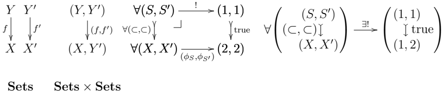
those $x \notin S_0$ with $\sigma x \in S_1$, and those x with $\sigma x \notin S_1$. Define $\phi_0 x = 0$, 1, or 2 accordingly. Then, ϕ_0 on S_0, with the usual characteristic function ϕ_1 of $S_1 \subset X_1$, is an arrow $\phi = (\phi_0, \phi_1)$ to the object Ω displayed below,

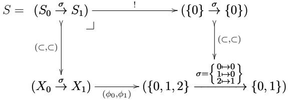


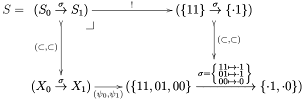
$\sigma 0 = 0$, $\sigma 1 = 0$, $\sigma 2 = 1$,

in $\mathbf{Sets}^{\mathbf{2}}$, and $S_0 \to S_1$ is the inverse image of $(\{0\} \xrightarrow{1} \{0\}) = 1 \rightarrowtail \Omega$.

In brief, this characteristic function $\phi = \langle\, \phi_0, \phi_1 \,\rangle$ is that arrow which specifies whether "x is in S" is "true" always, only at 1, or never. One may say that ϕ gives the "time till truth".







$x \in S_0$	$\sigma x \in S_1$	$\phi_0 x$	$\phi_1(\sigma x)$	$\psi_0 x$	$\psi_1(\sigma x)$
T	**T**	0	0	11	·1
F	**T**	1	0	01	·1
F	**F**	2	1	00	·0

See:
http://angg.twu.net/LATEX/2020clops-and-tops.pdf#page=29

Page 63, exercise 8:

8. Consider a small category $\mathbf{C}$. For each object B of $\mathbf{C}$ there is a functor $D_B\colon \mathbf{C}/B \to \mathbf{C}$ defined by taking the domain of each arrow to B. Hence, each $T\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$ yields $T_B = T \circ D^{\mathrm{op}}\colon (\mathbf{C}/B)^{\mathrm{op}} \to \mathbf{Sets}$. Define an exponential T^S by

$$T^S(B) = \mathrm{Hom}_{\widehat{(\mathbf{C}/B)}}(S_B, T_B),$$

with the evident evaluations $e_B\colon T^S(B) \times S(B) \to T(B)$. Show that T^S with this evaluation e is indeed the exponential in the functor category $\widehat{\mathbf{C}} = \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$.

Page 63, exercise 10:

10. Generalize Theorem 2 of Section 9 to presheaf categories. More precisely, prove that for a morphism (i.e., a natural transformation) $f\colon Z \to Y$ in $\widehat{\mathbf{C}} = \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, the pullback functor

$$f^*\colon \mathrm{Sub}_{\widehat{\mathbf{C}}}(Y) \to \mathrm{Sub}_{\widehat{\mathbf{C}}}(Z)$$

has both a left adjoint $\exists_f$ and a right adjoint $\forall_f$. [Hint: the left adjoint can be constructed by taking the pointwise image. Define the right adjoint $\forall_f$ on a subfunctor S of Z by $\forall_f(S)(C) = \{y \in Y(C) \mid \text{ for all } u\colon D \to C \text{ in } \mathbf{C} \text{ and } z \in Z(D),\ z \in S(D) \text{ whenever } f_D(z) = yu\}$.]