On a way to visualize some Grothendieck Topologies

(talk @ XX EBL)

http://angg.twu.net/math-b.html#2022-ebl

Eduardo Ochs

Salvador, 2022sep13

A definition

A *modern mathematician* is someone who is trying to learn proof assistants, and who is trying to translate its mathematical knowledge into the language of proof assistants.

I am a "modern mathematician" in that sense.
I am learning Agda (slowly).

Canonical Grothendieck Topology

The *canonical Grothendieck topology* on $\mathbb{R}$, J_{can}, is easy to define, but the definition takes several steps...

1. for each open set $U \in \mathcal{O}(\mathbb{R})$ a *sieve on* U is a subset of $\mathcal{O}(U)$ that is downward-closed;

2. for each $U \in \mathcal{O}(\mathbb{R})$ we write $\Omega(U)$ for the set of all sieves on U;

3. we say that a sieve $\mathcal{S} \in \Omega(U)$ is *covering* when $\bigcup \mathcal{S} = U$;

4. for each $U \in \mathcal{O}(\mathbb{R})$ we define $J_{\mathrm{can}}(U)$ as the set of covering sieves on U.

Canonical Grothendieck Topology (2)

The *canonical Grothendieck topology* $\boxed{\text{on } (X, \mathcal{O}(X))}$, J_{can}, is easy to define, but the definition takes several steps...

1. for each open set $U \in \mathcal{O}(X)$ a *sieve on U* is a subset of $\mathcal{O}(U)$ that is downward-closed;

2. for each $U \in \mathcal{O}(X)$ we write $\Omega(U)$ for the set of all sieves on U;

3. we say that a sieve $\mathcal{S} \in \Omega(U)$ is *covering* when $\bigcup \mathcal{S} = U$;

4. for each $U \in \mathcal{O}(X)$ we define $J_{\text{can}}(U)$ as the set of covering sieves on U.

To a “modern mathematician” these definitions feel very wrong.

In, ahem, “modern mathematics”, things have to be defined in a certain order.

That order is very similar to how we write “generators” and “filters” in set comprehensions. For example:

$$\{\,\underbrace{(x,y)}_{\text{result}} \mid \underbrace{x \in \{0,\ldots,5\}}_{\text{generator}}, \underbrace{y \in \{0,\ldots,x\}}_{\text{generator}}, \underbrace{x^2+y^2 \le 25}_{\text{filter}}\,\}$$

$$\Rightarrow \quad \{\,\underbrace{x \in \{0,\ldots,5\}}_{\text{generator}}, \underbrace{y \in \{0,\ldots,x\}}_{\text{generator}}, \underbrace{x^2+y^2 \le 25}_{\text{filter}}\;;\; \underbrace{(x,y)}_{\text{result}}\,\}$$

A *series of (four) generators*
is something like this:

$$\begin{array}{rcl} a & \in & A, \\ b & \in & B(a), \\ c & \in & C(a,b), \\ d & \in & D(a,b,c) \end{array}$$

where:

$$\begin{array}{clll} a & \text{is a variable,} & A & \text{is an expression that yields a set,} \\ b & \text{is a variable,} & B(a) & \text{is an expression that yields a set} \\ & & & \text{and that may depend on } a, \\ c & \text{is a variable,} & C(a,b) & \text{is an expression that yields a set} \\ & & & \text{and that may depend on } a \text{ and } b, \\ d & \text{is a variable,} & D(a,b,c) & \text{is an expression that yields a set} \\ & & & \text{and that may depend on } a,\ b, \text{ and } c \end{array}$$

A *context* with four declarations
in Dependent Type Theory (DTT)
is something like this:

$$\begin{array}{ccl} a & : & A, \\ b & : & B(a), \\ c & : & C(a,b), \\ d & : & D(a,b,c) \end{array}$$

Hints for beginners:
think that the ':'s are '∈'s,
and think that the types
$A, B(a), C(a,b), D(a,b,c)$
are sets.

Propositions

(From Martin-Löf's *Intuitionistic Type Theory*, p.6):

Classically, a proposition is nothing but a truth value, that is, an element of the set of truth values, whose two elements are the true and the false. Because of the difficulties of justifying the rules for forming propositions by means of quantification over infinite domains, when a proposition is understood as a truth value, this explanation is rejected by the intuitionists and replaced by saying that

> a proposition is defined by laying down what counts as a proof of the proposition,

and that

> a proposition is true if it has a proof, that is, if a proof of it can be given.

Thus, intuitionistically, truth is identified with provability, though of course not (because of Godel's incompleteness theorem) with derivability within any particular formal system.

Types in Physics

When I tried to study Physics I tried to assign types to variables.

An instant, t, would be an element of the set of all instants.
A horizontal position, x, would be an element of the set of all horizontal positions.
The set of all instants and the set of horizontal positions would be two "different" (?!?!) copies of $\mathbb{R}$. Suppose that these copies were called T and X.
The notation $[\![\cdot]\!]$ is pronounced "the space of".
There is a "dictionary of types" that assigned to most $\boxed{\text{names}}$ of variables their "types"... so:

$$\begin{array}{rcl} t, t_0, t_1 & \in & T \\ x, x_0, x_1 & \in & X \\ [\![t]\!], [\![t_0]\!], [\![t_1]\!] & = & T \\ [\![x]\!], [\![x_0]\!], [\![x_1]\!] & = & X \end{array}$$

When P is a proposition
I write $[\![P]\!]$ for the set/space of all proofs of P,
and I write $\langle\!\langle P \rangle\!\rangle$ for $\boxed{\text{a}}$ proof of P...

(Indefinite article! $\uparrow$)

A few months ago I finished a paper called
"On the Missing Diagrams in Category Theory",
and it explains these tricks with "the space of"
and indefinite articles.

Link:
`http://angg.twu.net/math-b.html#md`

Values for contexts

$$\{\underbrace{(x,y)}_{\text{result}} \mid \underbrace{x \in \{0,\ldots,5\}}_{\text{generator}}, \underbrace{y \in \{0,\ldots,x\}}_{\text{generator}}, \underbrace{x^2+y^2 \leq 25}_{\text{filter}}\}$$

$$\Rightarrow \quad \{\underbrace{x \in \{0,\ldots,5\}}_{\text{generator}}, \underbrace{y \in \{0,\ldots,x\}}_{\text{generator}}, \underbrace{x^2+y^2 \leq 25}_{\text{filter}} \,;\, \underbrace{(x,y)}_{\text{result}}\}$$

$$\Rightarrow \quad \{\underbrace{\underbrace{x \in \{0,\ldots,5\}}_{\text{generator}}, \underbrace{y \in \{0,\ldots,x\}}_{\text{generator}}, \underbrace{\langle\!\langle x^2+y^2 \leq 25\rangle\!\rangle}_{\text{filter}}}_{\text{context}} \,;\, \underbrace{(x,y)}_{\text{result}}\}$$

A *value* for the context above
is a triple like this:

$$(x, y, \langle\!\langle x^2+y^2 \leq 25\rangle\!\rangle)$$

Long names and telescopes

In Agda these are roughly equivalent:

$$
\begin{array}{rcl}
a & : & A, \\
b & : & B(a), \\
c & : & C(a,b), \\
\langle d,e,f\rangle & : & \Sigma d{:}D(a,b,c). \\
 & & \Sigma e{:}E(a,b,c,d), \\
 & & \Sigma f{:}F(a,b,c,d,e) \\
\multicolumn{3}{l}{\quad \textsf{open}\ \langle d,e,f\rangle,} \\
g & : & G(a,b,c,d,e,f), \\
h & : & H(a,b,c,d,e,f,g)
\end{array}
\qquad\qquad
\begin{array}{rcl}
a & : & A, \\
b & : & B(a), \\
c & : & C(a,b), \\
d & : & D(a,b,c), \\
e & : & E(a,b,c,d), \\
f & : & F(a,b,c,d,e), \\
\langle d,e,f\rangle & = & (d,e,f), \\
g & : & G(a,b,c,d,e,f), \\
h & : & H(a,b,c,d,e,f,g)
\end{array}
$$

See: "*25 Years of AutoMATH*"
and N.G. de Bruijn's paper from 1991:
"*Telescopic Mappings in Typed Lambda Calculus*"

Grothendieck Topologies via telescopes

Now the natural thing to do would be to define Grothendieck Topologies in Agda-ish pseudocode, and then close the parts of the telescope that are made of proof terms, like:

$$\langle\!\langle \mathcal{O}(X) \text{ is a topology on } X \rangle\!\rangle$$

I tried to do that, and it didn't help. What worked for me was to ignore all the proof terms, and to try to obtain values for contexts like:

$$(X, \mathcal{O}(X), U, \mathcal{S}, \mathcal{U}, V)$$

The bottle topology in $\mathbb{R}$
Let:

$$\begin{array}{rcl}
\mathcal{A}_5 & = & \left\{\begin{matrix} A_1, & \\ \downarrow & \searrow \\ A_2, & A_3, \\ \downarrow & \downarrow \\ A_4, & A_5 \end{matrix}\right\} = \left\{\begin{matrix} [2,3], & \\ \downarrow & \searrow \\ [1,2), & (3,4], \\ \downarrow & \downarrow \\ (-\infty,1), & (4,+\infty) \end{matrix}\right\}, \\
\mathcal{A}_{32} & = & \{ \bigcup \mathcal{A}' \mid \mathcal{A}' \subseteq \mathcal{A}_5 \}, \\
\mathcal{O}_B(\mathbb{R}) & = & \mathcal{O}(\mathbb{R}) \cap \mathcal{A}_{32},
\end{array}$$

Then $(\mathbb{R}, \mathcal{O}_B(\mathbb{R}))$ is a
topological space
with 10 open sets...

The bottle topology in $\mathbb{R}$ (2)

$$
\begin{tikzcd}
& (-\infty,+\infty) & & & \\
& & \begin{array}{c}(-\infty,2)\cup\\(3,+\infty)\end{array} \arrow[ul] & & \\
& \begin{array}{c}(-\infty,2)\cup\\(4,+\infty)\end{array} \arrow[ur] & & \begin{array}{c}(-\infty,1)\cup\\(3,+\infty)\end{array} \arrow[ul] & \\
(-\infty,2) \arrow[ur] & & \begin{array}{c}(-\infty,1)\cup\\(4,+\infty)\end{array} \arrow[ul] \arrow[ur] & & (3,+\infty) \arrow[ul] \\
& (-\infty,1) \arrow[ul] \arrow[ur] & & (4,+\infty) \arrow[ul] \arrow[ur] & \\
& & \emptyset \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

The bottle topology in $\mathbb{R}$ (3)

See my "Planar Heyting Algebras for Children":
http://angg.twu.net/math-b.html#zhas-for-children-2

$$\left(\mathcal{O}\left(\begin{tikzcd}[row sep=small, column sep=small] 3\_ \arrow[d] \arrow[dr] & \\ 2\_ \arrow[d] & \_2 \arrow[d] \\ 1\_ & \_1 \end{tikzcd}\right), \subseteq\right) \quad = \quad \begin{tikzcd}[row sep=small, column sep=small] & 32 & & & \\ & & 22 \arrow[ul] & & \\ & 21 \arrow[ur] & & 12 \arrow[ul] & \\ 20 \arrow[ur] & & 11 \arrow[ul] \arrow[ur] & & 02 \arrow[ul] \\ & 10 \arrow[ul] \arrow[ur] & & 01 \arrow[ul] \arrow[ur] & \\ & & 00 \arrow[ul] \arrow[ur] & & \end{tikzcd}$$

Why "bottle"?

From Bagpuss, episode 1:

Conventions on names

Here the diagram at the right *complements* the diagram on the left...

$$
\begin{array}{ccccccc}
a & \in & A & \in & \mathcal{P}(X) \\
 & & & & \rotatebox{90}{$\subseteq$} \\
 & & U & \in & \mathcal{O}(X) \\
 & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
x & \in & V & \in & \mathcal{O}(U)
\end{array}
$$

$$
\begin{array}{ccccc}
\left(\begin{array}{c}\text{a}\\ \text{point}\\ \text{of } X\end{array}\right) & \in & \left(\begin{array}{c}\text{a}\\ \text{subset}\\ \text{of } X\end{array}\right) & \in & \left(\begin{array}{c}\text{all}\\ \text{subsets}\\ \text{of } X\end{array}\right) \\
 & & & & \rotatebox{90}{$\subseteq$} \\
 & & \left(\begin{array}{c}\text{an open}\\ \text{subset}\\ \text{of } X\end{array}\right) & \in & \left(\begin{array}{c}\text{all open}\\ \text{subsets}\\ \text{of } X\end{array}\right) \\
 & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\left(\begin{array}{c}\text{a}\\ \text{point}\\ \text{of } V\end{array}\right) & \in & \left(\begin{array}{c}\text{an open}\\ \text{subset}\\ \text{of } U\end{array}\right) & \in & \left(\begin{array}{c}\text{all open}\\ \text{subsets}\\ \text{of } U\end{array}\right)
\end{array}
$$

Visualizing downward-closedness

A *sieve* $\mathcal{S}$ on U is a subset of $\mathcal{O}(U)$
that is "downward-closed", i.e.:

$$\forall V, W \in \mathcal{O}(U).\ \begin{pmatrix} V \\ \text{above} \\ W \end{pmatrix} \to \begin{pmatrix} V \in \mathcal{S} \\ \downarrow \\ W \in \mathcal{S} \end{pmatrix}$$

$$\forall V, W \in \mathcal{O}(U).\ \begin{pmatrix} V \\ \cup\!\! \mid \\ W \end{pmatrix} \to \begin{pmatrix} V \in \mathcal{S} \\ \downarrow \\ W \in \mathcal{S} \end{pmatrix}$$

Visualizing downward-closedness (2)

General case: $(X, \mathcal{O}(X), U, \mathcal{S})$

Particular case: $(\mathbb{R}, \mathcal{O}_B(\mathbb{R}), 21, \{00, 01, 10, 11, 20\})$

$$\mathcal{O}_B(\mathbb{R}) \quad \Rightarrow \quad \left(\begin{array}{ccccc} & 32 & & & \\ & & 22 & & \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{array}\right)$$

$$\mathcal{O}(21) \subseteq \mathcal{O}_B(\mathbb{R}) \quad \Rightarrow \quad \left(\begin{array}{ccccc} & 0 & & & \\ & & 0 & & \\ & 1 & & 0 & \\ 1 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array}\right)$$

$$\{00, 01, 10, 11, 20\} \subseteq \mathcal{O}(21) \quad \Rightarrow \quad \left(\begin{array}{ccccc} & \cdot & & & \\ & & \cdot & & \\ & 0 & & \cdot & \\ 1 & & 1 & & \cdot \\ & 1 & & 1 & \\ & & 1 & & \end{array}\right)$$

Everything is finite, so...

Every intersection of open sets is an open set.
Every intersection of covering sieves is a covering sieve.

$\uparrow$ Huh???

Mac Lane/Moerdijk, p.110, translated

Definition. A *Grothendieck Topology* on a category $\mathcal{O}(X)$ is a function J which assigns to each object U of $\mathcal{O}(U)$ a collection $J(U)$ of sieves on U, in such a way that:

(J1) The maximal sieve $t(U) = \{\, V \in \mathcal{O}(X) \mid V \subseteq U \,\}$ is in $J(U)$;

(J2) (stability axiom) if $\mathcal{U} \in J(U)$, then $(V \subseteq U)^*(\mathcal{U}) \in J(V)$ for any open set $V \in \mathcal{O}(U)$;

(J3) (transitivity axiom) if $\mathcal{U} \in J(U)$ and $\mathcal{S}$ is any sieve on U such that $(U \subseteq V)^*(\mathcal{S}) \in J(V)$ for all $V \in \mathcal{O}(U)$ in $\mathcal{U}$, then $\mathcal{S} \in J(U)$.

Grothendieck Topologies backwards

My favorite way to present GroTops —
obs: it was the way that worked for me —
is to start with the canonical GroTop, J_{can},
then check that it obeys the axioms J1, J2, and J3,
and only them look for other GroTops that also
obey J1, J2, and J3...

$$\begin{array}{rcl}
\Omega(U) & = & \{\, \mathcal{S} \in \mathcal{O}(U) \mid \mathcal{S} \text{ is a sieve on } U \,\} \\
 & = & \{\, \mathcal{S} \in \mathcal{O}(U) \mid \mathcal{S} \text{ is downward-closed} \,\} \\
J_{\text{can}}(U) & = & \{\, \mathcal{S} \mid \mathcal{S} \text{ is a sieve on } U, \mathcal{S} \text{ covers } U \,\} \\
 & = & \{\, \mathcal{S} \in \Omega(U) \mid \bigcup \mathcal{S} = U \,\} \\
J(U) & = & \{\, \mathcal{S} \mid \mathcal{S} \text{ is a sieve on } U, \mathcal{S} \ J\text{-covers } U \,\} \\
 & = & \{\, \mathcal{S} \in \Omega(U) \mid \mathcal{S} \in J(U) \,\}
\end{array}$$

Grothendieck Topologies backwards (2)

One of the consequences of J1, J2, J3, is this:
If $\mathcal{U}'$ and $\mathcal{U}''$ are J-covering sieves on U, i.e., if $\mathcal{U}', \mathcal{U}'' \in J(U)$,
then $\mathcal{U}' \cap \mathcal{U}''$ is a J-covering sieve on U, i.e., $\mathcal{U}' \cap \mathcal{U}'' \in J(U)$...

$\boxed{\text{If everything is finite}}$ then:

$$\begin{array}{rcl} \bigcap J(U) & \in & J(U) \\ \uparrow (\bigcap J(U)) & = & J(U) \end{array}$$

So every $J(U)$ has a minimal element, and is generated by it.

The topology $\mathcal{O}_B(\mathbb{R})$ has only 10 'U's.
We can try to draw these maps:

$$\begin{array}{rcl} U & \mapsto & \bigcap J(U) \\ U & \mapsto & \uparrow (\bigcap J(U)) \end{array}$$

All sieves of a certain form

I will write this (note the square brackets!!!):

$$\left[\begin{array}{ccccc} & \cdot & & & \\ & & \cdot & & \\ & 0 & & \cdot & \\ ? & & ? & & \cdot \\ & ? & & 1 & \\ & & 1 & & \end{array}\right]$$

for the set of all sieves of that form. So:

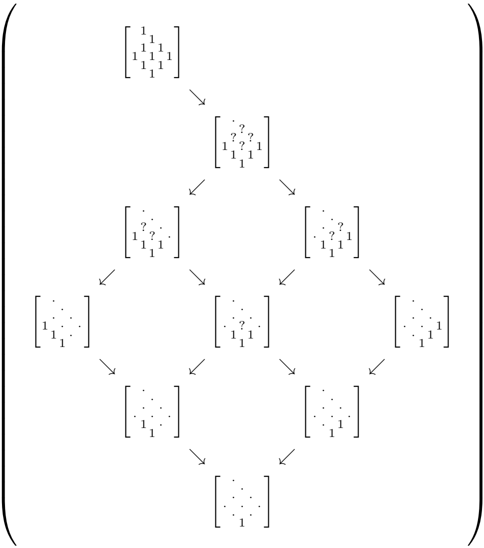
$$\left[\begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & 0 & & \cdot & \\ ? & & ? & & \cdot \\ & ? & & 1 & \\ & & 1 & & \end{smallmatrix}\right] = \left\{ \left(\begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & 0 & & \cdot & \\ 0 & & 0 & & \cdot \\ & 0 & & 1 & \\ & & 1 & & \end{smallmatrix}\right), \left(\begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & 0 & & \cdot & \\ 0 & & 0 & & \cdot \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix}\right), \left(\begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & 0 & & \cdot & \\ 0 & & 1 & & \cdot \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix}\right), \left(\begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & 0 & & \cdot & \\ 1 & & 0 & & \cdot \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix}\right), \left(\begin{smallmatrix} & \cdot & & & \\ & & \cdot & & \\ & 0 & & \cdot & \\ 1 & & 1 & & \cdot \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix}\right) \right\}$$

Some big figures
Next page: J_{can} on $\mathcal{O}_B(\mathbb{R})$.
After that: Ω on $\mathcal{O}_B(\mathbb{R})$.

For more figures — including a way to find all GroTops on a given finite topology — see these (very unfinished) notes:

http://angg.twu.net/math-b.html#2021-groth-tops

$J_{\mathrm{can}} =$



$\Omega =$