Cálculo C2 - 2023.2

Aula 31: revisão de números complexos

Eduardo Ochs - RCN/PURO/UFF
http://anggtwu.net/2023.2-C2.html

Links

StewPtCap17p6 (p.1020) Equações diferenciais de 2ª ordem
StewPtCap17p20 (p.1034) Caso 3: subamortecimento
StewPtApendiceHp5 (p.A51) Apêndice H: Números complexos

BoyceDip3p5 (p.105) Capítulo 3: Equações lineares de 2ª ordem
BoyceDip3p11 (p.111) Seção 3.2: o operador diferencial L
BoyceDip3p13 (p.113) Teorema 3.2.2: o princípio da superposição
BoyceDip3p21 (p.121) 3.3. Raízes complexas da equação característica
BoyceDip3p23 (p.123) Figura 3.3.1

BoyceDipEng3p4 (p.103) Chapter 3: Second-order linear ODEs
BoyceDipEng3p11 (p.110) Section 3.2: the differential operator L
BoyceDipEng3p13 (p.112) Theorem 3.2.2: principle of superposition
BoyceDipEng3p21 (p.120) 3.3 Complex Roots of the Characteristic Equation
BoyceDipEng3p24 (p.123) Figure 3.3.1
`https://en.wikipedia.org/wiki/Complex_number` (bom)
`https://pt.wikipedia.org/wiki/N%C3%BAmero_complexo` (ruim, cheio de erros)

2yT12 (Gabarito da P1 de 2019.2) A questão 3 usa o truque do E

HernandezP57 (p.47) principais identidades trigonométricas

$$
\begin{array}{rcll}
a, b, c, d & \in & \mathbb{R} & \\
z, w & \in & \mathbb{C} & \\
\theta & \in & \mathbb{R} & \text{(ângulo)} \\
k & \in & \mathbb{Z} & \\
\\
\mathsf{Re}(a+bi) & = & a & \text{(parte real)} \\
\mathsf{Im}(a+bi) & = & b & \text{(parte imaginária)} \\
z & = & \mathsf{Re}(z)+\mathsf{Im}(z)i & \text{(isto sempre vale)} \\
\overline{z} & = & \mathsf{Re}(z)-\mathsf{Im}(z)i & \text{(conjugado: definição fácil)} \\
\overline{a+bi} & = & a-bi & \text{(conjugado: definição difícil)} \\
|z| & = & \sqrt{\mathsf{Re}(z)^2+\mathsf{Im}(z)^2} & \text{(módulo/norma: definição fácil)} \\
|a+bi| & = & \sqrt{a^2+b^2} & \text{(módulo/norma: definição difícil)} \\
\\
180^\circ & = & \pi & \text{(}\leftarrow\text{ lembre)} \\
1^\circ & = & \frac{\pi}{180} & \text{(}\leftarrow\text{ lembre)} \\
42^\circ & = & 42\frac{\pi}{180} & \\
{}^\circ & = & \frac{\pi}{180} & \text{(podemos tratar o }^\circ\text{ como uma constante)} \\
\\
e^{i\theta} & = & \cos\theta + i\,\mathrm{sen}\,\theta & \text{(vamos entender isto aos poucos)} \\
E & = & c+is & \text{(abreviação pra igualdade acima)} \\
\\
z & = & |z|\,e^{i\mathsf{arg}(z)} & \text{(vamos entender isto aos poucos)} \\
1+i & = & |1+1i|\,e^{i\mathsf{arg}(1+i)} & \text{(}\leftarrow\text{ exemplo)} \\
 & = & \sqrt{1^2+1^2}\,e^{i45^\circ} & \\
 & = & \sqrt{2}\,e^{i\frac{\pi}{4}} &
\end{array}
$$

$$
\begin{array}{rcl}
(a+bi)(c+di) & = & a(c+di)+bi(c+di) \\
 & = & ac+adi+bic+bidi \\
 & = & ac+adi+bci+bd(i^2) \\
 & = & ac+adi+bci+bd(-1) \\
 & = & ac+adi+bci-bd \\
 & = & ac-bd+adi+bci \\
 & = & (ac-bd)+(ad+bc)i \\
\\
(ae^{i\alpha})(be^{i\beta}) & = & (ab)(e^{i\alpha}\,e^{i\beta}) \\
 & = & (ab)(e^{i\alpha+i\beta}) \\
 & = & (ab)(e^{i(\alpha+\beta)}) \\
 & = & (ab)(e^{i(\alpha+\beta)})
\end{array}
$$

“Partes de cima”

Fórmulas e definições:

$$\begin{array}{rcl}
e^{i\theta} &=& \cos\theta + i\,\mathrm{sen}\,\theta \\
e^{ik\theta} &=& \cos k\theta + i\,\mathrm{sen}\,k\theta \\
e^{-i\theta} &=& \cos -\theta + i\,\mathrm{sen}\,-\theta \\
&=& \cos\theta + i(-\,\mathrm{sen}\,\theta) \\
&=& \cos\theta - i(\mathrm{sen}\,\theta) \\
e^{i\theta} + e^{-i\theta} &=& \cos\theta + i\,\mathrm{sen}\,\theta \\
&+& \cos\theta - i\,\mathrm{sen}\,\theta \\
&=& 2\cos\theta \\
e^{i\theta} - e^{-i\theta} &=& \cos\theta + i\,\mathrm{sen}\,\theta \\
&-& (\cos\theta - i\,\mathrm{sen}\,\theta) \\
&=& 2i\,\mathrm{sen}\,\theta \\
\frac{e^{i\theta}+e^{-i\theta}}{2} &=& \cos\theta \\
\frac{e^{i\theta}-e^{-i\theta}}{2i} &=& \mathrm{sen}\,\theta \\
\frac{e^{ik\theta}+e^{-ik\theta}}{2} &=& \cos k\theta \\
\frac{e^{ik\theta}-e^{-ik\theta}}{2i} &=& \mathrm{sen}\,k\theta \\
\mathrm{ccos}\,\theta &=& e^{i\theta} + e^{-i\theta} \\
\mathrm{csen}\,\theta &=& e^{i\theta} - e^{-i\theta} \\
\mathrm{ccos}\,k\theta &=& e^{ik\theta} + e^{-ik\theta} \\
\mathrm{csen}\,k\theta &=& e^{ik\theta} - e^{-ik\theta}
\end{array}$$

$$\begin{array}{rcl}
E &=& c + is \\
e^{ik\theta} &=& \cos k\theta + i\,\mathrm{sen}\,k\theta \\
E^{-1} &=& \cos -\theta + i\,\mathrm{sen}\,-\theta \\
&=& c + i(-s) \\
&=& c - i(s) \\
E + E^{-1} &=& c + is \\
&+& c - is \\
&=& 2c \\
E - E^{-1} &=& c + is \\
&-& (c - is) \\
&=& 2is \\
\frac{E+E^{-1}}{2} &=& c \\
\frac{E-E^{-1}}{2i} &=& s \\
\frac{E^k+E^{-k}}{2} &=& \cos k\theta \\
\frac{E^k-E^{-k}}{2i} &=& \mathrm{sen}\,k\theta \\
\mathrm{ccos}\,\theta &=& E + E^{-1} \\
\mathrm{csen}\,\theta &=& E - E^{-1} \\
\mathrm{ccos}\,k\theta &=& E^k + E^{-k} \\
\mathrm{csen}\,k\theta &=& E^k - E^{-k}
\end{array}$$

O seno e o cosseno “são” frações.
O $\boxed{\text{c}}$sen é a “parte de cima” do seno.
O $\boxed{\text{c}}$cos é a “parte de cima” do cosseno.

Um exemplo do método:

$$\begin{array}{rcl}
(\cos\theta)^3 &=& (\frac{1}{2}\,\mathrm{ccos}\,\theta)^3 \\
&=& (\frac{1}{2})^3(\mathrm{ccos}\,\theta)^3 \\
(\mathrm{ccos}\,\theta)^3 &=& (E + E^{-1})^3 \\
&=& E^3 + 3E + 3E^{-1} + E^{-3} \\
&=& (E^3 + E^{-3}) + (3E + 3E^{-1}) \\
&=& \mathrm{ccos}\,3\theta + 3\,\mathrm{ccos}\,\theta \\
(\cos\theta)^3 &=& (\frac{1}{2})^3(\mathrm{ccos}\,\theta)^3 \\
&=& (\frac{1}{2})^3(\mathrm{ccos}\,3\theta + 3\,\mathrm{ccos}\,\theta) \\
&=& \frac{1}{4}(\frac{1}{2}\,\mathrm{ccos}\,3\theta + 3\frac{1}{2}\,\mathrm{ccos}\,\theta) \\
&=& \frac{1}{4}(\cos 3\theta + 3\cos\theta)
\end{array}$$

Pra mim a parte do meio é a parte legal das contas, e as partes de cima e de baixo são as partes chatas (por causa das frações).

Compare com o gabarito da questão 3 daqui: 2yT12 (Gabarito da P1 de 2019.2)

Exercício
Use a técnica acima pra integrar:
a) $\int (\cos\theta)^2\,d\theta$
b) $\int (\mathrm{sen}\,\theta)^2\,d\theta$
c) $\int (\mathrm{sen}\,\theta)(\cos\theta)\,d\theta$
d) $\int (\mathrm{sen}\,2\theta)(\cos 3\theta)\,d\theta$

(%i1) p : 4*x^2 + 5*x^1 + 6*x^0 + 7*x^-1 + 8*x^-2;
(%o1)
$$4\,x^2+5\,x+\frac{7}{x}+\frac{8}{x^2}+6$$

(%i2) q : 4*E^2 + 5*E^1 + 6*E^0 + 7*E^-1;
(%o2)
$$4\,E^2+5\,E+\frac{7}{E}+6$$

(%i3) lpdot(p, x);
(%o3)
$$\begin{pmatrix}4 & 5 & 6 & . & 7 & 8\end{pmatrix}$$

(%i4) lpdot(q, E);
(%o4)
$$\begin{pmatrix}4 & 5 & 6 & . & 7\end{pmatrix}$$

(%i5) f : cos(th)^3;
(%o5)
$$(\cos\theta)^3$$

(%i6) g : ccos(th)^3;
(%o6)
$$8\,(\cos\theta)^3$$

(%i7) lpe(f);
(%o7)
$$\frac{\cos(3\,\theta)}{4}+\frac{3\,\cos\theta}{4}$$

(%i8) lpe(g);
(%o8)
$$2\,\cos(3\,\theta)+6\,\cos\theta$$

(%i9)

(%i9) exponentialize(f);
(%o9)
$$\frac{\left(e^{i\,\theta}+e^{-i\,\theta}\right)^3}{8}$$

(%i10) expand(exponentialize(f));
(%o10)
$$\frac{e^{3\,i\,\theta}}{8}+\frac{3\,e^{i\,\theta}}{8}+\frac{3\,e^{-i\,\theta}}{8}+\frac{e^{-3\,i\,\theta}}{8}$$

(%i11) demoivre(expand(exponentialize(f)));
(%o11)
$$\frac{i\,\sin(3\,\theta)+\cos(3\,\theta)}{8}+\frac{\cos(3\,\theta)-i\,\sin(3\,\theta)}{8}+\frac{3\,(i\,\sin\theta+\cos\theta)}{8}+\frac{3\,(\cos\theta-i\,\sin\theta)}{8}$$

(%i12) expand(demoivre(expand(exponentialize(f))));
(%o12)
$$\frac{\cos(3\,\theta)}{4}+\frac{3\,\cos\theta}{4}$$

(%i13) subst(th_E,expand(exponentialize(f)));
(%o13)
$$\frac{E^3}{8}+\frac{3\,E}{8}+\frac{3}{8\,E}+\frac{1}{8\,E^3}$$

(%i14) subst(th_E,expand(exponentialize(g)));
(%o14)
$$E^3+3\,E+\frac{3}{E}+\frac{1}{E^3}$$

(%i15)
lpE(f);
(%o15)
$$\begin{pmatrix}\frac{1}{8} & 0 & \frac{3}{8} & 0 & . & \frac{3}{8} & 0 & \frac{1}{8}\end{pmatrix}$$

(%i16) lpE(g);
(%o16)
$$\begin{pmatrix}1 & 0 & 3 & 0 & . & 3 & 0 & 1\end{pmatrix}$$

```
(%i17) lpE(  ccos(th)^3);
(%o17)
```
$$\begin{pmatrix} 1 & 0 & 3 & 0 & . & 3 & 0 & 1 \end{pmatrix}$$

```
(%i18) lpE(  ccos(th));
(%o18)
```
$$\begin{pmatrix} 1 & 0 & . & 1 \end{pmatrix}$$

```
(%i19) lpE(3*ccos(th));
(%o19)
```
$$\begin{pmatrix} 3 & 0 & . & 3 \end{pmatrix}$$

```
(%i20) lpE(ccos(3*th));
(%o20)
```
$$\begin{pmatrix} 1 & 0 & 0 & 0 & . & 0 & 0 & 1 \end{pmatrix}$$

```
(%i21) lpE(ccos(3*th)+3*ccos(th));
(%o21)
```
$$\begin{pmatrix} 1 & 0 & 3 & 0 & . & 3 & 0 & 1 \end{pmatrix}$$

```
(%i22)
       lpE( ccos(th)   );
(%o22)
```
$$\begin{pmatrix} 1 & 0 & . & 1 \end{pmatrix}$$

```
(%i23) lpE( ccos(th)^2 );
(%o23)
```
$$\begin{pmatrix} 1 & 0 & 2 & . & 0 & 1 \end{pmatrix}$$

```
(%i24) lpE( ccos(th)^3 );
(%o24)
```
$$\begin{pmatrix} 1 & 0 & 3 & 0 & . & 3 & 0 & 1 \end{pmatrix}$$

```
(%i25) lpE( csin(th)     );
(%o25)
```
$$\begin{pmatrix} 1 & 0 & . & -1 \end{pmatrix}$$

```
(%i26) lpE( csin(th)^2   );
(%o26)
```
$$\begin{pmatrix} 1 & 0 & -2 & . & 0 & 1 \end{pmatrix}$$

```
(%i27) lpE( csin(th)^3   );
(%o27)
```
$$\begin{pmatrix} 1 & 0 & -3 & 0 & . & 3 & 0 & -1 \end{pmatrix}$$

```
(%i28) lpE( csin(2*th)   );
(%o28)
```
$$\begin{pmatrix} 1 & 0 & 0 & . & 0 & -1 \end{pmatrix}$$

```
(%i29) lpE( csin(2*th)^2 );
(%o29)
```
$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & . & 0 & 0 & 0 & 1 \end{pmatrix}$$

```
(%i30)
```