Cálculo 2 - 2024.1

Aula 31: revisão de números complexos

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http://anggtwu.net/2024.1-C2.html

Links

StewPtCap17p6 (p.1020) Equações diferenciais de 2ª ordem
StewPtCap17p20 (p.1034) Caso 3: subamortecimento
StewPtApendiceHp5 (p.A51) Apêndice H: Números complexos

BoyceDip3p5 (p.105) Capítulo 3: Equações lineares de 2ª ordem
BoyceDip3p11 (p.111) Seção 3.2: o operador diferencial L
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`https://en.wikipedia.org/wiki/Complex_number` (bom)
`https://pt.wikipedia.org/wiki/N%C3%BAmero_complexo` (ruim, cheio de erros)

2yT12 (Gabarito da P1 de 2019.2) A questão 3 usa o truque do E

HernandezP57 (p.47) principais identidades trigonométricas

$$\begin{array}{rcll}
a,b,c,d & \in & \mathbb{R} & \\
z,w & \in & \mathbb{C} & \\
\theta & \in & \mathbb{R} & \text{(ângulo)} \\
k & \in & \mathbb{Z} & \\
\\
\mathsf{Re}(a+bi) & = & a & \text{(parte real)} \\
\mathsf{Im}(a+bi) & = & b & \text{(parte imaginária)} \\
z & = & \mathsf{Re}(z)+\mathsf{Im}(z)i & \text{(isto sempre vale)} \\
\overline{z} & = & \mathsf{Re}(z)-\mathsf{Im}(z)i & \text{(conjugado: definição fácil)} \\
\overline{a+bi} & = & a-bi & \text{(conjugado: definição difícil)} \\
|z| & = & \sqrt{\mathsf{Re}(z)^2+\mathsf{Im}(z)^2} & \text{(módulo/norma: definição fácil)} \\
|a+bi| & = & \sqrt{a^2+b^2} & \text{(módulo/norma: definição difícil)} \\
\\
180^\circ & = & \pi & (\leftarrow \text{lembre}) \\
1^\circ & = & \frac{\pi}{180} & (\leftarrow \text{lembre}) \\
42^\circ & = & 42\frac{\pi}{180} & \\
{}^\circ & = & \frac{\pi}{180} & \text{(podemos tratar o }{}^\circ\text{ como uma constante)} \\
\\
e^{i\theta} & = & \cos\theta + i\,\mathrm{sen}\,\theta & \text{(vamos entender isto aos poucos)} \\
E & = & c+is & \text{(abreviação pra igualdade acima)} \\
\\
z & = & |z|\,e^{i\mathsf{arg}(z)} & \text{(vamos entender isto aos poucos)} \\
1+i & = & |1+1i|\,e^{i\mathsf{arg}(1+i)} & (\leftarrow \text{exemplo}) \\
 & = & \sqrt{1^2+1^2}\,e^{i45^\circ} & \\
 & = & \sqrt{2}\,e^{i\frac{\pi}{4}} &
\end{array}$$

$$\begin{array}{rcl}
(a+bi)(c+di) & = & a(c+di)+bi(c+di) \\
 & = & ac+adi+bic+bidi \\
 & = & ac+adi+bci+bd(i^2) \\
 & = & ac+adi+bci+bd(-1) \\
 & = & ac+adi+bci-bd \\
 & = & ac-bd+adi+bci \\
 & = & (ac-bd)+(ad+bc)i \\
\\
(ae^{i\alpha})(be^{i\beta}) & = & (ab)(e^{i\alpha}\,e^{i\beta}) \\
 & = & (ab)(e^{i\alpha+i\beta}) \\
 & = & (ab)(e^{i(\alpha+\beta)}) \\
 & = & (ab)(e^{i(\alpha+\beta)})
\end{array}$$

"Partes de cima"

Fórmulas e definições:

$$\begin{array}{rcl}
e^{i\theta} &=& \cos\theta + i\,\mathrm{sen}\,\theta \\
e^{ik\theta} &=& \cos k\theta + i\,\mathrm{sen}\,k\theta \\
e^{-i\theta} &=& \cos -\theta + i\,\mathrm{sen}\,-\theta \\
&=& \cos\theta + i(-\mathrm{sen}\,\theta) \\
&=& \cos\theta - i(\mathrm{sen}\,\theta) \\
e^{i\theta} + e^{-i\theta} &=& \cos\theta + i\,\mathrm{sen}\,\theta \\
&+& \cos\theta - i\,\mathrm{sen}\,\theta \\
&=& 2\cos\theta \\
e^{i\theta} - e^{-i\theta} &=& \cos\theta + i\,\mathrm{sen}\,\theta \\
&-& (\cos\theta - i\,\mathrm{sen}\,\theta) \\
&=& 2i\,\mathrm{sen}\,\theta \\
\frac{e^{i\theta} + e^{-i\theta}}{2} &=& \cos\theta \\
\frac{e^{i\theta} - e^{-i\theta}}{2i} &=& \mathrm{sen}\,\theta \\
\frac{e^{ik\theta} + e^{-ik\theta}}{2} &=& \cos k\theta \\
\frac{e^{ik\theta} - e^{-ik\theta}}{2i} &=& \mathrm{sen}\,k\theta \\
\mathrm{ccos}\,\theta &=& e^{i\theta} + e^{-i\theta} \\
\mathrm{csen}\,\theta &=& e^{i\theta} - e^{-i\theta} \\
\mathrm{ccos}\,k\theta &=& e^{ik\theta} + e^{-ik\theta} \\
\mathrm{csen}\,k\theta &=& e^{ik\theta} - e^{-ik\theta}
\end{array}$$

$$\begin{array}{rcl}
E &=& c + is \\
e^{ik\theta} &=& \cos k\theta + i\,\mathrm{sen}\,k\theta \\
E^{-1} &=& \cos -\theta + i\,\mathrm{sen}\,-\theta \\
&=& c + i(-s) \\
&=& c - i(s) \\
E + E^{-1} &=& c + is \\
&+& c - is \\
&=& 2c \\
E - E^{-1} &=& c + is \\
&-& (c - is) \\
&=& 2is \\
\frac{E + E^{-1}}{2} &=& c \\
\frac{E - E^{-1}}{2i} &=& s \\
\frac{E^k + E^{-k}}{2} &=& \cos k\theta \\
\frac{E^k - E^{-k}}{2i} &=& \mathrm{sen}\,k\theta \\
\mathrm{ccos}\,\theta &=& E + E^{-1} \\
\mathrm{csen}\,\theta &=& E - E^{-1} \\
\mathrm{ccos}\,k\theta &=& E^k + E^{-k} \\
\mathrm{csen}\,k\theta &=& E^k - E^{-k}
\end{array}$$

O seno e o cosseno "são" frações.
O c sen é a "parte de cima" do seno.
O c cos é a "parte de cima" do cosseno.

Um exemplo do método:

$$\begin{array}{rcl}
(\cos\theta)^3 &=& (\frac{1}{2}\,\mathrm{ccos}\,\theta)^3 \\
&=& (\frac{1}{2})^3(\mathrm{ccos}\,\theta)^3 \\
(\mathrm{ccos}\,\theta)^3 &=& (E + E^{-1})^3 \\
&=& E^3 + 3E + 3E^{-1} + E^{-3} \\
&=& (E^3 + E^{-3}) + (3E + 3E^{-1}) \\
&=& \mathrm{ccos}\,3\theta + 3\,\mathrm{ccos}\,\theta \\
(\cos\theta)^3 &=& (\frac{1}{2})^3(\mathrm{ccos}\,\theta)^3 \\
&=& (\frac{1}{2})^3(\mathrm{ccos}\,3\theta + 3\,\mathrm{ccos}\,\theta) \\
&=& \frac{1}{4}(\frac{1}{2}\,\mathrm{ccos}\,3\theta + 3\frac{1}{2}\,\mathrm{ccos}\,\theta) \\
&=& \frac{1}{4}(\cos 3\theta + 3\cos\theta)
\end{array}$$

Pra mim a parte do meio é a parte legal das contas, e as partes de cima e de baixo são as partes chatas (por causa das frações).

Compare com o gabarito da questão 3 daqui:
2yT12 (Gabarito da P1 de 2019.2)

Exercício
Use a técnica acima pra integrar:
a) $\int (\cos\theta)^2\,d\theta$
b) $\int (\mathrm{sen}\,\theta)^2\,d\theta$
c) $\int (\mathrm{sen}\,\theta)(\cos\theta)\,d\theta$
d) $\int (\mathrm{sen}\,2\theta)(\cos 3\theta)\,d\theta$

```
(%i1) as_33 : create_list(x+%i*y, y, seqn(2,0,2), x, seqn(0,2,2));
(%o1)
```

$$[2i, 2i+1, 2i+2, i, i+1, i+2, 0, 1, 2]$$

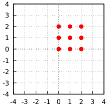
```
(%i2) as_55 : create_list(x+%i*y, y, seqn(2,0,4), x, seqn(0,2,4))$

(%i3) as    : as_33;
(%o3)
```

$$[2i, 2i+1, 2i+2, i, i+1, i+2, 0, 1, 2]$$

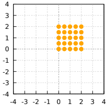
```
(%i4) xyrange(r) := [xr(-r,r), yr(-r,r), more(proportional_axes=xy)];
(%o4)
```

$$\mathrm{xyrange}(r) := [\mathrm{xr}(-r,r), \mathrm{yr}(-r,r), \mathrm{more}(\mathrm{proportional\_axes}=\mathrm{xy})]$$

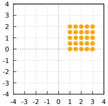
```
(%i5) myqdraw_nhr(n,h,r,[rest]) :=
        myQdraw(format("complex--a",n), format("height=~acm",h), xyrange(r), rest);
(%i6)  myqdraw_nhr(1,5,4, zpts(as_33,    myps(4),pc(red)));
(%o6)
```



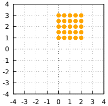

```
(%i7) myqdraw_nhr(2,5,4, zpts(as_55,    myps(4),pc(orange)));
(%o7)
```



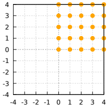

```
(%i8) myqdraw_nhr(3,5,4, zpts(as_55+1,  myps(4),pc(orange)));
(%o8)
```



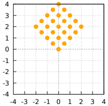

```
(%i9) myqdraw_nhr(4,5,4, zpts(as_55+%i, myps(4),pc(orange)));
(%o9)
```




```
(%i10) myqdraw_nhr(5,5,4, zpts(as_55*2,  myps(4),pc(orange)));
(%o10)
```

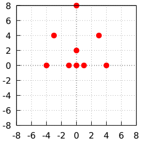



```
(%i11) myqdraw_nhr(6,5,4, zpts(as_55*(1+%i), myps(4),pc(orange)));
(%o11)
```

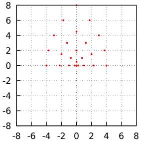



```
(%i12)
```

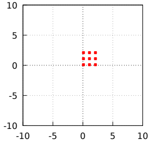
```
(%i12) asq_33 : makelist(z^2, z, as_33)$
(%i13) asq_55 : makelist(z^2, z, as_55)$
(%i14) myqdraw_nhr(7,5,8, zpts(asq_33, myps(4),pc(red)));
(%o14)
```



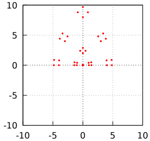

```
(%i15) myqdraw_nhr(8,5,8, zpts(asq_55, myps(1),pc(red)));
(%o15)
```




```
(%i16) as_22   : create_list(x+%i*y, y, seqn(0,1,1), x, seqn(0,1,1))$
(%i17) as_332  : create_list(z+w,    z, as_33, w, as_22*0.2)$
(%i18) asq_332 : makelist(z^2, z, as_332)$
(%i19) myqdraw_nhr( 9,5,10,  zpts(as_332, myps(1),pc(red)));
(%o19)
```




```
(%i20) myqdraw_nhr(10,5,10, zpts(asq_332, myps(1),pc(red)));
(%o20)
```




```
(%i21)
```

```
(%i1) p : 4*x^2 + 5*x^1 + 6*x^0 + 7*x^-1 + 8*x^-2;
```
(%o1)
$$4\,x^2+5\,x+\frac{7}{x}+\frac{8}{x^2}+6$$

```
(%i2) q : 4*E^2 + 5*E^1 + 6*E^0 + 7*E^-1;
```
(%o2)
$$4\,E^2+5\,E+\frac{7}{E}+6$$

```
(%i3) lpdot(p, x);
```
(%o3)
$$\begin{pmatrix}4 & 5 & 6 & . & 7 & 8\end{pmatrix}$$

```
(%i4) lpdot(q, E);
```
(%o4)
$$\begin{pmatrix}4 & 5 & 6 & . & 7\end{pmatrix}$$

```
(%i5) f :  cos(th)^3;
```
(%o5)
$$(\cos\theta)^3$$

```
(%i6) g : ccos(th)^3;
```
(%o6)
$$8\,(\cos\theta)^3$$

```
(%i7) lpe(f);
```
(%o7)
$$\frac{\cos(3\,\theta)}{4}+\frac{3\,\cos\theta}{4}$$

```
(%i8) lpe(g);
```
(%o8)
$$2\,\cos(3\,\theta)+6\,\cos\theta$$

(%i9)

```
(%i9) exponentialize(f);
```
(%o9)
$$\frac{\left(e^{i\,\theta}+e^{-i\,\theta}\right)^3}{8}$$

```
(%i10) expand(exponentialize(f));
```
(%o10)
$$\frac{e^{3\,i\,\theta}}{8}+\frac{3\,e^{i\,\theta}}{8}+\frac{3\,e^{-i\,\theta}}{8}+\frac{e^{-3\,i\,\theta}}{8}$$

```
(%i11) demoivre(expand(exponentialize(f)));
```
(%o11)
$$\frac{i\,\sin(3\,\theta)+\cos(3\,\theta)}{8}+\frac{\cos(3\,\theta)-i\,\sin(3\,\theta)}{8}+\frac{3\,(i\,\sin\theta+\cos\theta)}{8}+\frac{3\,(\cos\theta-i\,\sin\theta)}{8}$$

```
(%i12) expand(demoivre(expand(exponentialize(f))));
```
(%o12)
$$\frac{\cos(3\,\theta)}{4}+\frac{3\,\cos\theta}{4}$$

```
(%i13) subst(th_E,expand(exponentialize(f)));
```
(%o13)
$$\frac{E^3}{8}+\frac{3\,E}{8}+\frac{3}{8\,E}+\frac{1}{8\,E^3}$$

```
(%i14) subst(th_E,expand(exponentialize(g)));
```
(%o14)
$$E^3+3\,E+\frac{3}{E}+\frac{1}{E^3}$$

```
(%i15) lpE(f);
```
(%o15)
$$\begin{pmatrix}\frac{1}{8} & 0 & \frac{3}{8} & 0 & . & \frac{3}{8} & 0 & \frac{1}{8}\end{pmatrix}$$

```
(%i16) lpE(g);
```
(%o16)
$$\begin{pmatrix}1 & 0 & 3 & 0 & . & 3 & 0 & 1\end{pmatrix}$$

```
(%i17) lpE(  ccos(th)^3);
(%o17)
```
$$\begin{pmatrix}1 & 0 & 3 & 0 & . & 3 & 0 & 1\end{pmatrix}$$

```
(%i18) lpE(  ccos(th));
(%o18)
```
$$\begin{pmatrix}1 & 0 & . & 1\end{pmatrix}$$

```
(%i19) lpE(3*ccos(th));
(%o19)
```
$$\begin{pmatrix}3 & 0 & . & 3\end{pmatrix}$$

```
(%i20) lpE(ccos(3*th));
(%o20)
```
$$\begin{pmatrix}1 & 0 & 0 & 0 & . & 0 & 0 & 1\end{pmatrix}$$

```
(%i21) lpE(ccos(3*th)+3*ccos(th));
(%o21)
```
$$\begin{pmatrix}1 & 0 & 3 & 0 & . & 3 & 0 & 1\end{pmatrix}$$

```
(%i22)
       lpE( ccos(th)   );
(%o22)
```
$$\begin{pmatrix}1 & 0 & . & 1\end{pmatrix}$$

```
(%i23) lpE( ccos(th)^2 );
(%o23)
```
$$\begin{pmatrix}1 & 0 & 2 & . & 0 & 1\end{pmatrix}$$

```
(%i24) lpE( ccos(th)^3 );
(%o24)
```
$$\begin{pmatrix}1 & 0 & 3 & 0 & . & 3 & 0 & 1\end{pmatrix}$$

```
(%i25) lpE( csin(th)     );
(%o25)
```
$$\begin{pmatrix}1 & 0 & . & -1\end{pmatrix}$$

```
(%i26) lpE( csin(th)^2   );
(%o26)
```
$$\begin{pmatrix}1 & 0 & -2 & . & 0 & 1\end{pmatrix}$$

```
(%i27) lpE( csin(th)^3   );
(%o27)
```
$$\begin{pmatrix}1 & 0 & -3 & 0 & . & 3 & 0 & -1\end{pmatrix}$$

```
(%i28) lpE( csin(2*th)   );
(%o28)
```
$$\begin{pmatrix}1 & 0 & 0 & . & 0 & -1\end{pmatrix}$$

```
(%i29) lpE( csin(2*th)^2 );
(%o29)
```
$$\begin{pmatrix}1 & 0 & 0 & 0 & -2 & . & 0 & 0 & 0 & 1\end{pmatrix}$$

```
(%i30)
```