Notes about Jacobs's book

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Cartesianness (Jacobs)

From [Jac99, p.27]:

1.1.3. Definition. Let $p : \mathbb{E} \to \mathbb{B}$ be a functor.
(i) A morphism $f : X \to Y$ in $\mathbb{E}$ is **Cartesian over** $u : I \to J$ in $\mathbb{B}$ if $pf = u$ and every $g : Z \to Y$ in $\mathbb{E}$ for which one has $pg = u \circ w$ for some $w : pZ \to I$, uniquely determines an $h : Z \to X$ in $\mathbb{E}$ above w with $f \circ h = g$. In a diagram:

$$\begin{tikzcd}
 & Z \arrow[dr, dashed, "h"'] \arrow[drr, "g"] & & \\
\mathbb{E} \arrow[ddd, "p"'] & & X \arrow[r, "f"'] & Y \\
 & pZ \arrow[dr, "w"'] \arrow[drr, "u \circ w = pg"] & & \\
\mathbb{B} & & I \arrow[r, "u"'] & J
\end{tikzcd}$$

We call $f : X \to Y$ in the total category $\mathbb{E}$ Cartesian if it is Cartesian over its underlying map pf in $\mathbb{B}$.

Cartesianness (Edrx)

Let $p : \mathbb{E} \to \mathbb{B}$ be our projection functor.
$\beta : Q \to R$ is *cartesian* when:

$$Q \xrightarrow{\beta} R \qquad \Bigg| \qquad \forall \qquad \Bigg| \qquad \exists!$$

$$\begin{tikzcd}
P \arrow[drr, "\gamma"] & & \\
& Q \arrow[r, "\beta"'] & R \\
A \arrow[dr, "f"'] \arrow[drr, "h"] & & \\
& B \arrow[r, "g"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[dr, "\alpha"'] \arrow[drr, "\gamma"] & & \\
& Q \arrow[r, "\beta"'] & R \\
A \arrow[dr, "f"'] \arrow[drr, "h"] & & \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

The details:

$$\forall \left(\begin{matrix} P, A, B, C, \gamma, f, g, h, \\ A = pP, B = pQ, C = pR, \\ g = p\beta, h = p\gamma, h = g \circ f \end{matrix} \right) . \exists! \left(\begin{matrix} \alpha, \\ \gamma = \beta \circ \alpha, \\ f = p\alpha \end{matrix} \right)$$

Cloven and split fibrations (Jacobs)

The definition of a fibration is of the form "for every x and y there is a z such that ...". This does not imply that we are given for each pair x, y an explicit z, unless we make use of the Axiom of Choice. The differences in the way the structure of a fibration may be given will concern us in this section. Briefly, a fibration is called **cloven** if it comes together with a choice of Cartesian liftings; and it is called **split** if it is cloven and the given liftings are well-behaved in the sense that they satisfy certain functoriality conditions. These fibrations behave more pleasantly, and therefore we prefer to work with fibrations in split form (if this is possible). Cloven and split fibrations give rise to so-called indexed categories $\mathbb{B}^{\mathrm{op}} \to \mathbf{Cat}$. These generalise set-valued functors (or presheaves) $\mathbb{B}^{\mathrm{op}} \to \mathbf{Cat}$.

We recall that a functor $\left(\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{P}\end{smallmatrix}\right)$ is a fibration if for every map $u : I \to J$ in the base category $\mathbb{B}$ and every object $X \in \mathbb{E}$ above J in the total category, there is a Cartesian lifting $\bullet \to X$ in $\mathbb{E}$. Assume now we choose for every such u and X a specific Cartesian lifting and write it as

$$u^*(X) \xrightarrow{\overline{u}(X)} X$$

(By Proposition 1.1.4 we can only choose up-to vertical isomorphisms.)

We claim that, having made such choices, every map $u : I \to J$ in $\mathbb{B}$ determines a functor u^* from the fibre $\mathbb{E}_J$ over J to the fibre $\mathbb{E}_I$ over I. (Note the direction!) The recipe for $u^* : \mathbb{E}_J \to \mathbb{E}_I$ is as follows.

- for an object $X \in \mathbb{E}_J$ one has $pX = J$ and so we take $u{*}(X) \in \mathbb{E}_I$ to be the domain of the previously determined Cartesian lifting $\overline{u}(X) : u^*(X) \to X$;

- for a map $f : X \to Y$ in $\mathbb{E}_J$, consider the following diagram in $\mathbb{E}$.

$$\begin{tikzcd} u^*(X) \arrow[r, "\overline{u}(X)"] \arrow[d, dashed, "u^*(f)"'] & X \arrow[d, "f"] \\ u^*(Y) \arrow[r, "\overline{u}(Y)"] & Y \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

  The composite $f \circ \overline{u}(X) : u^*(X) \to Y$ is above u, since f is vertical. Because $u(Y)$ is by definition the *terminal* lifting of u with codomain Y, there is a unique map $u^*(X) \dashrightarrow u^*(Y)$, call it $u^*(f)$, with $\overline{u}(Y) \circ u^*(f) = f \circ \overline{u}(X)$.

By uniqueness, $u{*}$ preserves identities and composition. Thus one obtains a functor $u^* : \mathbb{E}_J \to \mathbb{E}_I$. Such functors $u{*}$ are known under various names: as **reindexing** functors, **substitution** functors, **relabelling** functors, **inverse image** functors or sometimes also as **change-of-base** or as **pullback functors**. We mostly use the first two names.

$$\begin{pmatrix} \{\, a \in A \mid P(f(a)) \,\} \\ \big\downarrow \\ A \end{pmatrix} \longleftarrow\!\!\mapstochar \begin{pmatrix} \{\, b \in B \mid P(b) \,\} \\ \big\downarrow \\ B \end{pmatrix} \qquad \{\, a \in A \mid P(f(a)) \,\} \longleftarrow\!\!\mapstochar \{\, b \in B \mid P(b) \,\} \qquad P(f(a)) \longleftarrow\!\!\mapstochar P(b)$$

$$A \xrightarrow{\quad f \quad} B \qquad\qquad A \xrightarrow{\quad f \quad} B \qquad\qquad A \xrightarrow{f} B$$

$$\begin{tikzcd}
A\times_B(B\times_C D) \arrow[r, "\pi'"] \arrow[d, "\pi"'] & B\times_C D \arrow[r, "\pi'"] \arrow[d, "\pi"'] & D \arrow[d, "h"] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C
\end{tikzcd}$$

Poor man's string diagrams

From "On the missing...", section 4.2:
(https://anggtwu.net/LATEX/2022on-the-missing.pdf#page=18)

Let me use an example to discuss this. This is the definition of universal arrow in [CWM2, p.55], including the original diagram, modulo change of letters:

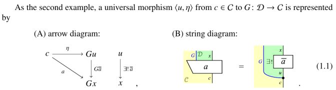
> **Definition.** If $R : \mathbf{B} \to \mathbf{A}$ is a functor and A an object of $\mathbf{A}$, a universal arrow from A to R is a pair (B, η) consisting of an object B of $\mathbf{B}$ and and arrow $\eta : A \to RB$ of $\mathbf{A}$ such that to every pair (B', g) with B' an object of $\mathbf{B}$ and $g : A \to RB'$ an arrow of $\mathbf{A}$, there is a unique arrow $f : B \to B'$ of $\mathbf{B}$ with $Rf \circ \eta = g$. In other words, every arrow h to R factors uniquely through the universal arrow η, as in the commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[d, equal] & RB \arrow[d, dotted, "Rf"] & B \arrow[d, dotted, "f"] \\ A \arrow[r, "g"] & RB', & B'. \end{tikzcd}$$

The definition itself goes only up to the "with $Rf \circ \eta = g$.", so let me ignore the part starting from "In other words", and draw a better "missing diagram" for the definition:

$$\begin{tikzcd} & A \arrow[d, "\eta"] \\ B \arrow[r, mapsto] & RB \\ \\ \mathbf{B} \arrow[r, "R"] & \mathbf{A} \end{tikzcd} \quad \Bigg| \; \forall \quad \begin{tikzcd} & A \arrow[d, "\eta"] \arrow[dd, bend left, "g"] \\ B \arrow[r, mapsto] & RB \\ B' \arrow[r, mapsto] & RB' \\ \mathbf{B} \arrow[r, "R"] & \mathbf{A} \end{tikzcd} \quad \Bigg| \; \exists! \quad \begin{tikzcd} & A \arrow[d, "\eta"] \arrow[dd, bend left, "g"] \\ B \arrow[r, mapsto] \arrow[d, "f"'] & RB \arrow[d, "Rf"] \\ B' \arrow[r, mapsto] & RB' \\ \mathbf{B} \arrow[r, "R"] & \mathbf{A} \end{tikzcd}$$

Compare that with the diagram 1.1 from Nakahira's "Diagrammatic category theory"
(https://arxiv.org/pdf/2307.08891),



and with my "Poor man's cell diagrams", from:
https://anggtwu.net/LATEX/2026riehl.pdf

The equation $\eta; Rf = g$ becomes:

$$\begin{bmatrix} & A \\ & \eta \\ R & \begin{bmatrix} B \\ f \\ B' \end{bmatrix} \end{bmatrix} = \begin{bmatrix} A \\ g \\ RB' \end{bmatrix}$$