

Notes on Valeria de Paiva's Technical Report (1991)

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<https://anggtwu.net/math-b.html>

Proposition 1

Proposition 1, p.4,

a.k.a: “composition needs checking”.

Lets suppose that we are in **Set**.

Let’s omit the types when they can be inferred from the diagrams, and let’s use dependent variables.

So $A \equiv (U, X, \alpha)$ is a triple

$(U \in \mathbf{Sets}, X \in \mathbf{Sets}, \alpha : U \times X \rightarrow \{0, 1\})$,

So $B \equiv (V, Y, \beta)$ is a triple

$(V \in \mathbf{Sets}, Y \in \mathbf{Sets}, \beta : V \times Y \rightarrow \{0, 1\})$,

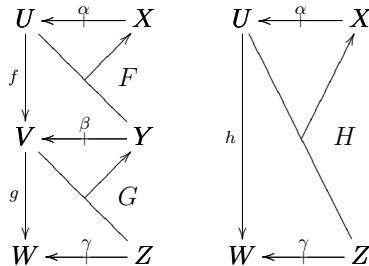
and a morphism $(f, F) : A \rightarrow B$

is actually a triple

$$\left(\begin{array}{l} f : U \rightarrow V, \\ F : U \times Y \rightarrow X, \\ [\forall(u, y). \alpha \rightarrow \beta] : \dots \end{array} \right)$$

in which the $[\forall(u, y). \alpha \rightarrow \beta]$ is a guarantee/witness for:

$$\begin{aligned} & \forall(u, y). \alpha \rightarrow \beta \\ \equiv & \forall(u, y). \alpha(u, x) \rightarrow \beta(v, y) \\ \equiv & \forall(u, y). \alpha(u, x(u, y)) \rightarrow \beta(v(u), y) \\ \equiv & \forall(u, y). \alpha(u, F(u, y)) \rightarrow \beta(f(u), y) \end{aligned}$$



We want to compose the

$(f, F, [\forall(u, y). \alpha \rightarrow \beta])$ with the

$(g, G, [\forall(v, z). \beta \rightarrow \gamma])$,

and obtain a:

$$(h, H, [\forall(u, z). \alpha \rightarrow \gamma]).$$

Proposition 1: compatibility conditions

Let's forget α , β and γ for a while,
and let's consider f , F , g and G as
compatibility conditions...

I will write $\begin{pmatrix} u, x, \\ v, y, \\ w, z \end{pmatrix}$, **with** commas,
for a tuple that only obeys

$u \in U$, $x \in X$, \dots , $z \in Z$,

and will write $\begin{pmatrix} u, x \\ v, y \\ w, z \end{pmatrix}$, **without** commas,

for $\begin{pmatrix} u, x, \\ v, y, \\ w, z \end{pmatrix}$ plus guarantees for the
"obvious" compatibility conditions,
that in this case are:

$$\begin{array}{l} [v = v(u)], \\ [x = x(u, y)], \\ [w = w(v)], \\ [y = y(v, z)] \end{array} \quad \text{i.e.,} \quad \begin{array}{l} [v = f(u)], \\ [x = F(u, y)], \\ [w = g(v)], \\ [y = G(v, z)] \end{array}$$

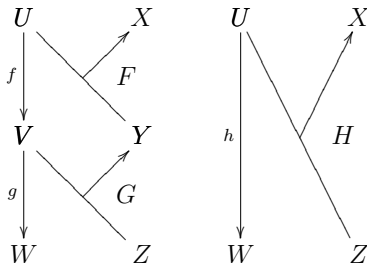
So:

$\begin{pmatrix} u, x \\ v, y \\ w, z \end{pmatrix}$ has 4 compatibility conditions,

$\begin{pmatrix} u, x \\ v, y \end{pmatrix}$ has 2 compatibility conditions,

$\begin{pmatrix} v, y \\ w, z \end{pmatrix}$ has 2 compatibility conditions,

$\begin{pmatrix} u, x \\ v, y \end{pmatrix}$ has 0 compatibility conditions...



...and $\begin{pmatrix} u, x \\ w, z \end{pmatrix}$ has these compatibility conditions:

$$\begin{aligned} w &= w(v(u)) \\ &= g(f(u)) \\ x &= x(u, y) \\ &= x(u, y(v, z)) \\ &= x(u, y(v(u), z)) \\ &= F(u, G(f(u), z)) \end{aligned}$$

Proposition 1: basic pullbacks

Now let's write $\begin{Bmatrix} u & x \\ v & y \\ w & z \end{Bmatrix}$
 for the space of all (compatible)
 tuples of the form $\begin{pmatrix} u & x \\ v & y \\ w & z \end{pmatrix}$,
 and do the same for all other
 combinations of letters...

We have the maps at the right,
 and the diamond at the middle
 is a pullback.

Note that:

α is defined on $\begin{Bmatrix} u & x \\ v & y \\ w & z \end{Bmatrix}$,

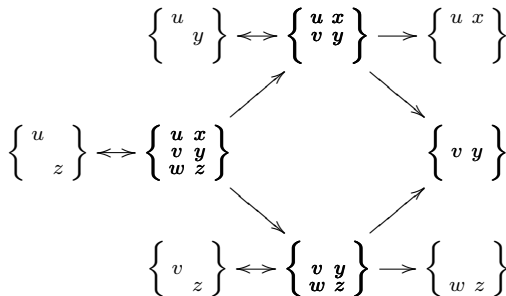
β is defined on $\begin{Bmatrix} v & y \\ w & z \end{Bmatrix}$,

γ is defined on $\begin{Bmatrix} u & x \\ w & z \end{Bmatrix}$,

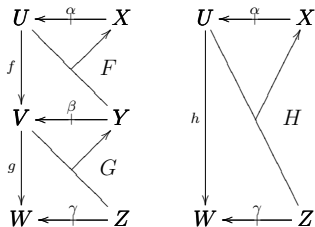
$\alpha \rightarrow \beta$ is defined on $\begin{Bmatrix} u & y \\ v & z \end{Bmatrix}$,

$\beta \rightarrow \gamma$ is defined on $\begin{Bmatrix} v & z \\ w & z \end{Bmatrix}$, and

$\alpha \rightarrow \gamma$ is defined on $\begin{Bmatrix} u & z \\ w & z \end{Bmatrix}$...



Proposition 1: the implication



Remember that want to compose the $(f, F, [\forall(u, y).\alpha \rightarrow \beta])$ with the $(g, G, [\forall(v, z).\beta \rightarrow \gamma])$, and obtain a: $(h, H, [\forall(u, z).\alpha \rightarrow \gamma])$.

We saw that

$h(u) = g(f(u))$ and

$H(u, z) = F(u, G(f(u), z))$.

We still need to check this:

$$\left(\begin{array}{l} (\forall ({}^u y).\alpha \rightarrow \beta) \wedge \\ (\forall ({}^v z).\beta \rightarrow \gamma) \end{array} \right) \rightarrow (\forall ({}^u z).\alpha \rightarrow \gamma)$$

Sketch of the proof:

$$\frac{\frac{\forall ({}^u y).\alpha \rightarrow \beta}{\forall ({}^u x y).\alpha \rightarrow \beta}}{\frac{\forall ({}^u x y).\alpha \rightarrow \beta \quad \frac{\forall ({}^v z).\beta \rightarrow \gamma}{\forall ({}^v w z).\beta \rightarrow \gamma}}{\forall ({}^u x y w z).\alpha \rightarrow \beta}} \rightarrow \frac{\forall ({}^u x y w z).\alpha \rightarrow \beta}{\forall ({}^u z).\alpha \rightarrow \gamma}$$