1 Examples of J-operators

2 The sandwich lemma

The *sandwich lemma* says that if $P \leq Q \leq P^*$, then $P =^* Q$:

$$\dfrac{P \leq Q \quad Q \leq P^*}{P^* = Q^*}\,Sa \quad := \quad \dfrac{\dfrac{P \leq Q}{P^* \leq Q^*}\,Mo \quad \dfrac{\dfrac{Q \leq P^*}{Q^* \leq P^{**}}\,Mo}{Q^* \leq P^*}\,M2}{P^* = Q^*}$$

Let's use the notation of closed interval, $[P, R]$, for the set of all points between P and R (in the partial order $\leq$ of the ZHA):

$$[P, R] := \{\, Q \in \Omega \mid P \leq Q \leq R \,\}$$

We saw that $Q \in [P, P^*]$ implies $P =^* Q$, which means that all points in $[P, P^*]$ are *-equal, and that the equivalence class $[P]^*$ contains at least the set $[P, P^*]$ – which is never empty, because $M1$ tells us that $P \leq P^*$, so

$$P \leq P \leq P^* \to P \in [P, P^*],$$
$$P \leq P^* \leq P^* \to P^* \in [P, P^*].$$

Let's call a non-empty set of the form $[P, R]$ a *lozenge*, even though it may inherit some dents from the ambient ZHA, as here:

$$[11, 33] = (diagram)$$

If $P_1^* = P_2^*$, then the equivalence class $[P_1]^*$ $(= [P_2]^*)$ contains the union of the lozenges $[P_1, P_1^*]$ and $[P_2, P_2^*] = [P_2, P_1^*]$. If $42^* = 33^* = 14^* = 56$, then

$$[42]^* = [33]^* = [14]^* \supseteq$$
$$[42, 56] \cup [33, 56] \cup [14, 56]$$

so each equivalence class $[P]^*$ is a union of lozenges or the form $[\_, P^*]$; and if $[P_1]^* = \{\, P1, P_2, \ldots, P_n \,\}$ then $[P_1]^* = [P_1, P_1^*] \cup [P_2, P_1^*] \cup \ldots \cup [P_n, P_1^*]$.

Let's go back to the example above, where $42^* = 33^* = 14^* = 56$. We have

$$\begin{array}{rcl} ((42\&33)\&14)^* &=& \\ (42\&33)^*\&14^* &=& \\ (42^*\&33^*)\&14^* &=& \\ (56\&56)\&56 &=& 56, \end{array}$$

so 42&33&14 is in the same equivalence class as 42, 33, and 14. We have $[12, 56] \subseteq [42]^* = [33]^* = [14]^*$, which means this shape:

$$(diagram)$$

(...)

$$\dfrac{P =^* Q}{P\&Q =^* P}\ ECC\& \quad := \quad \dfrac{\dfrac{P^* = Q^*}{P^*\&Q^* = P*}}{(P\&Q)^* = P^*}\ M3$$

$$\dfrac{P =^* Q}{P =^* P \vee Q}\ ECC\vee \quad := \quad \dfrac{\dfrac{}{P \leq P \vee Q} \quad \dfrac{\dfrac{}{P \leq P^*}\ M1 \quad \dfrac{\dfrac{}{Q \leq Q^*}\ M1 \quad Q^* = P^*}{Q \leq P^*}}{P \vee Q \leq P^*}}{P^* = (P \vee Q) *}\ Sa$$

3 How J-operators interact with connectives

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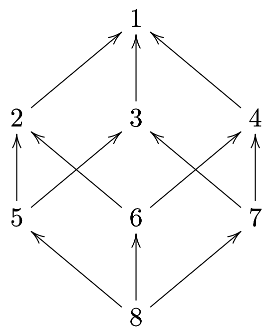
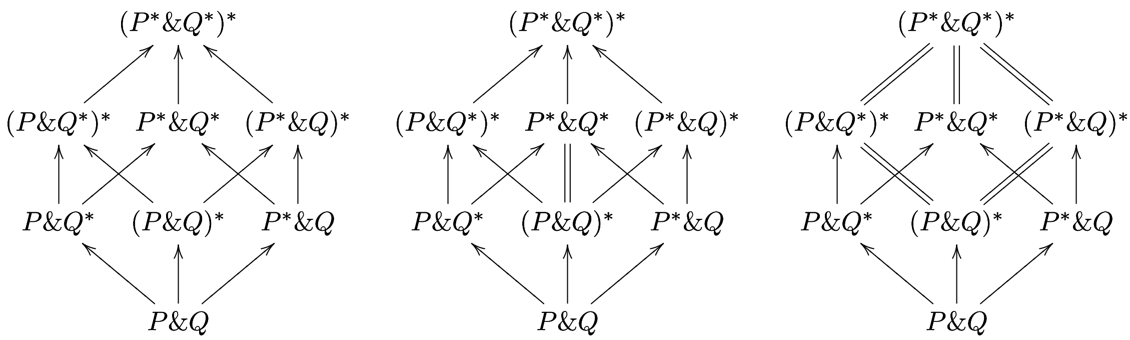
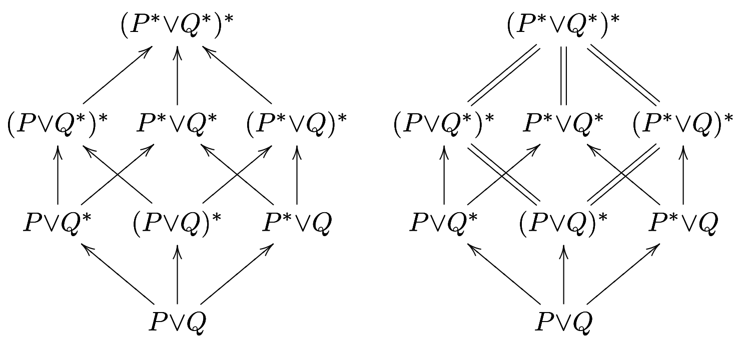
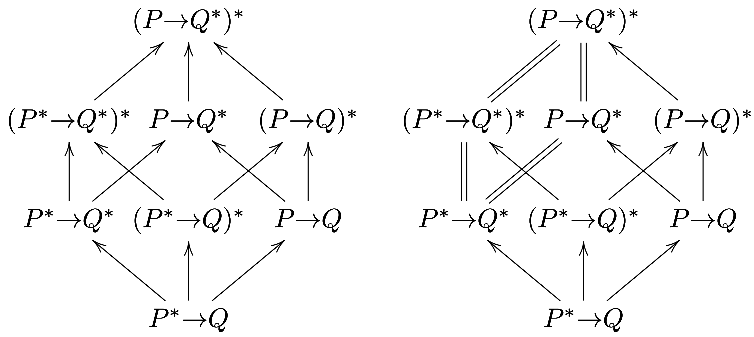
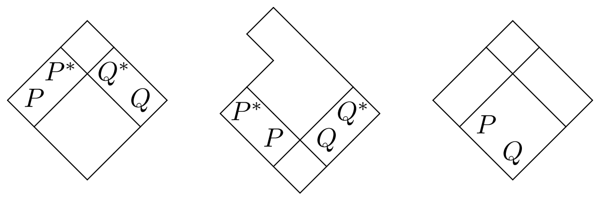










$$P^* \& Q^* \overset{(M3)}{=} (P\&Q)^*$$
$$(P^*\&Q^*)^* \overset{(M3)}{=} (P\&Q)^{**} \overset{(M2)}{=} (P\&Q)^*$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \le P\lor Q}\,\iota}{P^* \le (P\lor Q)^*}\,Mo \qquad \dfrac{\dfrac{}{Q \le P\lor Q}\,\iota'}{Q^* \le (P\lor Q)^*}\,Mo}{\dfrac{P^*\lor Q^* \le (P\lor Q)^*}{\dfrac{(P^*\lor Q^*)^* \le (P\lor Q)^{**}}{(P^*\lor Q^*)^* \le (P\lor Q)^*}\,M2}\,Mo}\,[,]}{}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P{\to}Q^* \le P{\to}Q^*}\,\text{id}}{(P{\to}Q^*)\&P \le Q^*}\,\flat}{((P{\to}Q^*)\&P)^* \le Q^{**}}\,Mo}{((P{\to}Q^*)\&P)^* \le Q^*}\,M2}{(P{\to}Q^*)^*\&P^* \le Q^*}\,M3}{(P{\to}Q^*)^* \le P^*{\to}Q^*}\,\sharp$$

4 There are no Y-cuts

$a\,{}^{\,12}_{34\ 56}{}_{78}\,b$

$a\,{}^{\,12}_{34\ 56}{}_{78}\,b$

A *small lozenge* in a ZHA is four points like {...} or {...} in the figure below; we will write these small lozenges as ${}^{\,12}_{34\ 56}{}_{78}$ and ... .

When the points in a small lozenge L belong to four different equivalence classes, like in ..., we say that L is an *X-cut*; when they belong to three different equivalence classes, like ... or ..., a *Y-cut*; two different equivalence classes, like ... or ..., an *I-cut*; one equivalence class, a *no-cut*.

We saw in section 2 that equivalence classes are lozenges. Let's now see that the frontiers between equivalence classes can't be more irregular than the examples in section 1.

Every way of dividing a ZHA A into lozenges induces an operation $\cdot^* : A \to A$ that takes each element to the top element of its equivalence class; this '$\cdot^*$' obeys axiom M1 and M2...

We need two lemmas:

5 Partitions into contiguous classes

A partition of $\{0,\ldots,n\}$ into contiguous classes (a "picc") is one in which this holds: if $a,b,c \in \{0,\ldots,n\}$, $a<b<c$ and $a \sim c$, then $a \sim b \sim c$. So, for example, $\{\{0,1\},\{2\},\{3,4,5\}\}$ is a partition into contiguous classes, but $\{\{0,2\},\{1\}\}$ is not.

A partition of $\{0,\ldots,n\}$ into contiguous classes induces an equivalence relation $\cdot \sim_P \cdot$, a function $[\cdot]_P$ that returns the equivalence class of an element, a function

$$\begin{array}{rrcl} \cdot^P: & \{0,\ldots,n\} & \to & \{0,\ldots,n\} \\ & a & \mapsto & \max\,[a]_P \end{array}$$

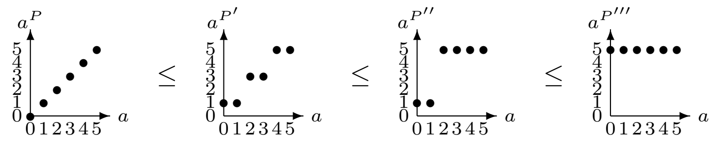
that takes each element to the top element in its class, a set $\mathsf{St}_P := \{\, a \in \{0,\ldots,n\} \mid a^P = a \,\}$ of the "stable" elements of $\{0,...,n\}$, and a graphical representation with a bar between a and $a+1$ when they are in different classes:

$$01|2|345 \quad \equiv \quad \{\{0,1\},\{2\},\{3,4,5\}\},$$

which will be our favourite notation for partitions into contiguous classes from now on.

6 The algebra of piccs

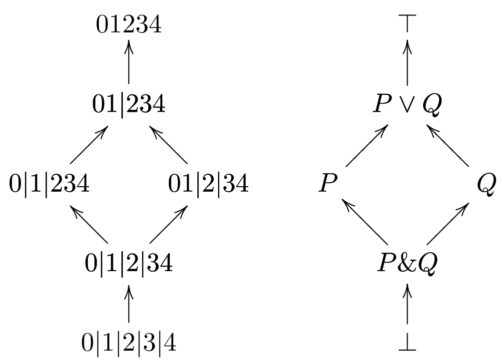
When P and P' are two piccs on $\{0,\ldots,n\}$ we say that $P \le P'$ when $\forall a \in \{0,\ldots,n\}.a^P \le a^{P'}$. The intuition is that $P \le P'$ means that the graph of the function $\cdot^P$ is under the graph of $\cdot^{P'}$:



$$\begin{array}{ccccccc} 0|1|2|3|4|5 & \le & 01|23|45 & \le & 01|2345 & \le & 012345 \\ P & \le & P' & \le & P'' & \le & P''' \end{array}$$

This yields a partial order on piccs, whose bottom element is the identity function $0|1|\ldots|n$, and the top element is $01\ldots n$, that takes all elements to n.

It turns out that the piccs form a (Heyting!) algebra, in which we can define $\top$, $\bot$, &, $\vee$, and even $\to$.



The operations & and $\vee$ are easy to understand in terms of cuts (the "|"s):

$$\begin{array}{rcl} \mathsf{Cuts}(P \vee Q) & = & \mathsf{Cuts}(P) \cap \mathsf{Cuts}(Q) \\ \mathsf{Cuts}(P \& Q) & = & \mathsf{Cuts}(P) \cup \mathsf{Cuts}(Q) \end{array}$$

The stable elements of a picc on $\{0,\ldots,n\}$ are the ones at the left of a cut plus the n, so we have:

$$\begin{array}{rcl} \mathsf{St}(P \vee Q) & = & \mathsf{St}(P) \cap \mathsf{St}(Q) \\ \mathsf{St}(P \& Q) & = & \mathsf{St}(P) \cup \mathsf{St}(Q) \end{array}$$

Here is a case that shows how $\cdot^{P\vee Q}$ can be problematic. The result of $a^{P\vee Q}$ must be stable by both P and Q. Let:

$$\begin{array}{rcl} E &:=& 01|2|34|56|789 \\ O &:=& 01|23|45|6|789 \\ E\vee O &=& 01|23456|789 \\ E\&O &=& 01|2|3|4|5|6|789 \end{array}$$

We can define $a^{P\&Q} := a^P \& a^Q$, and this always works. But $a^{P\vee Q} := a^P \vee a^Q$ does not, we may have to do something like iterating the two functions many times:

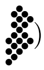
$$\begin{array}{rcl} O\vee E &=& 0123456 \\ 2^{O\vee E} &=& 6 \\ a^{O\vee E} &=& (((a^O)^E)^O)^E \end{array}$$

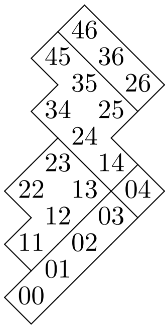
$$O = 01|23|45|6 \qquad\qquad E = 0!12|34|56$$

$$\begin{array}{rcl} O\&E &=& 0|1|2|3|4|5|6 \\ 2^{O\&E} &=& 2^O \& 2^E = 3\&2 = 2 \\ a^{O\&E} &=& a^O \& a^E \end{array}$$

The "$\to$" in the algebra of piccs will not be relevant to us, so we will not discuss it.

7 ZQuotients

A *ZQuotient* for a ZHA with top element 46 is a partition of $\{0,\ldots,4\}$ into contiguous classes (a "partition of the left wall"), plus a partition of $\{0,\ldots,6\}$ into contiguous classes (a "partition of the right wall"). Our favourite short notation for ZQuotients is with "/"s and "\"s, like this, "4321/0 0123\45\6", because we regard the cuts in a ZQuotient as diagonal cuts on the ZHA. The graphical notation is this (for 4321/0 0123\45\6 on ):



which makes clear how we can adapt the definitions of $\cdot \sim_P \cdot$, $[\cdot]_P$, $\cdot^P$, St_P, which were on (one-dimensional!) PICCs in section 5, to their two-dimensional counterparts on ZQuotients. If P is the ZQuotient of the figure above, then:

$$\begin{array}{rcl} 34 \sim_P 25 & \text{is} & \text{true}, \\ 23 \sim_P 24 & \text{is} & \text{false}, \\ [12]_P &=& \{11,12,13,22,23\}, \\ 22^P &=& 23, \\ \mathsf{St}_P &=& \{03,04,23,45,46\}. \end{array}$$

8 The algebra of ZQuotients

The ideas of the last section apply to ZQuotients, with a few adjustments. The

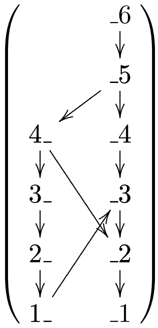
$$\mathsf{St}(P\&Q) = \mathsf{St}(P) \cup \mathsf{St}(Q)$$

would mean, for $P = 54/32/10$ and $Q = 01/23/45$, that

$$\mathsf{St}(P_cuts) = St($$

9 2-Column graphs

A 2-column graph is a graph like the one in the picture below, composed of a *left column* $4_- \to 3_- \to 2_- \to 1_- \to 0_-$, a *right column* $6_- \to 5_- \to 4_- \to 3_- \to 2 \to 1_- \to 0_-$, and some intercolumn arrows: $4_- \to {}_-2$, $1_- \to {}_-3$ going from the left column to the right column, ${}_-5 \to 4_-$ going from the right column to the left column.



A compact way to specify a 2-column graph is this: (left height, right height, left-to-right arrows, right-to-left arrows). In the graph of the example this is $(5, 7, \{4_- \to {}_-2, 1_- \to {}_-3\}, \{4_- \leftarrow {}_-5\})$.

We need to attribute a precise meaning for 0_-, …, 4_- and ${}_-0$, …, ${}_-6$. For the moment let's use this one, in which they are points in $\mathbb{N}^2$ with $x = 0$ or $x = 2$:

$$\begin{array}{cc}
\vdots & \vdots \\
3_- := (0,3) & {}_-4 := (2,4) \\
2_- := (0,2) & {}_-2 := (2,2) \\
1_- := (0,1) & {}_-1 := (2,1) \\
0_- := (0,0) & {}_-0 := (2,0)
\end{array}$$

Let's write $(5, 7, \{\ldots\}, \{\ldots\})^*$ for the transitive-reflexive closure of of the graph, and $\mathcal{O}(5, 7, \{\ldots\}, \{\ldots\})$ for its order topology.

Every open set of a 2-column graph is made of a pile of a elements at the bottom of the left column and a pile of b elements at the bottom of the right column. If we write these "2-piles" as ab,

$$\left(\begin{smallmatrix} & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \equiv \{1_-, {}_-3, {}_-2, {}_-1\} \equiv 13$$

the connection with ZHAs becomes clear:

$$\mathcal{O}(3, 4, \{\}, \{\}) \quad = \quad \begin{array}{c}
34 \equiv \left(\begin{smallmatrix} & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \\
33 \equiv \left(\begin{smallmatrix} & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \quad 24 \equiv \left(\begin{smallmatrix} & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \\
32 \equiv \left(\begin{smallmatrix} & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \quad 23 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \quad 14 \equiv \left(\begin{smallmatrix} & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \\
31 \equiv \left(\begin{smallmatrix} & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \quad 22 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \quad 13 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \quad 04 \equiv \left(\begin{smallmatrix} & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{smallmatrix}\right) \\
30 \equiv \left(\begin{smallmatrix} & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{smallmatrix}\right) \quad 21 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \quad 12 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \quad 03 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{smallmatrix}\right) \\
20 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{smallmatrix}\right) \quad 11 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{smallmatrix}\right) \quad 02 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{smallmatrix}\right) \\
10 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \quad 01 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \\
00 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)
\end{array}$$

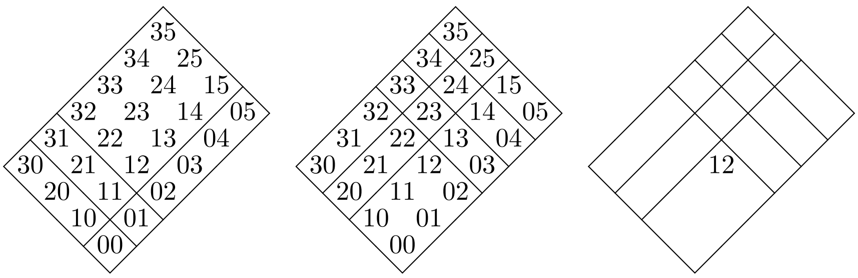
What happens if we add a left-right arrow, for example $2\_ \to \_2$?

$$(3,4,\{2\_ \to \_2\},\{\}) \qquad = \qquad \left(\begin{array}{ccc} & & \_4 \\ & & \downarrow \\ 3\_ & & \_3 \\ \downarrow & & \downarrow \\ 2\_ & \longrightarrow & \_2 \\ \downarrow & & \downarrow \\ 1\_ & & \_1 \end{array}\right)$$

Its effect is to make all the 2-piles of the form $\left(\begin{smallmatrix} & ? \\ ? & ? \\ 1 & 0 \\ ? & ? \end{smallmatrix}\right)$ *non*-open — more simply, the ones of the form $\left(\begin{smallmatrix} & 0 \\ ? & 0 \\ 1 & 0 \\ 1 & ? \end{smallmatrix}\right)$, which are $20 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $21 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, $30 \equiv \left(\begin{smallmatrix} & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $31 \equiv \left(\begin{smallmatrix} & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. We have:

$$\mathcal{O}(3,4,\{2_ \to _2\},\{\}) \qquad = \qquad \begin{array}{c}
34 \equiv \left(\begin{smallmatrix} & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \\
33 \equiv \left(\begin{smallmatrix} & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \quad 24 \equiv \left(\begin{smallmatrix} & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \\
32 \equiv \left(\begin{smallmatrix} & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \quad 23 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \quad 14 \equiv \left(\begin{smallmatrix} & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \\
22 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \quad 13 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \quad 04 \equiv \left(\begin{smallmatrix} & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{smallmatrix}\right) \\
12 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \quad 03 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{smallmatrix}\right) \\
11 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{smallmatrix}\right) \quad 02 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{smallmatrix}\right) \\
10 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \quad 01 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \\
00 \equiv \left(\begin{smallmatrix} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)
\end{array}$$

Graphica
like this,
we see that a topology like $\mathcal{O}(3,4,\{\},\{\})$ is a ZHA:



34
33 24
32 23 14
31 22 13 04
30 21 12 03
20 11 02
10 01
00